BURKARD POLSTER

YEA WHY TRY HER RAW WET HAT

A Tour of the Smallest Projective Space

Remember our dear colleague the Rev. Thomas P. Kirkman, best known for the following classical problem in combinatorics?

> **Kirkman's Schoolgirls Problem**
> Fifteen schoolgirls walk each day in five groups of three. Arrange the girls' walks for a week so that, in that time, each pair of girls walks together in a group just once.

Recently, I received a letter from Thomas which I feel obliged to share with you. It contains the reasons why Thomas decided to flee our world, a pictorial solution to his problem, and a lot of illustrations which cannot be found in any textbook. It also contains some of Thomas's most recent insights into the nature of his problem, and some sinister implications of his results of which I think all of you should be aware.

The Nightmare

Dear friend, . . . For many years I had a suspicion that there is something fundamentally wrong with our—that is, your—universe. In 1851 I finally figured out what! I woke up in the middle of the night and the only thing I could remember was this nightmare of me falling into some kind of bottomless pit (Fig. 1). Sounds familiar? As usual, I had fallen asleep thinking about geometry. Probably it was because of this that I woke up in a mathematical frame of mind, thinking, "Let us assume that I am a flatlander and I wake up from the flat equivalent of my falling nightmare. Then the last picture from my dream that I remember will look like this."

At this point, it dawned on me that the fact that parallel lines do not meet is the reason for my flat counterpart having this terrible dream. Similarly, it is because there are parallel planes that I keep waking up in the middle of the night. I had heard rumours of certain *projective spaces* in which two planes always intersect in a line, and I realized that I had to travel to one of these distant worlds to escape the terrible nightmare. As you know, my journey was successful, and I have been living a carefree life in the smallest projective space; a life dedicated to research in combinatorial mathematics. That is, carefree until 2 weeks ago, when I woke up from a nightmare again! Not the same as before. It all has to do with my research connected with the problem which is named after me. Let me explain.

The Smallest Projective Plane

Remember that a projective plane is a point-line geometry that satisfies the following axioms.

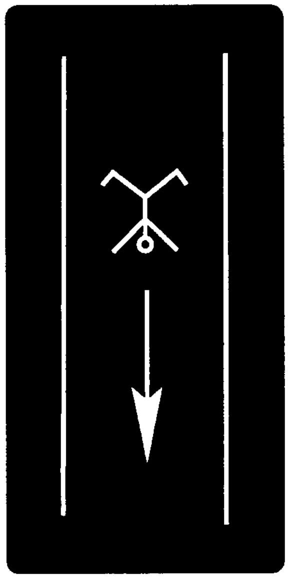

Figure 1. The dream.

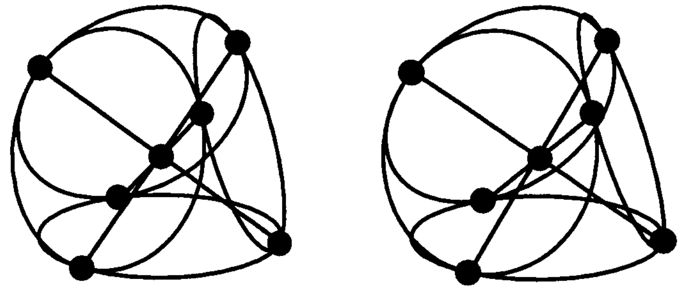

Figure 3. A stereogram of the Fano plane.

- Two distinct points are contained in a unique line.
- Two distinct lines intersect in a unique point.
- Every point is contained in at least three lines and every line contains at least three points.

Associated with every field K is a *classical projective plane* whose points and lines can be identified with the 1- and 2-dimensional subspaces of the 3-dimensional vector space over the field K. In these classical projective planes, the three axioms are easily verified. For example, the first axiom corresponds to the fact that two 1-dimensional subspaces of a 3-dimensional vector space are contained in exactly one 2-dimensional subspace.

For completeness's sake, I should remark that there are nonclassical projective planes.

The smallest projective plane is the *Fano plane,* that is, the projective plane associated with the field $\mathbf{Z}_2$. It has seven points and seven lines. Every line contains exactly three points and every point is contained in exactly three lines. Figure 2 is a well-known picture of this plane. In fact, it seems to be the only picture of this fundamental geometry of which most people are aware. Remember that the "circle" counts as a line.

