

Mathematical Rugby

Burkard Polster and Marty Ross

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1 Introduction

The December 1978 issue of the *Mathematical Gazette* [8] contains an elegant and humorous contribution from Anthony Hughes. His note gives a recipe for determining the optimal spot from which to make a conversion attempt in rugby. Others then elaborated on Hughes's idea; see [12] and [14], in particular. In 1996, Isaksen [9] rediscovered Hughes's results while investigating the kicking of extra points in American gridiron. There are also a number of popularisations and summaries; the ones of which we are aware are listed in the references.

Hughes's approach to locating the optimal spot is to maximise the angle subtended by the bases of the two goalposts. However, this does not take into account that the ball must also go over the crossbar. In 1999, Michael de Villiers [13] attempted to incorporate the crossbar into modelling the optimal conversion kick.

We recently used this material for one of our columns in our hometown newspaper, *The Age*, in Melbourne, Australia. In the course of reviewing the original papers, we decided that it would be useful to provide a survey of this material, including some new ideas and proofs of our own.

In particular, in this survey we would like to: (1) introduce a hyperboloid as an alternative and elegant method of deriving de Villiers' results; (2) present a model incorporating the crossbar that is more realistic than the one given by de Villiers'; (3) introduce an alternative approach, leading back to Hughes's hyperbola; and (4), begin a discussion of how the mathematical models compare to the actual practices of rugby kickers. The presentation follows quite closely our own path of discovery.¹

¹We would like to thank Cameron Smith, star of the NRL team Melbourne Storm, for being so helpful and so generous with his time. Ditto, to the Storm's kicking coach, biomechanist Dr. Kevin Ball of Victoria University, Melbourne.

2 Kicking between the goalposts: the rugby hyperbola

As a refresher, here is how conversions are scored in rugby.² One of the players first has to score a *try*, by carrying the ball across the opposition’s goal line and touching the ground with the ball at some point before the dead ball line. Following the try, the kicker can then place the ball anywhere in the field along a line through this grounding point and perpendicular to the goal line: we’ll call this the *conversion line*. To make a *conversion*, the kicker must then kick the ball between the goalposts and above the crossbar.

We first describe Hughes’s geometrical solution to this problem. Critically, Hughes simply ignores the crossbar and then determines the optimal point on the conversion line from which to kick between the goalposts. In essence, Hughes is determining the optimal point from which to putt a golf ball.

In particular, if the ball has been grounded between the goalposts, then with Hughes’s formulation the decision is a no-brainer: we simply kick/putt from the goal line. Very difficult to miss!

²We don’t mean to be condescending: we’re from an Aussie Rules town. As well, and at the risk of inflaming cultural passions, we use “rugby” here to refer to both rugby union and rugby league.

Hughes deals with the non-trivial situation, when the try is scored beyond one of the goalposts. Figure 1 shows a rugby field, together with pieces of circles ending at the goalposts.³

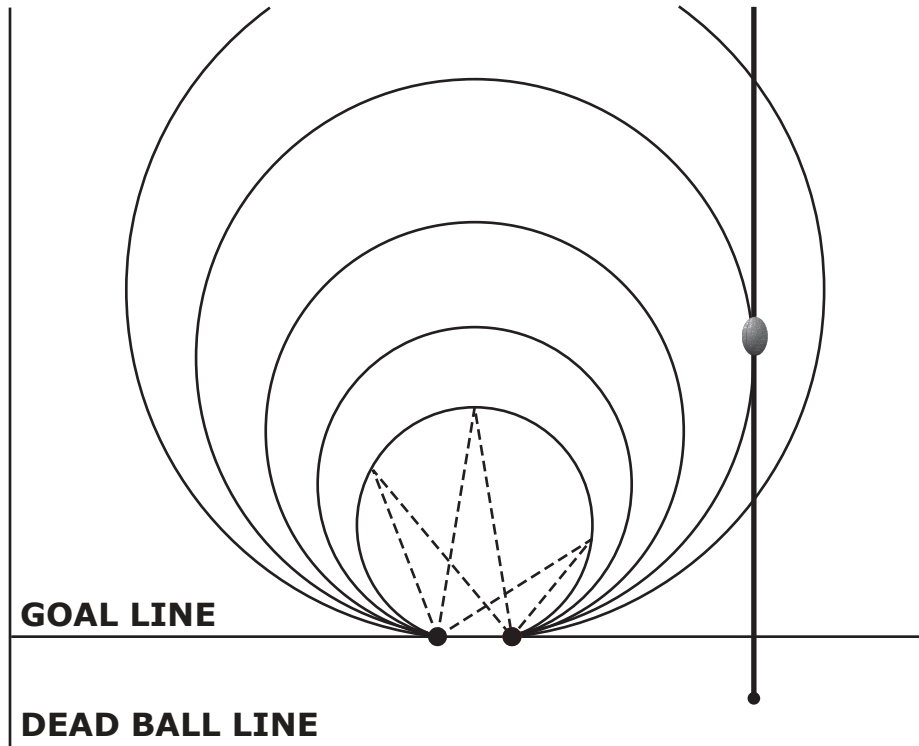


Figure 1: The circle tangent to the conversion line provides the maximum subtended angle.

