A case of continuous hangover

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Abstract

We consider a continuous analogue of the classic problem of stacking identical bricks to construct a tower of maximal overhang.

1 Introduction

How much of an overhang can we produce by stacking identical rectangular blocks at the edge of a table? Most mathematicians know that the overhang can be as large as desired: we arrange the blocks in the form of a staircase as shown in Figure 1. This stack will (just) fail to topple over, and with n blocks of length 2 the overhang sums to

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}.$$

Since the harmonic series diverges, it follows that the overhang can be arranged to be as large as desired, simply by using a suitably large number of blocks.

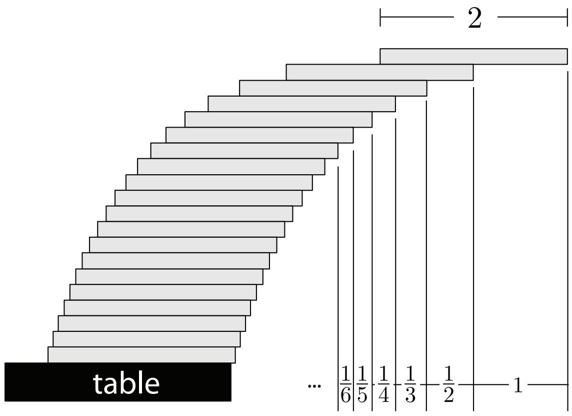

Figure 1: The total overhang of this tower of twenty blocks is $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{20}$.

In practice, these special stacks are constructed from top to bottom: the top block is placed so that its middle, balancing point is at the upper right corner of the second block. Then the top two blocks are placed together so that their combined balancing point is at the upper right corner of the third block, and so on.[1]

Okay, review is over; now for something new. Let's rescale our special stacks in the vertical direction, so that each stack has height 1; the resulting stacks resemble decks of playing cards, as indicated in Figure 2. We'll call these stacks the *harmonic staircases*. Notice that we've arranged for the top right corner of the table to coincide with the origin of our coordinate system.

We will prove (Theorem 2.1) that the sequence of harmonic staircases converges to the *harmonic stack*, determined by the function $1 - e^{-x}$. And, just like the harmonic staircases, the harmonic stack won't topple over. In fact, we will show (Theorem 3.2) that the harmonic stack is stable in a correspondingly stricter sense.

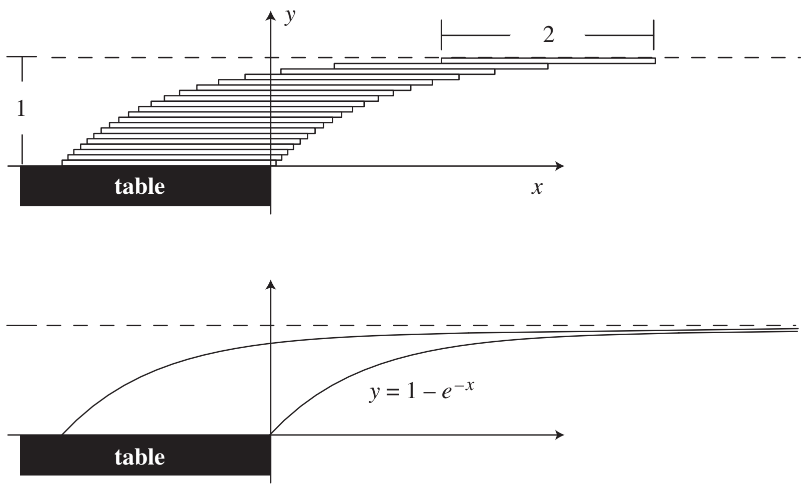

Figure 2: The harmonic staircases converge to the harmonic stack.

