# Viviani à la Kawasaki: Take Two

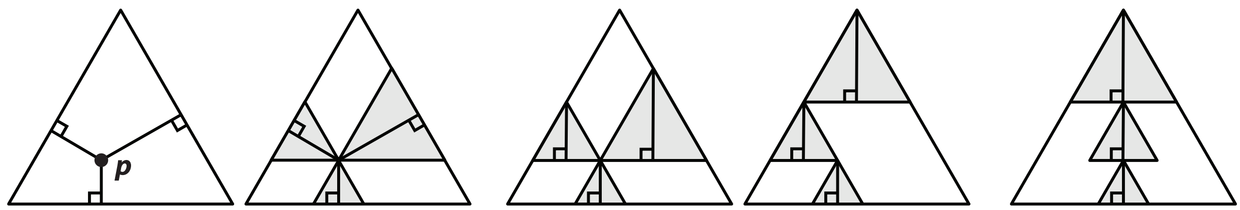
Burkard Polster

June 12, 2014

Viviani's theorem highlights a surprising property of equilateral triangles.

*In an equilateral triangle, the sum of the distances from any interior point $p$ to the three sides of the triangle is equal to the height of the triangle.*

Here is a slightly extended version of a very pretty proof without words by Kawasaki [4].

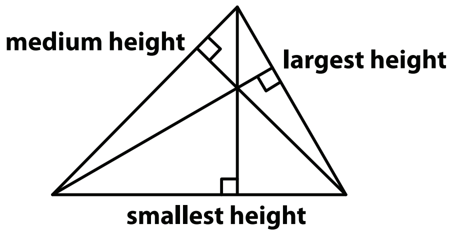



In Harold Jacobs' classic geometry textbook [3] Viviani's theorem forms the basis for a puzzle. Jacobs imagines a surfer stranded on an island in the shape of an equilateral triangle. (Happens all the time!) Our surfer dude enjoys all of the island's beaches and plans to spend an equal amount of time at each. That suggests that his hut would be most conveniently located so that the sum of the distances to the three beaches is as small as possible. Where would that be?

Of course, Viviani's theorem tells us that it does not matter where the surfer builds his hut. But what if the surfer is stranded on an island of a different shape? We'd like to present you with a natural twist to Kawasaki's argument that provides the answer for all triangles. We start by proving the following:

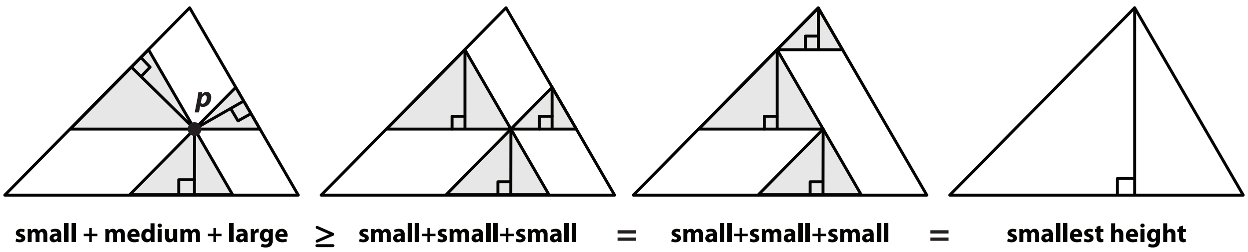
*Let $S$ be the sum of the distances from a point $p$ inside a triangle to its three sides. Then*

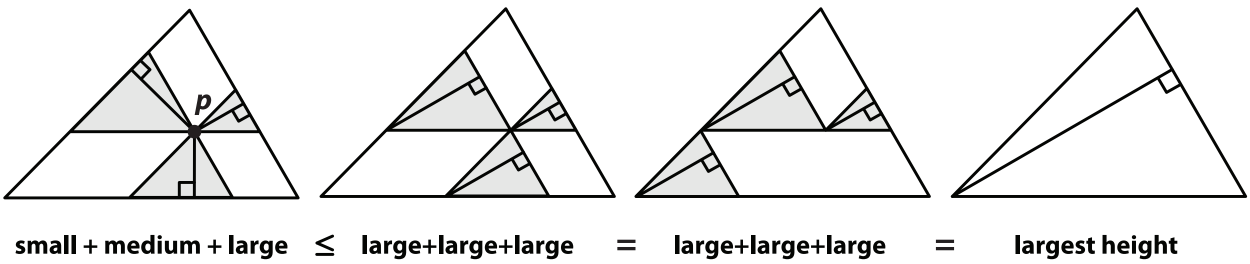
$$\text{the smallest height of the triangle} \leq S \leq \text{the largest height of the triangle.}$$




(For everything we say to also work for obtuse triangles the *distance to a side of a triangle* has to be interpreted as the *distance to the line that the side is contained in.*)

Proof (almost without words).