Fix one of these circles, take any point on that circle, and consider the angle subtended by the goalposts from that point. It is (or at least used to be) well-known that the angle is the same for every point. Furthermore, the larger the circle, the smaller the associated angle. It follows that, along a given conversion line, the maximum subtended angle is obtained by choosing the circle which just touches the line. So, with the Hughes formulation, this tangent point is the optimal spot from which to kick.

³The dimensions of rugby fields vary, and can be a source of controversy. In our calculations and diagrams we have taken the width of the field to be 70 metres, the maximum permissible in rugby union, and greater than the 68 metres maximum in international rugby league. We have also taken the width of the goals to be 5.6 metres as in rugby union (it is 5.5 metres in rugby league). Finally, we have taken the height of the crossbar to be 3 metres, the same as is fixed in both codes.

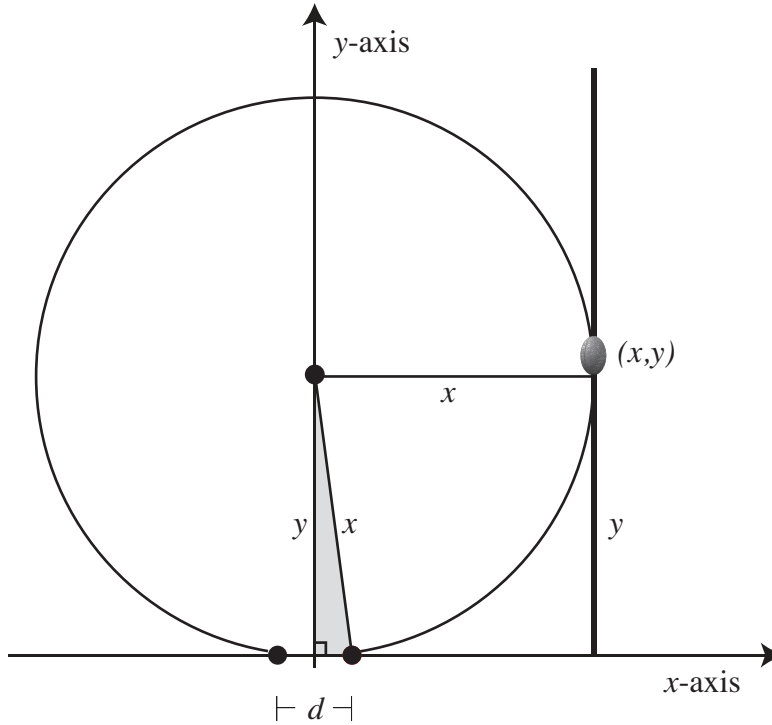


Figure 2: Pythagoras's Theorem leads to Hughes's hyperbola.

We now introduce coordinates (x, y) as indicated in Figure 2, with $(0, 0)$ the middle of the goals. If (x, y) are the coordinates of one of our critical points, then the critical circle has radius x , and this is also the distance from the centre of the circle to the goalposts. So, if the goals are d metres wide, then Pythagoras's Theorem tells us that the optimal point is a distance $y = \sqrt{x^2 - (d/2)^2}$ along the conversion line. Therefore, the optimal points for all different conversion lines form the right-angled hyperbola

$$x^2 - y^2 = (d/2)^2.$$

Hughes summarises his results: “The locus of points from which to take conversions is thus half of each of the branches of a rectangular hyperbola, and on this the kicker should place his ellipsoid of revolution before sending it on its parabolic way.”

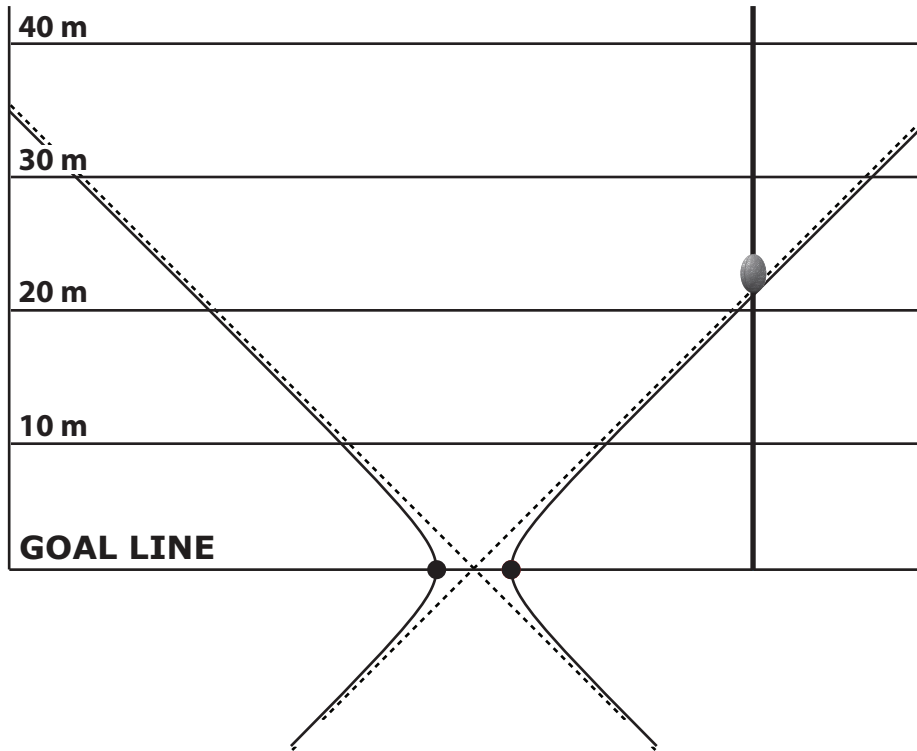


Figure 3: Hughes’s hyperbola, giving the optimal kicking point for each conversion line.