Motivated by the limiting process above, in this article we shall consider *general stacks* of width 2 and height 1. A general stack is not one solid piece, but rather consists of infinitely many infinitely thin and unconnected horizontal blocks. Similar to a tower of finite blocks, a general stack is capable of toppling at any level: to avoid toppling at a given height, the center of mass of the stack above that height must lie directly above the cross-section of the stack at that height. (By comparison, a solid stack is safe from toppling just as long as its total center of mass lies above its base.)

As indicated above, the harmonic staircases are distinguished in the framework of the original problem by each block being extended as far as possible.

[1] Recently, stacks have been investigated for which it is permitted to place two blocks upon any lower block. Stacking in this way, one can use some blocks as counterweights and thus achieve significantly greater overhangs than with the staircases. See [5] for these recent results and a comprehensive bibliography for the stacking problem.

We will show (Theorem 3.2) that the harmonic stack is similarly characterized among the stable stacks: cutting the stack horizontally at any height into two pieces, the center of mass of the top piece lies directly above the upper right corner of the lower piece.

The harmonic staircase consisting of n blocks has maximum overhang within a natural class of stacks made up of the same blocks. Similarly, we will show (Theorem 6.1) that the harmonic stack is a fastest growing stable stack. What may be surprising is that the harmonic stack is not the uniquely fastest growing stable stack (Theorem 6.2).

Other results include various methods of transforming stable stacks into new stable stacks, and further characterizations of the harmonic stack amongst stable stacks. All the arguments employed are elementary.

Getting ready to stack

The original stacking problem is posed in terms of three-dimensional blocks. However, the harmonic staircases and all stacks that we are interested in are simply figures in the xy-plane, orthogonally extended in the z-direction. Clearly, the two-dimensional stacks will be stable if and only if their extensions are. So, we lose nothing by restricting ourselves to discussing and drawing two-dimensional stacks. (In Section 7, we make a short excursion into the world of 3D blocks).

For a further simplification, note that scaling a stack horizontally or vertically cannot affect its stability. It follows that we can normalize, making all stacks one unit high and two units wide. There will be one further normalization in Section 3, once we introduce the notion of the *gravity curve* of a stack: effectively, all stacks considered will *just* balance at the table level.

We can now formally define stacks and the stability of stacks.

2 What is a stack?

The *stack* S given by the *stack function* $f : [0,1) \to \mathbb{R}$ is the region in the plane bordered by the graphs of $x = f(y)$ (*not* $y = f(x)$!) and $x = f(y) - 2$:

$$S = \{(x,y) : 0 \leqslant y < 1,\ f(y) - 2 \leqslant x \leqslant f(y)\}.$$

Though more general functions can be considered, it is natural to restrict to stack functions that are integrable,[2] and piecewise continuous and right continuous: that is, the one-sided limits of f exist at any $y \in [0,1)$, and $\lim_{t\to y^+} f(t) = f(y)$. We shall assume this throughout.[3]

[2]The stack function being either Riemann or Lebesgue integrable suffices. It is also sufficient that the stack have an improper Riemann integral as $y \to 1^-$.

[3]For some of what follows it is only required that f be Lebesgue integrable. However, such considerations are a bit arcane, even for us.

As the simplest example, the constant stack function $f(y) = 1$ gives a vertical stack. The harmonic stack HAR, pictured in Figure 2, has stack function $har(y)$ given by the inverse of $1 - e^{-x}$:

$$har(y) = -\log(1 - y)\,.$$

The *n-block harmonic staircase* HAR_n has piecewise constant stack function

$$har_n(y) = \sum_{m=n-\lfloor ny \rfloor}^{n} \frac{1}{m}\,,$$

where the floor function $\lfloor ny \rfloor$ is the largest integer $m \leqslant ny$. We've shaded the graph of $har_6(y)$ in the picture of HAR_6 in Figure 3. Essentially, it is comprised of the vertical right-hand borders of the rectangular blocks in the stack.