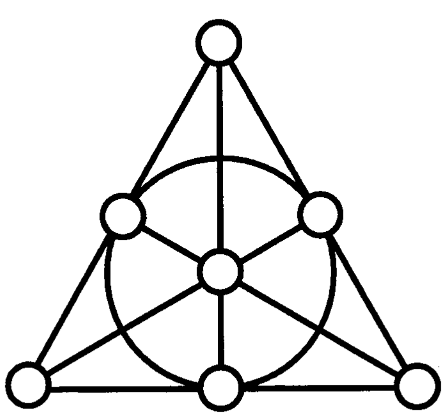

Figure 2. The traditional picture of the Fano plane.

I have to admit that it is a nice picture, but is it really the only picture worth drawing, and is it even the best model of this plane? Well, all the planes in the world that I am living in are Fano planes, and I have to show you at least two more beautiful pictures with which you are probably not familiar.

The stereogram in Figure 3 shows a spatial model of the Fano plane. The stereogram can be viewed with either the parallel or the cross-eyed technique; that is, one of the techniques that you had to master a couple of years ago to be able to view random-dot stereograms that were in fashion in your world. You can think of this model as being inscribed in the tetrahedron as follows. The points are the centers of the six edges of the tetrahedron plus the center of the tetrahedron. The lines are the three line segments connecting the centers of opposite edges plus the circles inscribed in the four sides of the tetrahedron. Every symmetry of the tetrahedron translates into an automorphism of the geometry. The symmetry group of the tetrahedron has order 24.

The picture of the Fano plane in Figure 4 shows that none of the points of the plane is distinguished among the points and that no line is distinguished among the lines. In fact, the rotation through 360/7 degrees around the center of the diagram corresponds to an automorphism of order 7, which generates a cyclic group of automorphisms acting transitively on the point and line sets of the plane.

Together with an automorphism of order 7 like the one underlying Figure 4, the 24 automorphisms apparent in the

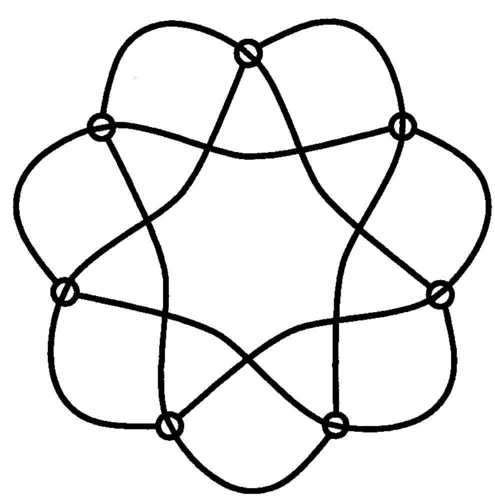

Figure 4. The Fano plane: all points are equal!

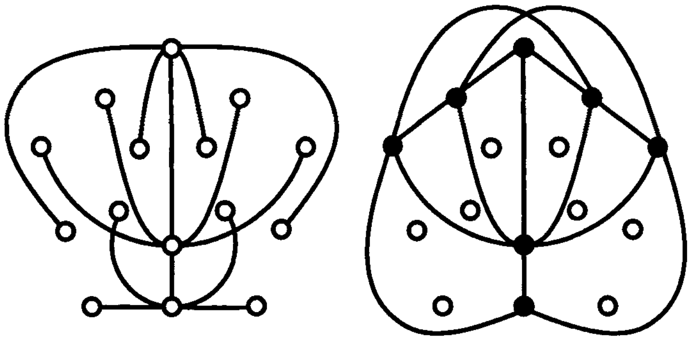

Figure 5. The smallest projective space.

spatial model generate the full automorphism group of the Fano plane. It has order $24 \times 7 = 168$.

The Smallest Perfect Universe

Associated with every field K is a (3-dimensional) projective space whose points, lines, and planes can be identified with the 1-, 2-, and 3-dimensional subspaces of the 4-dimensional vector space over the field K. Of course, there is also a set of axioms for projective spaces. I will not bother reminding you of these axioms, as, essentially, there are no examples of projective spaces apart from the classical ones associated with fields.

The smallest projective space over the field $\mathbf{Z}_2$ has 15 points, 35 lines, and 15 planes. Each of the 15 planes contains 7 points and 7 lines; as geometries, they are isomorphic to the Fano plane. Every point is contained in 7 lines and every line contains three points. Furthermore, two distinct points are contained in exactly one line and two planes intersect in exactly one line.

