Book Review

Essential Mathematical Methods 3&4 CAS

Enhanced TI-N/CP Versions (2012)

Cambridge University Press
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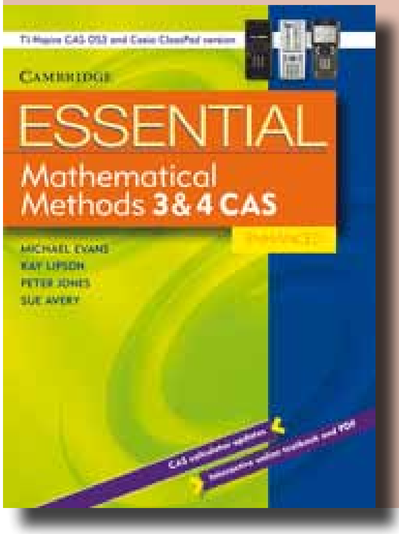

Burkard Polster, Marty Ross and David Treeby —
Burkard.Polster@monash.edu, martinirossi@gmail.com
david.treeby@gmail.com

Burkard Polster currently lectures at Monash University. He is the author of *Q.E.D.: Beauty in Mathematical Proof*, and together with Marty Ross writes the Maths Masters column for *The Age* newspaper. Marty Ross is a mathematical nomad. His hobby is smashing calculators with a hammer. David Treeby currently teaches at Presbyterian Ladies College in Melbourne. He delights in exploring beautiful mathematics with his students.

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In 2008, two of us, Marty and David, wrote a review of Year 9 Victorian mathematics texts (*Vinculum*, Issue 4, 2008). Our purpose was to highlight the poor treatment of mathematics in these texts, and the manner in which this treatment reflected and created a poor curriculum. Our goal here is similar.

For Year 9, there was no accepted superior text, making a comparative review appropriate. By contrast, for the VCE subject of Mathematical Methods 3 & 4 (Methods), the Cambridge Essentials text (*Cambridge*) is widely recognised as mathematically the most sound. Consequently, our review is devoted to this single text.

We shall discuss the treatment of various topics in *Cambridge*. We then make some general comments about *Cambridge* and its relation to the Methods curriculum. In particular, we shall indicate why, despite our strong criticisms, we regard *Cambridge* as the best available text.

Functions

This material is intended largely as revision, but our concerns here are indicative of our general concerns with *Cambridge*: the approach is too formal and pedantic to result in a helpful summary, and not sufficiently formal or consistent to serve as a rigorous presentation.

For example, for the composition $f \circ g$ of two functions to be defined, *Cambridge* follows the Methods syllabus in demanding that the range of g be a subset of the domain of f. This is needlessly restrictive, since one can always simply define $f \circ g$ where possible, on $\{x \colon g(x) \in \text{dom}(f)\}$.

Indeed, *Cambridge* adopts such simple conventions for the algebra of functions. The sum $f + g$, for instance, is taken to be defined on the intersection of the domains of f and g. However, *Cambridge* then muddies the waters by demanding that this intersection be non-empty. This trivial case, which would be better left unmentioned, requires no exceptional treatment: one might simply obtain the empty function, with empty domain.

A preoccupation with trivial cases also clutters the treatment of applied problems. It is repeatedly emphasised that the width of a rectangle or whatnot must be strictly positive, resulting in the domain of the relevant function being taken as an open interval. There is seldom any point to this; in most instances the function can be naturally (i.e. continuously) extended to the endpoints, which permits cleaner working on a closed domain.

Linear equations and matrices

Linear algebra is reasonably well presented in *Cambridge*, but there is no satisfactory approach to a topic that the Methods syllabus makes fundamentally pointless: there is sufficient material to confuse, and not enough to demonstrate why one would bother. The focus is on supposedly practical applications of matrices, but the practicality is more imagined than real.

Emphasis is placed upon the use of matrix inverses to solve linear systems of equations, even though this is not a practical method. The impracticality is highlighted and heightened by using CAS to magically produce the inverses.

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Matrices are later used to analyse linear transformations of relations and functions, resulting in material that is ugly and aimless. Everything done here with matrices can be done more simply and elegantly without.

Polynomials and powers

The revision of polynomials in *Cambridge* is overly formal but standard. There is included the confected guess-a-root method for solving cubics, and it is given a remarkable twist: to factorise $x^3 - 11x^2 - 125x + 1287$ by first employing a calculator to find a root is almost perfect in its pointlessness.

The material on general powers tends to fussy detail at the expense of clear and general principles. For instance, we fail to understand the concern for the intersections of the graphs of x^r and x^s, which seems neither interesting nor illuminating.