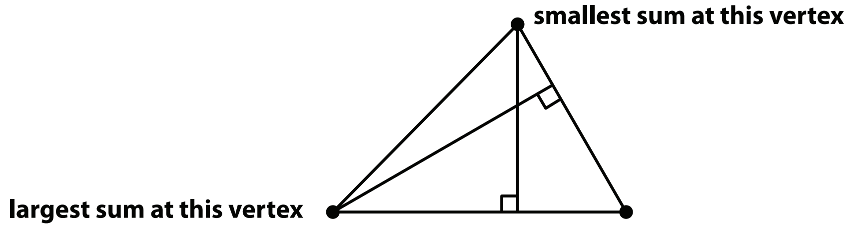


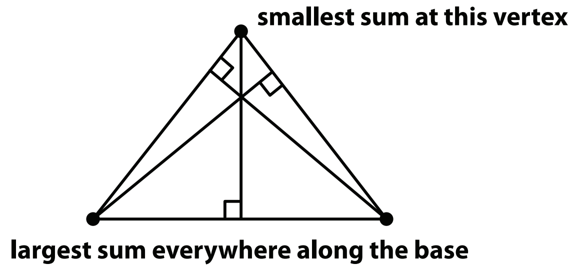

In the case of an equilateral triangle, all three heights are equal. Hence, in this case, our result implies Viviani's theorem.

Now, let's have a close look at the diagrams to figure out where the surfer should build his hut on a given triangular island, and where he should definitely not build his hut. In other words, let's figure out at which points of our triangle the sum of the distances to the sides is minimal and maximal. There are three cases to consider.

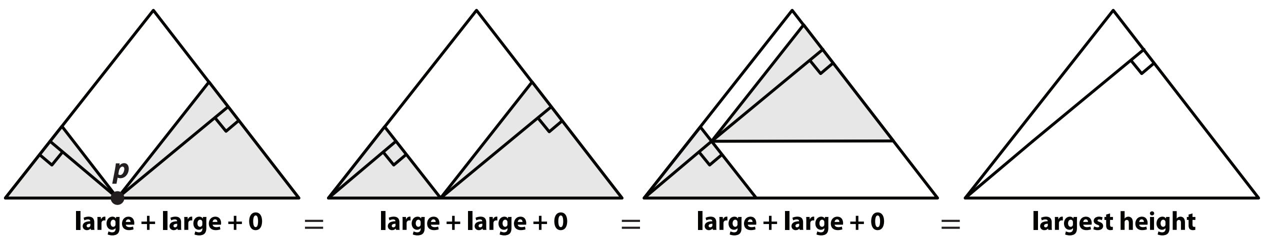
*Case 1: Generic triangle in which the three heights are different.* In such a triangle the first equality is strict except if the point $p$ is the vertex at the end of the smallest height. As well, the second equality is strict unless the point $p$ is the vertex at the end of the longest height. This means that the sum $S$ takes on its minimal and maximal values at these two vertices.



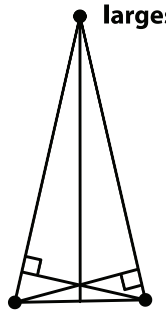

*Case 2: Proper isosceles triangle in which the smallest height is the symmetry axis.* As in the generic case we conclude that the vertex at the end of the smallest height corresponds to the smallest sum. The largest sum occurs everywhere along the base of the triangle.




To see the latter note that there are two longest heights. This means that the second inequality is strict unless the point $p$ is contained in the base of the triangle. Let's redraw the sequence of diagrams with the point $p$ on the longest side to see at a glance that in this case the sum of the distances is equal to the longest height; see also [2].



*Case 3: Proper isosceles triangle in which the largest height is the symmetry axis.* As in the previous case, we conclude that the minimal and maximal sums occur at the points highlighted in the following diagram.



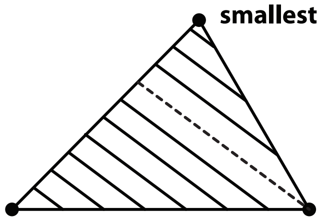

All right, ready to get stranded on a triangular island?

For the sake of completeness we mention the following result which was derived in [1]. (The arguments in this article can also be tweaked easily to yield a second proof of our results.)

*Let $S$ be the sum of the distances from a point $p$ inside a triangle that is not equilateral to the three sides of the triangle. Then (1) the triangle can be divided into parallel line segments, called isosum segments, on which $S$ is constant and (2) the sums associated with different isosum segments are different.*

We've seen that in isosceles triangles points on the base all have the same distance sum. Therefore, the isosum segments in (proper) isosceles triangles are just the segments parallel to the base.

The following picture shows the isosum segments for our example of a generic triangle.




For generic triangles we don't know of any easy rule to pin down the exact direction of the isosum segments. However, what we can say is that in a generic triangle, as in the one shown above, the isosum segment that passes through the "medium sum" vertex will intersect the edge that connects the smallest and largest sum vertices in an interior point. This is an easy consequence of (2).

We've prepared an interactive *Mathematica* CDF file that you can use to visualize everything we've been talking about in this note. Download the file from `www.qedcat.com/isosum.cdf` and open it with either *Mathematica* or the freely available *Wolfram CDF player*.

# References


[1] Elias Abboud, On Viviani's theorem and its extensions. *The College Math. Journal* **41** (2010) 203–211.

[2] Viviani in isosceles triangle: What is it? A Mathematical Doodle. An applet with accompanying explanation on the Cut-the-knot site: `http://www.cut-the-knot.org/Curriculum/Geometry/VivianiIsosceles.shtml`

[3] Harold Jacobs, *Geometry: Seeing, Doing, Understanding*, W.H. Freeman & Co., 2003, pp. 2–6.

[4] Ken-Ichiroh Kawasaki, Proof without words: Viviani's theorem. *Math. Mag.* **78** (2005) 213.