Having arrived at the optimal solution, Figure 3 suggests that in practice we needn’t actually bother with the hyperbola. It is much easier to follow along the asymptote, which very well approximates the hyperbola; this was pointed out by Worsnop [14]. Note that the asymptote starts from the middle of the goals, and travels at a 45 degrees angle, intersecting the sideline at the 35 metre line.

In practice, of course, the asymptote and its intersection with the conversion line are very easy to find, even for your average rugby player. Table 1 compares the location and associated angles of the hyperbola and asymptote, for several conversion lines.

	distance from goal centre			
	5m	10m	20m	35m
hyperbola	4.14m/34.06°	9.60m/16.26°	19.80m/8.05°	34.89m/4.59°
asymptote	5.00m/33.59°	10.00m/16.25°	20.00m/8.05°	35.00m/4.59°

Table 1: Distances and associated angles for Hughes’s hyperbola and its asymptotes.

3 Kicking over the crossbar: the rugby hyperboloid

As we remarked above, Hughes’s approach fails to take into account the need to clear the crossbar. This raises two related but quite distinct issues. First, it makes it obvious that no matter the optimal kicking location for angle width, the player does not want to kick when too close to the goals. In practice, when a try is scored directly in front of goals, players choose to kick a minimum of about 10 metres from the goals. So, we should probably imagine Hughes’s hyperbola and asymptote cut short by a suitable exclusion zone; see Figure 4.

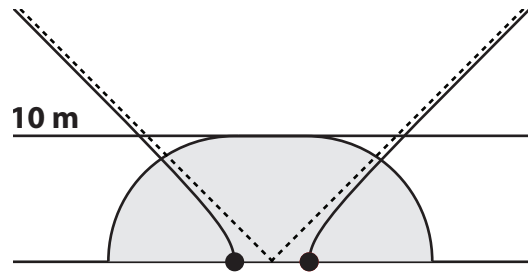


Figure 4: Hughes’s hyperbola and asymptotes, and the 10m exclusion zone around the goal.

Secondly, and more importantly, Hughes’s angle is simply the incorrect angle to consider. Given that the ball must clear the crossbar, it is some raised angle that should be optimised. In his paper, de Villiers [13] suggests optimising the angle subtended by two points on the goalposts at equal heights: for example, the ends of the crossbar.

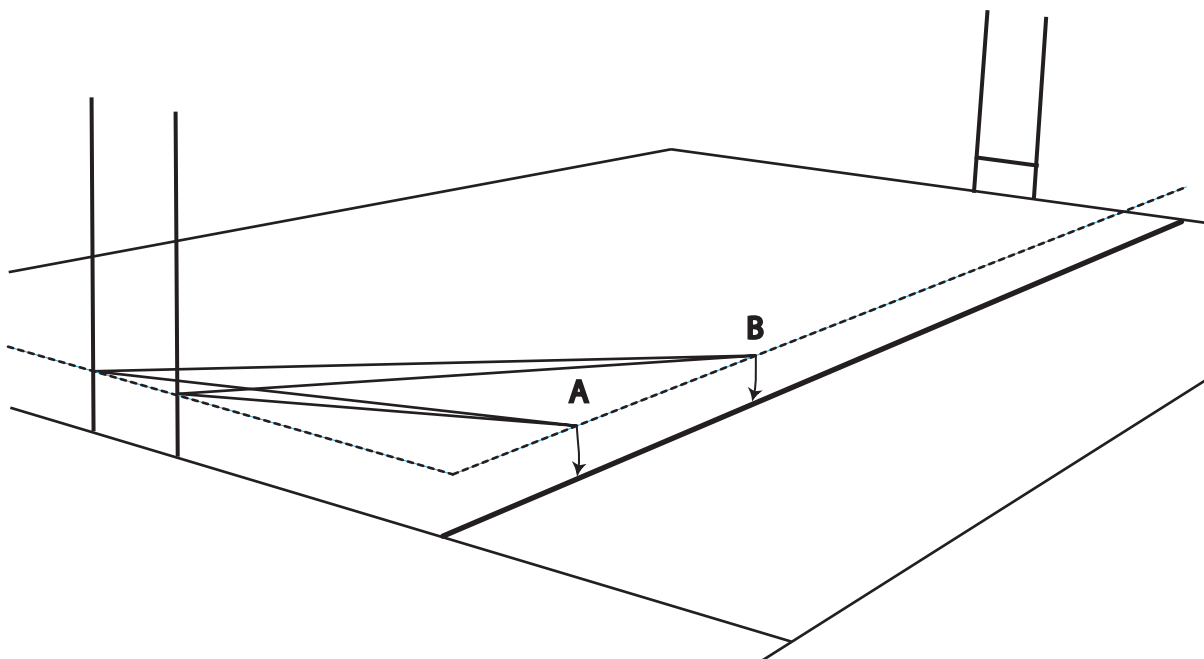


Figure 5: A perspective drawing of a rugby field.