There is insufficient emphasis that $x^{\frac{1}{n}}$ for $n \in \mathbb{N}$ has no independent meaning, that it is simply useful notation for the nth root of x. Then, the definition of $x^{\frac{p}{q}}$ for $\frac{p}{q} \in \mathbb{Q}$ leads to one of the gravest errors in *Cambridge*. It is implied that $x^{\frac{p}{q}}$ can only be defined if the fraction $\frac{p}{q}$ is in lowest terms. To support this, it is purportedly proved that $(-1)^{\frac{2}{6}} \neq (-1)^{\frac{1}{3}}$: supposedly $(-1)^{\frac{2}{6}} = ((-1)^2)^{\frac{1}{6}} = 1^{\frac{1}{6}} = 1$, whereas $(-1)^{\frac{1}{3}} = -1$.

All of this is nonsense, fundamentally ignoring the nature of equality: since $\frac{2}{6} = \frac{1}{3}$, it simply must be true that $(-1)^{\frac{2}{6}}$ is defined and equal to $(-1)^{\frac{1}{3}}$. The fallacious proof above illustrates the danger in treating "index laws" as God-given commandments, rather than as the consequence of carefully crafted definitions.

Finally, there is, or at least there should be, the issue of non-rational powers. After the rational gymnastics, one would at least expect the acknowledgment that general powers are difficult to define. However, all that is provided is a misleading graph of a^x, which appears out of nowhere.

The number *e*

Cambridge introduces the number e as "Euler's number", apparently named after "an eighteenth century Swiss mathematician"; we feel compelled to ask which one. In fact, until recently the number has never been referred to as Euler's number, it is seldom if ever referred to as such by research mathematicians, and there is no particular reason to do so. Though the great Leonhard Euler was the first to use the symbol e to refer to the number, use of the number itself long predated Euler.

The number e is correctly defined as $\lim_{n\to\infty}(1+\frac{1}{n})^n$ but the limit is poorly motivated. It is natural and beautiful to connect the limit to the concept of continuously compounded interest, but instead *Cambridge* offers an arcane example of growing marbles. Later *Cambridge* attempts to motivate e with models of continuous population growth, but no proper definition is provided and the motivation is flawed: though continuous models can be simple and illuminating, population growth is fundamentally a discrete phenomenon, in which e plays no intrinsic role.

Circular functions

Though the material is generally well presented, *Cambridge* overemphasises the graphs of circular functions at the expense of, well, the circles. Little connection is made between the symmetries of the graphs and the unit circle. Then, there is the inappropriate suggestion to solve an equation such as $\sin\theta = \frac{1}{2}$ by referring to the graph. Similarly, the formulas for the general solution of an equation such as $\sin x = a$ border on unreadable; the simple symmetries of the unit circle have been lost in excessive notation.

Differentiation

The derivative in *Cambridge* is well motivated, and the limit definition of the derivative and the first principles calculations are clearly presented. However, this good introduction is followed by less satisfactory material.

The algebraic aspect of differentiation is simple: one has the derivatives of basic functions (x^n, $\sin x$, etc.), together with the rules for the differentiation of newly constructed functions (the product, quotient, chain and inverse rules). All that ever changes is the occasional appearance of a new basic function (such as e^x).

This simple structure is lost in *Cambridge*. What is presented instead is a morass of needless special

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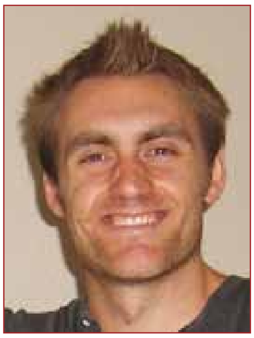

David Treeby

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cases, the derivatives of cos kx, $e^{f(x)}$ and so on. This gives the false impression that the student must be familiar with numerous special formulas.

It is pleasing that *Cambridge* includes proofs of most of the differentiation formulas. However, with the exception of the proof of the product rule, which is nicely geometric, the proofs tend to be poorly organised, inelegant and incomplete. The quotient rule, for example, is more naturally proved by first computing the derivative of $\frac{1}{g(x)}$, which is easy, and then applying the product rule to $f(x)\cdot\frac{1}{g(x)}$.

The establishment of the derivative of x^r is long, painful and incomplete; moreover, the simple statement of the result should be made at the outset, not after special cases scattered over twenty pages.

It is disappointing if unsurprising to see calculators dominate discussion of the derivative of e^x, but it is astonishing to see a similar treatment of the circular functions. The fundamental limit $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ can be established by simple geometry, and there is no justification for replacing this beautiful argument with calculator games.