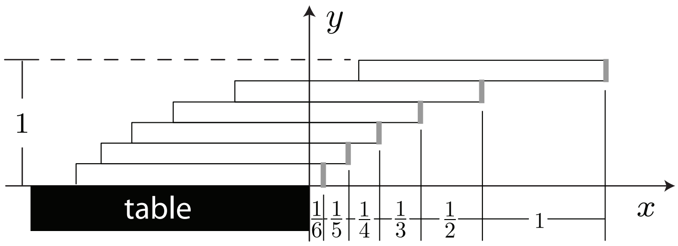

Figure 3: The 6-block harmonic staircase.

As indicated in the picture, all stacks are resting on the x-axis, and the table extends from $-\infty$ to 0, making the upper right corner of the table the origin. The *weight* of part of a stack is simply its area. Since stacks are of height 1 and constant width 2, it follows that all stacks have total weight 2.

Theorem 2.1 (The harmonic staircases converge to the harmonic stack) *The harmonic stack function $har(y)$ is the pointwise limit as $n \to \infty$ of the harmonic staircase functions $har_n(y)$.*[4]

Proof. We use the estimate

$$\sum_{m=1}^{n} \frac{1}{m} = \log(n+1) + \gamma + \epsilon_n\,,$$

[4]The proof actually shows that, for $h < 1$, $har_n(y)$ converges uniformly to $har(y)$ on $[0, h]$. Note also that Theorem 2.1 can be related to the Maclaurin expansion of the harmonic stack function:

$$har(y) = -\log(1 - y) = y + \frac{y^2}{2} + \frac{y^3}{3} + \frac{y^4}{4} + \cdots.$$

For the limiting value $y = 1$ this identity says that the overhang of the harmonic stack is equal to the limit of the harmonic series.

where γ is the Euler-Mascheroni constant and $\epsilon_n \to 0$. Now let $y \in [0,1)$. Then

$$\begin{aligned} har_n(y) &= \sum_{m=1}^{n} \frac{1}{m} - \sum_{m=1}^{n-1-\lfloor ny \rfloor} \frac{1}{m} \\ &= (\log(n+1) + \gamma + \epsilon_n) - \big(\log(n - \lfloor ny \rfloor) + \gamma + \epsilon_{n-1-\lfloor ny \rfloor}\big) \\ &= -\log\left(1 - \frac{\lfloor ny \rfloor}{n}\right) + \log\left(\frac{n+1}{n}\right) + \epsilon_n - \epsilon_{n-1-\lfloor ny \rfloor}. \end{aligned}$$

Now, $0 \leqslant ny - \lfloor ny \rfloor < 1$, from which it follows that $\frac{\lfloor ny \rfloor}{n} \to y$. Also, since $y < 1$,

$$n - 1 - \lfloor ny \rfloor = ny - \lfloor ny \rfloor - 1 + n(1-y) \to \infty\,.$$

Consequently

$$\lim_{n\to\infty} har_n(y) = -\log(1-y) = har(y)\,.$$

□

3 The stability of a stack and its gravity curve

Consider the stack in Figure 4 below. Thought of as a stack of six unconnected rectangles, it is clear that the top two rectangles will fall down: the center of mass of the top rectangle is not above the second rectangle, and the center of mass of the combined top two rectangles is not above the stack consisting of the remaining four rectangles. However, considered as one solid piece, the center of mass of this stack is above the table, and so it will not topple over.

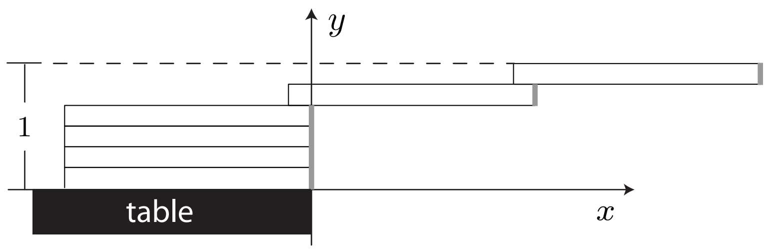

Figure 4: As one solid piece, this stack has a huge overhang and will not topple over.