The diagram on the left in Figure 5 is a partial picture of this space. It shows all 15 points and 7 "generator lines." The other lines are the images of these generator lines under four successive rotations of the diagram through 360/5 degrees. Given a point p and a line l not through this point, form the union of all points of the lines connecting p with points of l. This union is the point set of one of the planes of the space. All planes are generated in this way. The diagram on the right shows one such plane. Note that all lines connecting different points in such a plane are fully contained in the plane. As a point-line geometry, every such plane is really a Fano plane.

Hall's Magical Labelling

Figure 6 is a construction of the smallest projective space due to my friend Hall. Let SEVEN and EIGHT be the sets $\{1, 2, \ldots, 7\}$ and $\{1, 2, \ldots, 8\}$, respectively. Label the points of the Fano plane with the numbers in SEVEN in all possible ways. Remember that the automorphism group of the Fano plane has order 168. This means that there are $7!/168 = 30$ essentially different such labellings. On close inspection, it turns out that 2 among these 30 labellings have either 0, 1, or 3 lines (=triples of labels) in common. There is a unique partition of the 30 labelled Fano planes into 2 sets X and Y of 15 each such that any 2 Fano planes in 1 of the sets have exactly 1 line in common. Now, the 15 points of the projective space can be identified with the 15 labelled Fano planes in either X or Y, and the lines with

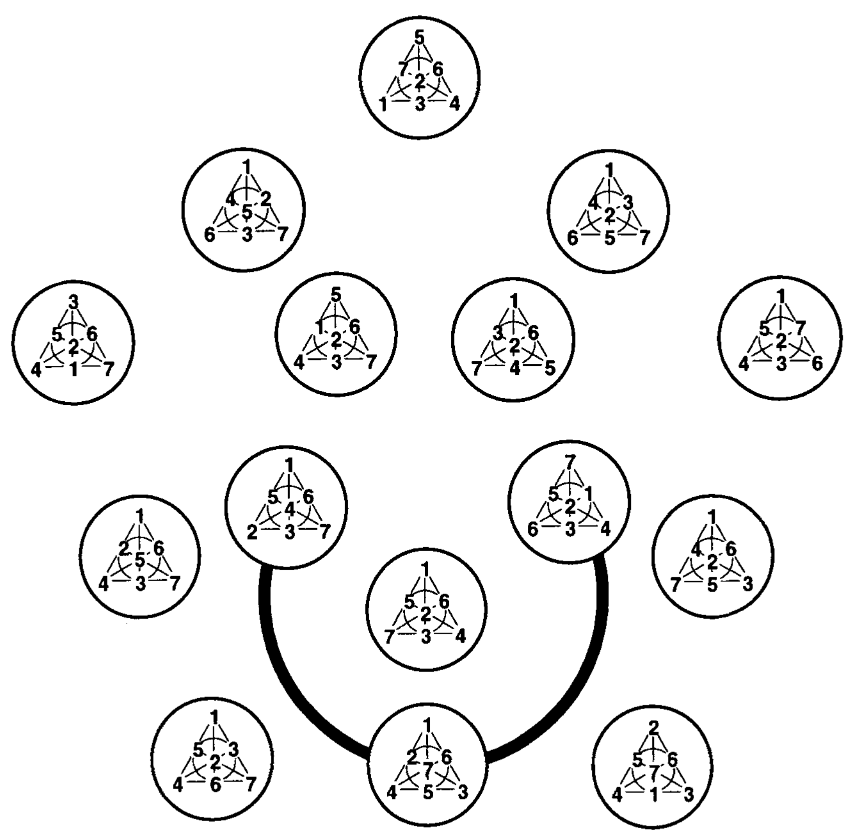

Figure 6. Hall's magical labelling.

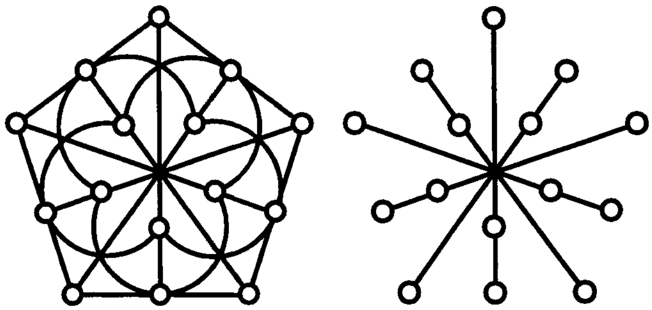

Figure 7. Generalized quadrangle and spread.

the $\binom{7}{3} = 35$ triples of distinct numbers in SEVEN. A point (=labelled Fano plane) is contained in a line (=triple) if the triple is a line in the labelled Fano plane. Figure 6 is a labelling of the above model of the projective space with labelled Fano planes. The highlighted line corresponds to the triple 237.