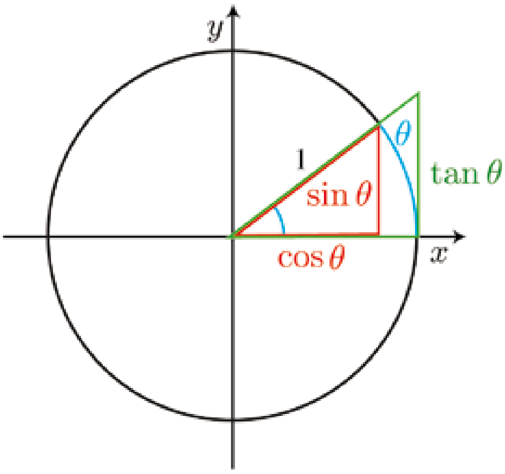

Figure 1. Comparing the area of the sector with the areas of the right-angled triangles shows that $\frac{\sin\theta\cos\theta}{2}\le\frac{\theta}{2}\le\frac{\tan\theta}{2}$. *This leads to* $\cos\theta\le\frac{\sin\theta}{\theta}\le\frac{1}{\cos\theta}$, *from which it follows that* $\frac{\sin\theta}{\theta}\to 1$.

Limits

It is fundamental that $\lim_{x\to a} f(x)$ is independent of the value of (or the existence of) $f(a)$. However, *Cambridge* suggests the opposite: "With many functions $f(a)$ is defined, so to evaluate the limit we simply substitute the value a into the rule for the function". The statement is false, and the examples that follow the statement are correspondingly confused.

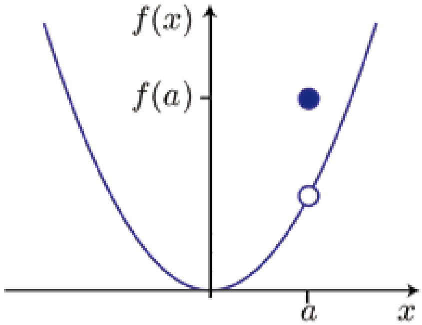

Figure 2. The limit $\lim_{x\to a} f(x)$ *needn't bear any relationship to* $f(a)$.

With limits in hand, the definition of continuity is simple: $f(x)$ is continuous at a if $\lim_{x\to a} f(x)=f(a)$.[1] Unfortunately, *Cambridge* defines continuity in terms of left and right limits. Though logically equivalent, this definition is badly misleading; it encourages the automatic consideration of one-sided limits when, outside of the artificial world of Methods, this is typically not required.

Cambridge never clarifies the importance of continuity. It would have been natural, for example, to mention continuity when considering roots of (at least) odd degree polynomials. There is a short discussion of maxima and minima of continuous functions on a closed interval $[a, b]$, but there is no statement of the theorem that such extrema exist. Moreover, the definitions given of the extrema are incorrect; what are actually defined are upper and lower bounds.

Cambridge remarks that a differentiable function is necessarily continuous, but the simple proof is

[1] That is, continuity exactly characterises those limits that can be evaluated by substitution. However, this provides no method of evaluating a limit unless we already know (or assume) that the relevant function is continuous.

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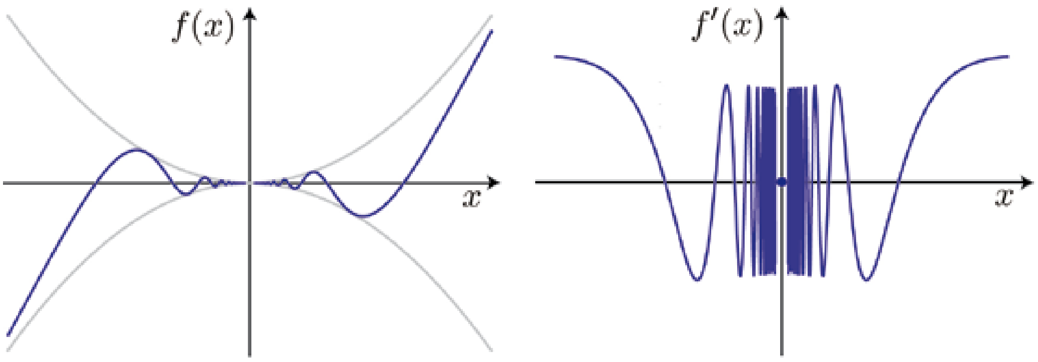

Figure 3. An everywhere differentiable function for which $\lim_{x\to 0} f'(x)$ *does not exist.*

absent. This is followed by a confused section on the differentiability of "hybrid functions".[2] Differentiability of a function $f(x)$ at a requires the existence of $\lim_{h\to 0}\frac{f(a+h)-f(h)}{h}$, which can be broken down to two one-sided limits if required. However, for a hybrid function *Cambridge* never properly considers these fundamental limits, offering a "test" for differentiability: if $f(x)$ is continuous at a and if the one-sided limits $\lim_{x\to a^{\pm}} f'(x)$ are equal.