Figure 6 is a perspective drawing of the field. The dashed line extends the crossbar. At the same height we've included a second dashed line, parallel to and hovering above the conversion line. We now view the ends of the crossbar from points A and B on the hovering line. The angles formed are exactly those considered by Hughes, as described above.

Think of the subtended angles as being made from inflexible rods, attached with hinges to the ends of the crossbar. Now let A and B fall to the ground. Because of the way the rods are hinged, both A and B will come to rest on the conversion line. Furthermore, A will be closer to the goal line than B , just as on the hovering line.

In their fallen positions, the angles are exactly of the type proposed by de Villiers. The conclusion is that as we move along the conversion line, we will come across the same angles in the same order as in the Hughes scenario. In particular, the optimal angle will be exactly the same as before. However, because of the swiveling, the optimal point to achieve that angle will be closer to the goal line.

There is a serious shortcoming to the de Villiers approach, which we address in the next section. For now, it suffices to note that if A is closer to the goal line than the height at which we're aiming, then A will not reach the ground when dropped. This means that if the angle at A is the optimal Hughes angle, then we never get to witness this angle in the de Villiers setting. In such a case, the "optimal" angle for de Villiers is the angle obtained by placing the ball on the goal line. However, it is clearly impossible to make a conversion from the goal line, and so something has obviously gone awry. But, for now, we continue on.

An equivalent method to obtain the new conversion points is to start with a point A on the

ground, anchor our rods at the bases of the goalposts, and swivel the rods upwards until A hits the hovering line. Then the optimal de Villiers point will be directly below the point of intersection.

We can now analyse all the conversion lines at once, by swivelling Hughes's hyperbola around the goal line and considering how the rotated curve intersects each hovering line. This amounts to taking the hyperboloid of revolution generated by Hughes's hyperbola, and intersecting this hyperboloid with the horizontal plane at the height of the crossbar. The result is a second hyperbola: just lower it to the field and you have the curve consisting of all optimal conversion points for the de Villiers scenario.

If you wish to aim higher than the crossbar, simply intersect the hyperboloid with the horizontal plane at the desired height, and this will result in the relevant hyperbola of optimal points. For a general height h , the optimising hyperbola is then given by the equation

$$x^2 - y^2 = (d/2)^2 + h^2.$$

Figure 6 shows the hyperbolas corresponding to four different heights.

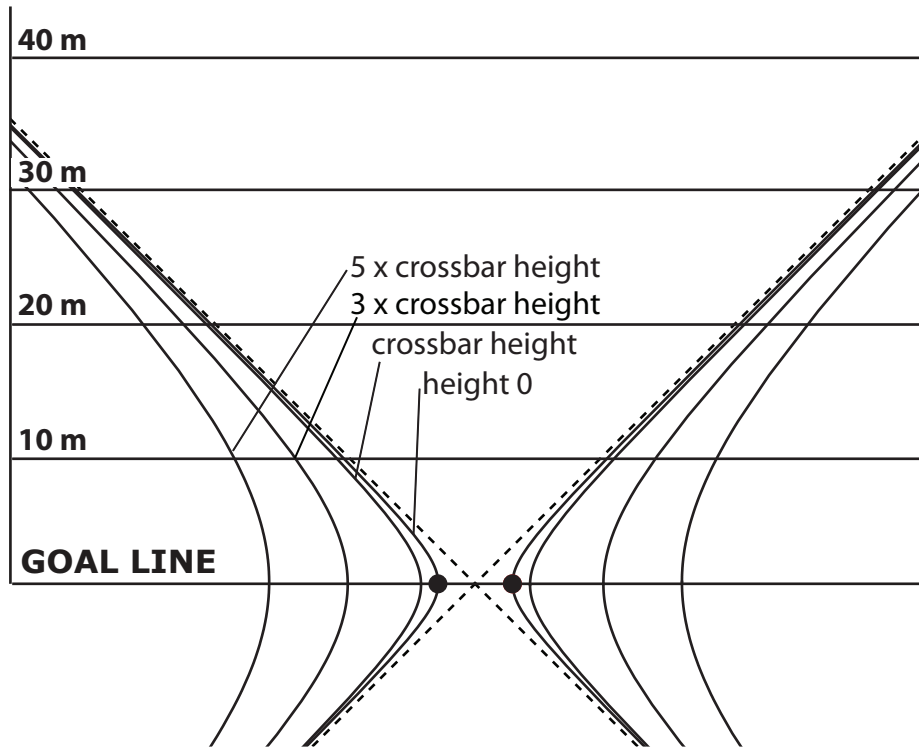


Figure 6: de Villiers' hyperbolas, corresponding to different target heights.

Note that all the hyperbolas are right-angled, and are therefore approximated by their common asymptotes. The approximation is best for Hughes's hyperbola ($h = 0$), and worsens as one aims higher.

4 A second kick over the crossbar

The maths in the previous section is elegant. However, as we noted, there is a serious problem with the setup. Let's have a look at the goals from a different perspective.