Generalized Quadrangles

Fix a number n in SEVEN. Then, there are $\binom{6}{2} = 15$ triples containing this number and 2 of the remaining 6 numbers in SEVEN. The 15 points of the projective space together with these 15 lines make a so-called *generalized quadrangle*; that is, a geometry satisfying the following axioms:

- Two points are contained in at most one line.
- Given a point p and a line l that does not contain p, there is a unique line k through p which intersects l.

For example, the generalized quadrangle which corresponds to the number 7 is the geometry depicted in Figure 7 on the left. Note that an ordinary quadrangle with its four vertices considered as the points and its four edges considered as the lines of a point-line geometry is a generalized quadrangle. Furthermore, just as in this prototype, the smallest n for which an n-gon can be drawn in a generalized quadrangle using only lines of the geometry is 4.

The Nightmare Continues

A *spread* of a geometry is a partition of its point set into disjoint lines or planes. Two parallel lines are quite scary, but the mere thought of a spread makes me want to hide somewhere. Fortunately, there are no spreads of planes in this world, and I can handle the fact that there are disjoint lines in this space. Still, I discovered that there are 56 spreads of lines!

Fix two of the elements of SEVEN. Then, there are five triples containing these two numbers. Every such set of five triples corresponds to a spread in the space. For example, the numbers 1 and 7 correspond to the spread in Figure 7. We can construct $\binom{7}{2} = 21$ of the 56 spreads contained in our space in this way.

Here is a natural identification of the 56 spreads in our space with the $\binom{8}{3} = 56$ triples of numbers contained in the set EIGHT. Let $xy8$ be such a triple. Then, the spread associated with it is the spread associated with the two numbers x and y. Let xyz be a triple in SEVEN. Then, the spread associated with it consists of xyz itself plus the four triples in SEVEN which are disjoint from xyz.

Packings—Solutions to Kirkman's Schoolgirls Problem

Now consider any labelling of the Fano plane with elements of EIGHT. Then the seven spreads corresponding to the seven lines (=triples in EIGHT) of the Fano plane are pairwise disjoint. In fact, every line in the projective space is contained in exactly one of these seven spreads. Any set of seven spreads of our space which has this property is called a *packing* of the space and is, oh horror of horrors, just a "spread of spreads." Because every packing of the space corresponds to such a labelling of the Fano plane, there are 8!/168 = 240 packings of our space. Figure 8 shows one packing and the labelled Fano plane associated with it.

Ironically, every packing corresponds to a solution of the problem which is named after me. Just identify the 15 girls with the 15 points, the "groups of 3" occurring during a week with the lines of the space, and the 7 walks with the 7 spreads of a packing. For a long time I thought that things cannot get any worse. I was mistaken.

Hyperpackings—the One-Point Extension of the Fano Plane

Let us play the following game. Remove the 7 spreads corresponding to a packing from the 56 spreads of our space. Try to find a packing among the remaining 49 spreads. If you find one, put it aside and try to find yet another one among the remaining 42 spreads, and so on until no more packings can be found. If this happens when no spread is left, you have constructed a *hyperpacking*; that is, a partition of the 56 spreads into 8 disjoint packings. What a horror! Clearly, every hyperpacking corresponds to a set of

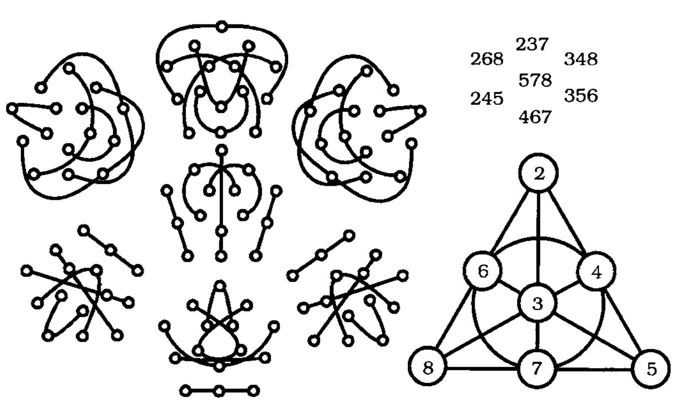

Figure 8. Packing.