We want to consider our stacks as unconnected, and thus capable of toppling, at any height; in effect, we are thinking of stacks as consisting of infinitely many infinitely thin blocks. It is then natural to define a stack to be stable if it balances at every possible level.

The gravity function and the gravity curve of a stack

More formally, consider a general stack S, and fix a height y. Consider the slab of S lying above y, and let $g(y)$ be the x-coordinate of the center of mass of this slab. Then we call $(g(y), y)$ the *gravity point* of S at height y. Notice that a gravity point always lies directly below the center of mass of the slab defining it, as pictured in Figure 5. We also call the function $g(y)$ the *gravity function* of the stack S, and the graph of $x = g(y)$ is the *gravity curve* of S.

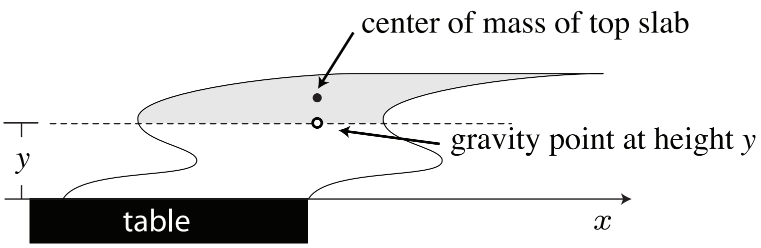

Figure 5: The gravity point at height y is directly below the center of mass of the top slab.

Proposition 3.1 (Equation of the gravity function) *Suppose S is a stack with stack function f. Then the gravity function of S is continuous and is given by*

$$g(y) = \frac{\int_y^1 (f(t) - 1)\,\mathrm{d}t}{1 - y}.$$

Wherever $f(y)$ is continuous, the gravity function $g(y)$ is differentiable and

$$f(y) = 1 - [g(y)(1 - y)]'.$$

At every point, g is differentiable from the right, and the above equation for f holds everywhere with the derivative so interpreted. Consequently, the stack function uniquely determines the gravity function and vice versa.[5]

Proof. If we consider the stack S to be made of infinitesimally thin blocks, then the block at height t has mass $2\mathrm{d}t$, and the x-coordinate of its center of mass is $f(t) - 1$. It then follows that

$$\begin{aligned} g(y) &= \frac{\int\limits_y^1 (f(t) - 1) \times 2\,\mathrm{d}t}{\text{mass of the slab above height } y} \\ &= \frac{\int\limits_y^1 (f(t) - 1)\,\mathrm{d}t}{(1 - y)}. \end{aligned}$$

[5]In the more general setting of Lebesgue integrable functions, the gravity function g is absolutely continuous, and determines f almost everywhere.

The rest of the proposition follows by multiplying by $1 - y$ and applying the fundamental theorem of calculus, recalling that we are only considering stack functions that are continuous from the right. $\square$

From here on, $g'(y)$ shall always denote, if need be, the right derivative of the gravity function g. The previous result then promises that this right derivative always exists.

Normalizing the tables

We say that a stack S is *stable* if S contains its gravity curve. If f is the stack function of S, this is the case exactly when

$$f(y) - 2 \leqslant g(y) \leqslant f(y) \qquad \text{for all } y \in [0, 1).$$

Further, we say that a stack is *balanced at* 0 if $g(0) = 0$: that is, if the center of mass of the whole stack is above the top right corner of the table.

It is easy to check that all harmonic staircases are stable stacks balanced at 0. In fact, a harmonic staircase is constructed exactly so that its gravity curve will contain the top right corner of the table, as well the top right corners of all but the topmost block; through the top block, the gravity curve is simply a vertical line directly up the middle.