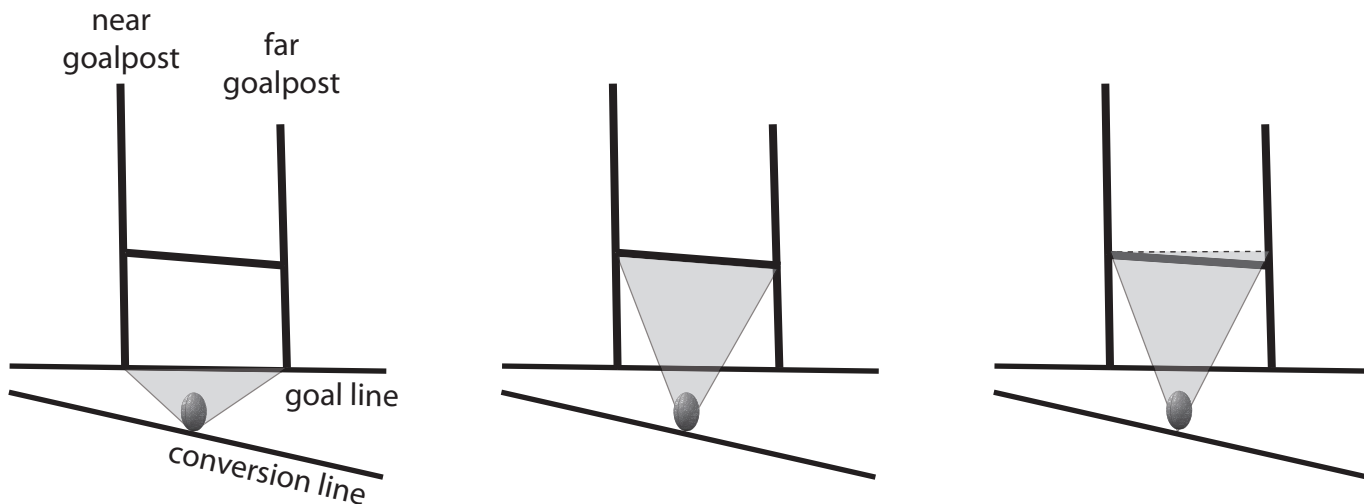


Figure 7: Hughes's angle, de Villiers' angle and our modification of de Villiers' angle.

The first and second pictures shows the Hughes and de Villiers angles. An important feature of the de Villiers angle is that the two segments making the angle will not typically have the same elevation: the segment connecting the ball to the nearer goalpost will form a greater slope with the horizontal. We already stumbled across this property, when we considered kicking the ball from the goal line: here, the de Villiers angle is vertical and non-zero, and gives no indication that the conversion is in fact impossible.

How can we fix this? In practice, the kicker kicks hard through the ball, and does not worry too much about the vertical deviation; his concern is the deviation to the left or right. This means, as in the Hughes scenario, he should really be worried about maximizing some horizontal angle: that is, the two segments forming the angle should have the same slope from the field. Such an angle is illustrated in the third picture of Figure 7.

Fix a point P on the field. To construct the new angle from P corresponding to crossbar height, we need the two segments forming the angle. For the first segment, we simply start at P and connect it to the end of the crossbar on the near goalpost. The second segment is then taken to have the same slope with the field, connecting P to the far goalpost.⁴ Note that the second segment will always hit its goalpost at a higher point.

Now it is business as usual, and we have to draw the curves of optimal conversion points

⁴The exact construction is necessarily somewhat arbitrary. But the arbitrariness is not significant for our purposes.

corresponding to the different heights; see Figure 8. It is probably no surprise that these curves are no longer hyperbolas. In fact, the only practical method of getting a sense of them is by using a software package such as *Mathematica*.⁵

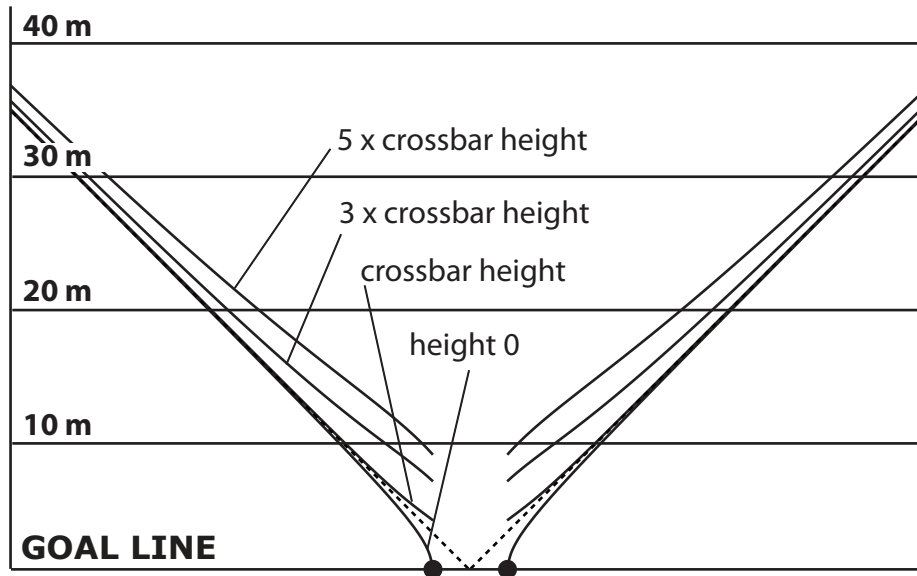


Figure 8: Curves for the adjusted de Villiers angle, corresponding to different heights.