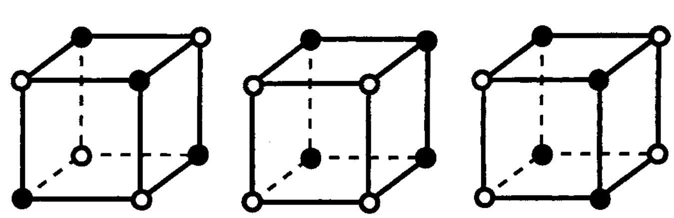

Figure 9. One-point extension of the Fano plane.

eight labelled Fano planes whose labels are contained in EIGHT and which are pairwise line-disjoint. Figure 9 is one way to construct such a set of labelled Fano planes.

The *one-point extension of the Fano plane* has eight points; the seven points of the Fano plane plus one additional point. It has 14 lines containing 4 points each. These are the complements of the lines of the Fano plane in its points set, plus the lines of the Fano plane which have all been extended by the additional point. Note that any 3 distinct points of the geometry are contained in exactly 1 of the 14 lines. The points of the one-point extension can be identified with the eight vertices of the cube such that the lines turn into the following sets:

- The vertex sets of the regular two tetrahedrons inscribed in the cube.
- The vertex sets of the six faces of the cube.
- The vertex sets of the six "diagonal rectangles."

The points of the *derived geometry at a point p* of the one-point extension are the seven points different from *p*. The lines are the lines of the one-point extension containing *p* which have been punctured in *p*. Clearly, every such derived plane is a Fano plane. Label the one-point extension with the elements of EIGHT. This labelling induces a labelling of the eight derived Fano planes, and it is easy to see that any such set of eight labelled Fano planes derived like this has the "desired" property. Figure 10 shows the eight packings of a hyperpacking which corresponds to the labelling of the cube in the middle of the diagram. The derived Fano planes at the points 1, 2, ... correspond to the packings in the upper left corner, in the middle above, and so on in the clockwise direction. Up to automorphisms, there are 30 different labellings of the one-point extension corresponding to the 30 essentially different labellings of the Fano plane. Unfortunately, not all hyperpackings can be constructed like this. In fact, there are 27,360 different hyperpackings!

Hyperhyperpackings

Let us play another game. Remove the 8 packings corresponding to a hyperpacking from the 240 packings of our space. Try to find a hyperpacking among the remaining 232 packings. If you find one, put it aside and try to find yet another one among the remaining 224 packings, and so on until no more hyperpackings can be found. If this happens when no packing is left, you have constructed a *hyperhyperpacking*; that is, a partition of the 240 packings into 30 disjoint hyperpackings. Unfortunately, these *hyperhyperpackings* do exist. In fact, the 30 hyperpackings corresponding to the essentially different labellings of the one-point extension form a hyperhyperpacking.

Hyperhyperhyperpackings?

I do not know what other monsters are lurking in the shadows. Conceivably, it might be possible to construct hyperhyperhyperpackings, hyperhyperhyperhyperpackings, and so on *ad infinitum*. I do not know, and I do not dare to investigate any further. I think it is time to flee again. I just