This test is needlessly circuitous and, importantly, is not equivalent to the definition of differentiability. It is true, though far from obvious, that a function satisfying *Cambridge*'s test will be differentiable at a.[3] However, the converse is false. For example, consider the function $f(x)=x^2\sin\left(\frac{1}{x}\right)$, with $f(0) = 0$. It can readily be shown that $f'(0) = 0$, but $\lim_{x\to 0} f'(x)$ is undefined.

Integration

Our main concerns with *Cambridge*'s presentation of integration mirrors the concerns we raised with the differentiation material: there is a jungle of particular integrals, where a few general principles would be preferable.

The standard introduction to integration in terms of Riemann approximations is well presented, although we would have preferred the iconic and historically significant function x^2 to the needlessly exotic $9 - 0.1x^2$. The marriage of antidifferentiation with integration is confusing, with the same integral notation used many pages before the connection is clarified. It is pleasing to see a proof of the fundamental theorem included, even if it could be more intuitively and elegantly presented.

Probability and statistics

Here, we shall consider just one example, representative of our general concerns: the mean of the standard normal distribution. Calculating the mean amounts to evaluating the integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{\frac{-x^2}{2}}\,dx,$$

and there are at least three reasonable approaches to this integral. First of all, one can integrate by substituting $u = x^2$, though that approach is forbidden in Methods. A second approach is to simply "guess" the antiderivative and then check by differentiation. However, if we are willing to accept the existence of the improper integral, there is a third, very simple approach: we just note that the integrand is an odd function and is integrated over a symmetric domain.

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[2] It should be noted that the concept of a 'hybrid function' is psychologically helpful but mathematically meaningless. The point is, a function is formally defined by its outputs, not by whatever rule is used to obtain those outputs.

[3] This can be proved by an application of the mean value theorem.

In our opinion, definitions are not consistently labelled as such; calculators and calculations supplant arguments; proofs tend to be flawed and inelegant; clear and general principles are lost in jungles of particular examples; and the small attempts at developing intuition are tripped up by pedantry and clumsy language.

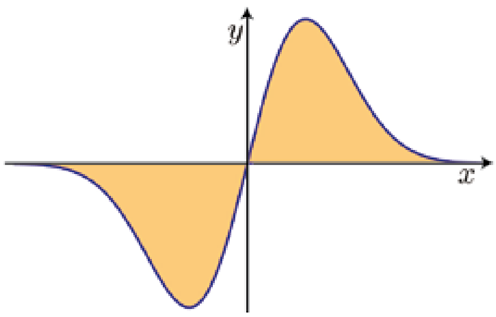

Figure 4. Evaluating the mean of the standard normal distribution.

Cambridge approaches the integral differently. To begin, a calculator is employed to draw the graph of the probability density function, and then "it can be seen from the graph" (somehow) that the mean is 0. *Cambridge* then directly considers the integral for the mean: they have a calculator perform the integration.

Conclusion

A textbook is constrained by the subject for which it is written, and VCE Methods is a poor subject. It is unmotivated and aimless, and dominated by canned techniques. The little formal mathematics it contains is prissy and pointless, more ritual than rigour. And then there's CAS, injecting a perverse religiosity into everything it touches.

How should texts (and teachers) deal with this depressing reality? Definitely not with idealism: students' VCE results are too important for anyone to do other than play the game. So, the fundamental responsibility of a text is to teach to the game, to provide solid practice on the predictable techniques required in SACs and exams.

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To this end, and despite our criticisms, we consider *Cambridge* to be the best text available. The exercises are in practice the most important part of a text, and, as a whole, the exercises in *Cambridge* are excellent.

But a Methods text can seek to offer more, either from attempting to promote a more solid understanding of the material, or out of a general sense of integrity. A text can provide intuition or rigorous background, or both. Our belief is that *Cambridge* is the only Methods text that makes any such serious attempt, and we praise *Cambridge* for the attempt.

However, we also believe that *Cambridge* fundamentally fails in the attempt. The problems detailed above are, in our opinion, representative of systemic problems: definitions are not consistently labelled as such; calculators and calculations supplant arguments; proofs of theorems tend to be flawed and inelegant; clear and general principles are lost in jungles of particular examples; and the small attempts at developing intuition are tripped up by pedantry and clumsy language.

As we have suggested, to some extent these flaws are probably inevitable, the consequence of attempting to follow the directionless Methods curriculum. Other deficiencies appear to be the result of insufficient thought or care.

Whatever the causes, our fundamental concern is whether VCE students will see and study good mathematics through exposure to this best text. Our belief is that they will not. Good mathematics is characterised by its beauty and clarity: *Cambridge* possesses too little of either.