As in the de Villiers scenario, the asymptotes to the original Hughes hyperbola are also asymptotes to these new curves. It is also interesting to note that the curve corresponding to the height of the crossbar very quickly approaches this asymptote. However, unlike the Hughes and de Villiers proposals, the new curves are located beyond the asymptote.

Of course the kicker has a choice as to how high to kick the ball. We assume that, whatever height is chosen, the kick is taken from the optimal conversion point for that height. Table 2 then shows how these optimal angles vary with height and with distance from the goal centre.

	distance from goal centre			
	5m	10m	20m	35m
0 x crossbar	4.14m/34.06°	9.60m/16.26°	19.80m/8.05°	34.89m/4.59°
1 x crossbar	5.29m/29.39°	9.98m/15.78°	19.95m/8.00°	34.96m/4.58°
3 x crossbar	8.23m/19.48°	12.10m/13.35°	21.03m/7.63°	35.55m/4.51°
5 x crossbar	10.51m/14.34°	14.50m/11.01°	22.80m/7.05°	36.64m/4.38°

Table 2: Distances from the goal line and subtended angles for optimal conversion points.

It is clear from this table that the higher the kicker aims, the smaller the angle he is permitted to deviate. This suggests that it is wise to aim close to the crossbar height. In absolute terms

⁵There are a few properties of these curves of optimal points that can be readily deduced without switching on the computer. For example, it is easy to see that the higher you aim, the further out the optimal conversion point will be.

this advice is more important if the conversion line is close to the goalpost and less important further out. However, the relative difference is always significant.

5 A new angle, and a familiar hyperbola

At this stage, we were quite pleased with our survey. We decided to polish things off by having a chat with players from our local rugby league team, the Melbourne Storm. We'll discuss this more in the next section, but one particular fact struck home, substantially changing view of the problem.

Apparently, rugby kickers do not generally alter their manner of kicking in accordance with their position on the field. That is, rather than aiming for a given height above the crossbar, or similar, they simply kick with approximately the same force and same initial angle of elevation, no matter from where the kick is taken.⁶ What this suggests is that we should be considering a single (approximately) parabolic flight path, just moved from point to point. If we do so, the angle to consider becomes very simple, and is easily optimised.

Fix a point P on the field. Take the two segments from P forming the Hughes angle, and take the two vertical planes lying above these segments. These planes form a wedge, and a conversion attempt will be successful if the initial velocity vector of the ball is within this wedge (assuming the ball is also flying high enough and far enough to clear the crossbar); see Figure 9.

Suppose now that the ball is kicked with an initial elevation angle E . The collection of all such directions forms a cone with apex at P , and a successful kick is one which begins in that portion of the cone within the vertical wedge. Specifically then, it is the raised angle R within the cone that interests us. If H is the Hughes angle at P , then this raised angle is exactly given by

$$R = H \cos E .$$

It follows that, no matter the fixed elevation E , the curve of optimal kicking spots is still exactly Hughes's hyperbola.⁷

⁶A semi-exception to this constancy is when players kick into a strong wind. In this situation, some players kick with a lower angle of elevation, punching the ball into the wind. But still, this new wind-adjusted angle is not generally further adjusted according to the position on the field.

⁷There are other ways to consider this scenario, which lead to slightly different definitions of the raised angle R . However, any reasonable definition leads to the conclusion that R depends only upon H , and in a monotonic manner. Thus one can still conclude that the Hughes hyperbola is optimal for the raised angle.