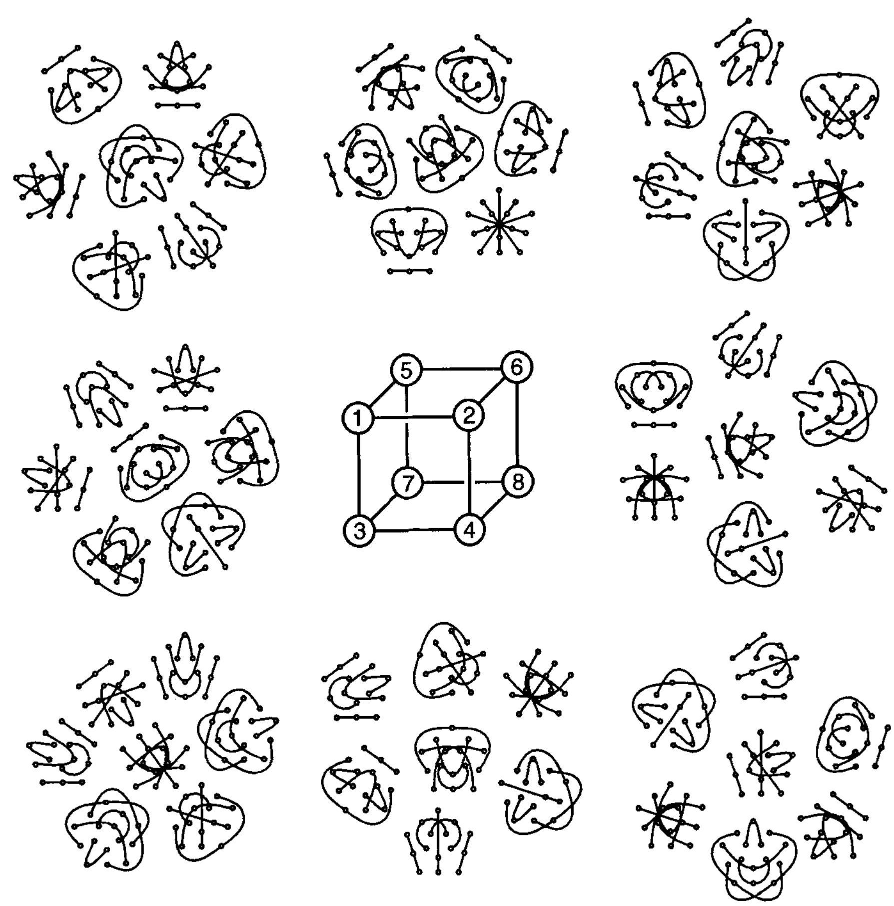

Figure 10. Hyperpacking.

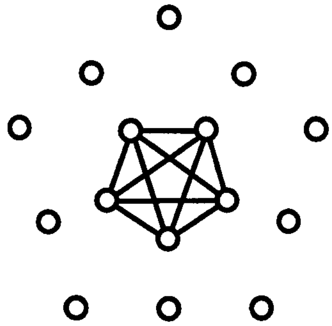

Figure 11. Inversive plane—the sign of the devil.

found out that other projective spaces also contain spreads and packings. So, I think I will try to turn myself into a flatlander and move to a projective plane.

Wish me luck that no other nightmares are waiting for me there.

Yours, apprehensively,
Thomas

P.S.: YEA WHY TRY HER RAW WET HAT! I just discovered the sign of the devil right in the middle of this universe while investigating the counterpart of the geometry of circles on the sphere in this space. A *sphere* in this world is a set of five points such that any line of the space intersects the set in one or two points. Every point of a sphere is contained in exactly one tangent plane. Hence there are five planes intersecting the sphere only in one point. The remaining 10 planes intersect the sphere in 3 points each. The sets of points of the three nested regular pentagons of points visible in our model of the space are three such spheres. The points of the geometry of circles associated with such a sphere are the points of the sphere. Its circles are the intersections with the sphere of all those planes that intersect the sphere in three points. Just like the one-point extension of the Fano plane, this geometry has the property that three distinct points are contained in exactly one circle. If you draw the circles of the geometry associated with the inner pentagon, you arrive at the following picture. (See Figure 11.) Ominous.

Acknowledgements, Further Readings, and Some Remarks

I would like to thank Gordon Royle for conducting an exhaustive computer search to calculate the numbers of the different hyperpackings and for suggesting the names for these new structures. By the way, Gordon and his colleague Rudi Mathon have classified a large number of nonclassical projective planes. If you are interested in investing in real estate in one of these planes, or if you want to have one named after you, get in touch with them. Thanks are due to Keith Hannabuss for the title of this article.

For an accessible introduction to combinatorics related to Kirkman's schoolgirls problem, see [2]. For more information about the smallest projective space, see [1–7]. In constructing Figure 6, I used the different labellings in [5]. The identification of the different packings with the different labellings of the Fano plane with elements of EIGHT can be found in [3] and [7]. The diagram of the generalized quadrangle in Figure 7 is called the *doily* and is due to Payne. Lots of stereograms and other pictures of spatial and plane models of the smallest projective space and many other finite and topological geometries can be found in [6].

Finally, I should acknowledge that Kirkman is not known to have fled our world in horror.

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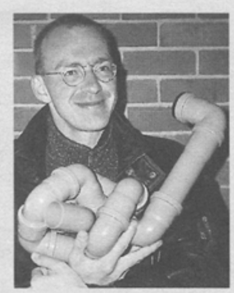

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Burkard Polster studied mathematics in Germany and in the United States and received his doctorate from Erlangen in 1993. He specializes in topological and finite geometry. Apart from mathematics, he is also obsessed with traveling, foreign languages, and practicing a variety of mind and motor skills. Just now, he wishes he could master the 7-ball cascade (juggling), wheel walking on a unicycle, and throwing the "perfect" teapot. He is shown here brandishing a knotted didgeridoo made from 13 elbow pieces of PVC drainage pipe.