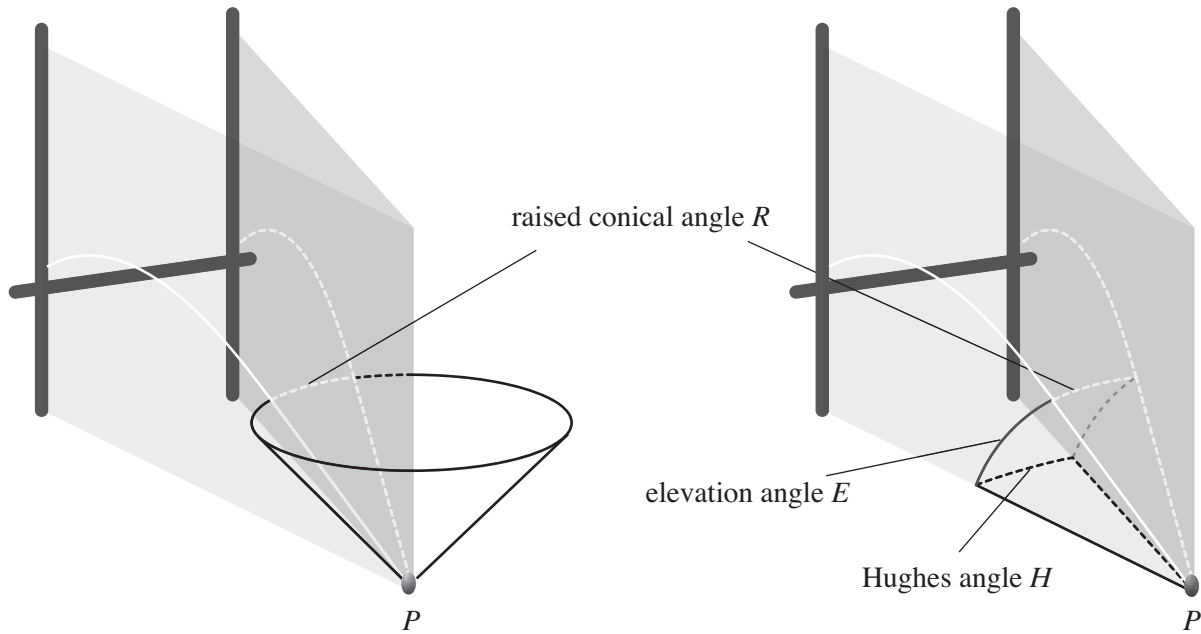


Figure 9: A new raised angle.

6 Optimising the new angle

That is all well and good, but we should be able to do better. True, a player wishes to be on the Hughes hyperbola, to optimise the Hughes angle. But your smarter rugby player will also consider lowering his kicking angle E , at least to the extent that he can.⁸

Though the Hughes hyperbola is still optimal for fixed E , the effective kicking angle R is increased as we flatten the trajectory, in proportion to $\cos E$. For example, kicking with an elevation of $E = 35^\circ$, the effective angle is 18% smaller than the Hughes angle. By comparison, kicking with an elevation $E = 20^\circ$, the effective angle is only 6% smaller. The moral is, all else being equal, and as long as the goals can comfortably be reached, the player should kick at as shallow an angle as possible.

Now, along any conversion line there is a point where E is minimised while still permitting a successful kick: this is the point from where the flattest feasible kick will be flying horizontally as it clears the crossbar. We then have a trade-off, between wanting both to minimise E and to maximise H . The precise point where the raised angle R is maximised will be somewhere inbetween.

Exactly how the trade-off works will depend significantly upon the ball's trajectory for different kicking angles. That is too far afield for us to get into here: to get a sense of things,

⁸It is actually not obvious that a player can reasonably adjust the kicking angle to whatever he wishes, as biomechanical considerations come into play. See our discussion in the following section.

we'll simply assume a standard parabolic flight path.⁹

With the assumption of parabolicity, we can now use the familiar equations for projectile motion to determine the function to be maximised. As in Figure 2, we consider coordinates (x, y) and goal width d . We assume the ball is kicked with initial speed V , and $g = 9.8$ is the acceleration due to gravity. As usual, $H(x, y)$ is the Hughes angle at (x, y) . Finally, we let $E(x, y)$ be the lowest angle of elevation such that kicking from (x, y) results in clearing the middle of the crossbar at a fixed height h above the ground.

Fixing x determines the conversion line. We then want to maximize $R = H \cos E$ as a function of y , where

$$\left\{ \begin{array}{l} \tan H = \frac{4yd}{4y^2 + 4x^2 - d^2}, \\ \sin(2E - \alpha) = \frac{v^2 h + gx^2 + gy^2}{v^2 \sqrt{x^2 + y^2 + h^2}}, \\ \tan \alpha = \frac{h}{\sqrt{x^2 + y^2}}. \end{array} \right.$$

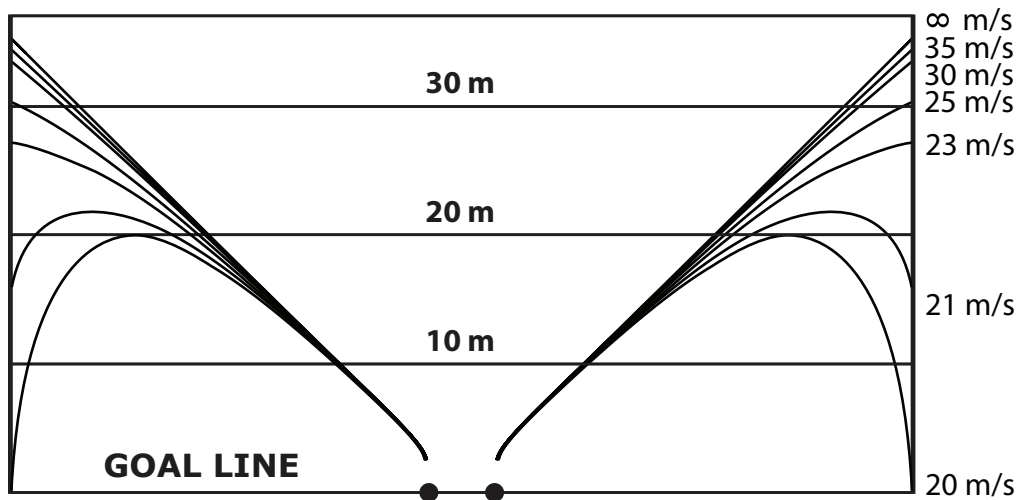


Figure 10: Optimal conversion lines for R , when the elevation angle E is permitted to vary.

Naturally, we again resort to *Mathematica* to do the heavy lifting. We take $d = 5.6$ metres and (allowing for some vertical deviation) $h = 6$ metres. In accord with [7] and [2], we consider initial speeds ranging from 20 m/s to 35 m/s, together with the limiting case, where the speed is infinite and the ball rockets along a straight line. The resulting optimal conversion curves are pictured in Figure 10.

⁹There are various analyses of the dynamics of a football in flight: see, for example, [3]. However, we are not aware of a simple summary of the actual paths of actual rugby balls. In any case, the parabolic approximation is perhaps not so bad for our rough purposes. However, as would be expected, air resistance does tend to bring the ball down quite steeply near the end of its flight.

For realistic kicking speeds (25–30 m/s), the curves lie quite close to the Hughes hyperbola and its asymptote. The limiting curve for infinite speed differs by only 14cm at the sideline, and nowhere by more than 50 cm.

On any conversion line, the smaller the initial speed the closer the optimal point will be to the goal line. If the initial speed is too low, then the goals cannot be reached at all; from the sidelines, the minimum permissible speed is approximately 20 m/s.

In the following table, we consider kicking from the sideline and clearing the crossbar at height 6 meters, with varying initial kicking speeds. We compare the overall optimal cone angle R to the optimal cone angle when shooting from the Hughes point. At the Hughes point, the minimum initial speed to cross at this point is approximately 23.4 m/s.

	optimal cone angle R overall				
optimal angle R	3.54°	3.82°	4.20°	4.34°	4.56°
elevation angle E	37.53°	32.74°	23.67°	18.86°	6.90°
optimal distance y	27.40m	30.28m	33.40m	34.31m	35.14m
	optimal cone angle R at the Hughes point				
optimal angle R	3.01°	3.76°	4.19°	4.34°	4.56°
elevation angle E	47.5°	34.93°	23.93°	18.91°	6.92°
constant distance y	34.88m	34.88m	34.88m	34.88m	34.88m
initial speed	23.4m/s	25m/s	30m/s	35m/s	∞ m/s

Table 3: Optimisation of the cone angle R overall, versus optimising at the Hughes point.

The table suggests that, unless the initial kicking speed is low, there is no significant gain in moving from the Hughes point, equivalently its approximating asymptote.

7 Conclusions: Rugby in the real world

So, finally, what should a rugby kicker actually decide? If the kicker can comfortably reach the goals, our suggestion is that he should come in slightly from the Hughes asymptote, and kick as shallowly as possible. If distance is an issue, then he needs to come closer to the goal line. However, if kicking from the sideline, a good professional player should seemingly still kick from at least the 30 meter line.

The natural question, then, is: how do our suggestions compare to the actual practice of rugby kickers? The answer is: not very closely. Professional players seem to follow the Hughes hyperbola quite closely when near the goals, but then deviate to be nearer the goal line when the conversion line is further away. From the sidelines, professional players typically kick from around the 25 meter mark. They also tend to kick with an initial elevation angle of around 35°, significantly higher than the 30° we might generally suggest. From all points,

the ball tends to clear the crossbar quite high, definitely at least 15 meters and more on occasion.

We are not sure of the reasons for this disagreement. Possibly, rugby players simply have it wrong: a rather lordly suggestion, but not without historical precedent.¹⁰ But it is also wise to be hesitant about our conclusions. We'll close by indicating some of our thoughts in this regard.

There are at least three good reasons to question our conclusions. First of all, kicking distance may be more of an issue than we are suggesting, especially from near the sideline. In particular, the flight of rugby balls is not truly parabolic, implying that we have been overestimating kicking distances. Coming in along the sidelines does not reduce the kicking distance by much (Pythagoras at work), but that small saving may still be significant.

Secondly, though mathematics may suggest the desirability of a low kicking angle, biomechanics may suggest otherwise. Thus, it may simply be physically unnatural to kick at an an “optimally” low angle.¹¹

Finally, one may wish to focus upon two distinct sources of error in kicking direction. The obvious source of error is simply launching the ball in slightly the wrong direction. However, a second source of error is from the ball actually deviating in flight, curling away from the original direction. This latter error is more significant for longer kicks, and may be a significant argument for kicking from closer in.

We are not sure of the practical weight of any of these concerns. We think it best to leave such investigations to those more qualified. However, we will be attending many more rugby games, and we will be keeping our eyes and our thoughts open.

¹⁰Kevin Ball has pointed out to us that kicking styles in rugby have changed dramatically over the years, more than once, and with significant gain in accuracy. We're not sure the extent to which such improvements are due to significant changes in the manufacture of the ball.

¹¹Such considerations are significant, for example, with the field event of shot putting. Mathematically, the optimal angle launch angle for the shot is about 42°. However, biomechanical considerations result in most shot putters launching the shot at around 35°.

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Burkard Polster
School of Mathematical Sciences
Monash University, Victoria 3800
Australia

e-mail: Burkard.Polster@sci.monash.edu.au

Marty Ross
PO Box 83
Fairfield, Victoria 3078
Australia

e-mail: martiniross@gmail.com.au

web: www.qedcat.com