As part of the next result, we prove that HAR is balanced at 0. As well, it is obvious that any stack can be translated to be balanced at 0. This justifies the following normalization:

From here on, we shall consider only those stacks that are balanced at 0.

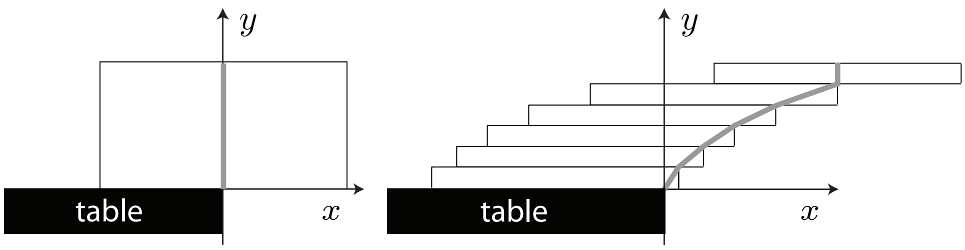

Figure 6: The gravity curves of the vertical stack and a harmonic staircase.

The gravity curve of the harmonic stack

Intuitively, the stack functions of HAR_n approximate the gravity functions of HAR_n, suggesting that the stack and gravity functions of HAR should coincide. This is indeed the case.

Theorem 3.2 (The harmonic stack and gravity functions coincide) *The harmonic stack HAR is the unique stack whose gravity function and stack function coincide. In particular, HAR is stable.*

Proof. Suppose S is a stack with stack function f. By Proposition 3.1, the stack and gravity functions of S will coincide if and only if f is continuous and

$$\frac{\int_y^1 (f(t)-1)\,\mathrm{d}t}{1-y} = f(y)\,.$$

It is easy to verify that the harmonic stack function satisfies this equation. Conversely, suppose the equation holds. It follows that the integral is differentiable, and thus f must be differentiable (in fact infinitely differentiable). Multiplying the equation by $1-y$ and differentiating,

$$1 - f(y) = f'(y)(1-y) - f(y)\,,$$

and so

$$f'(y) = \frac{1}{1-y}\,.$$

Antidifferentiating gives

$$f(y) = -\log(1-y) + C\,.$$

Since we have normalized to have all stacks balanced at 0, the only possibility is $C = 0$, giving the harmonic stack.

□

For completeness, we also prove:

Proposition 3.3 *The gravity functions of the stacks HAR_n converge pointwise to har.*

Proof. Let g_n be the gravity function of HAR_n and let g be the gravity function of HAR. We prove that, as suggested above, $g_n(y) - har_n(y) \to 0$ pointwise on $[0,1)$. The proposition then follows immediately from Theorem 2.1 and Theorem 3.2.

Fix $y \in [0,1)$, suppose $n \in \mathbb{N}$, and set $m = \lfloor ny \rfloor$. Then $y \in \left[\frac{m}{n}, \frac{m+1}{n}\right)$. If n is large then $n > m+1$, and therefore

$$g_n\left(\frac{m}{n}\right) \leqslant g_n(y) \leqslant har_n\,(y) = g_n\left(\frac{m+1}{n}\right) = g_n\left(\frac{m}{n}\right) + \frac{1}{n-m}\,.$$

From the proof of Theorem 2.1, we know that $\frac{1}{n-m} = \frac{1}{n-\lfloor ny \rfloor} \to 0$. It follows that $g_n(y) - har_n(y) \to 0$, as desired. □

4 Cut and paste

The following diagram shows two stacks S and T together with their gravity curves. We now slice both stacks at height t, and then combine them to make a new stack as pictured: the bottom slab of S stays fixed, and the top slab of T is horizontally translated so that the ends of the gravity curves coincide. We will denote this new stack by $S \setminus_t T$. From Proposition 3.1 we know that gravity functions are continuous. It then follows immediately from the definition of the gravity curve that the gravity curve of $S \setminus_t T$ is exactly the union of the two part-curves.

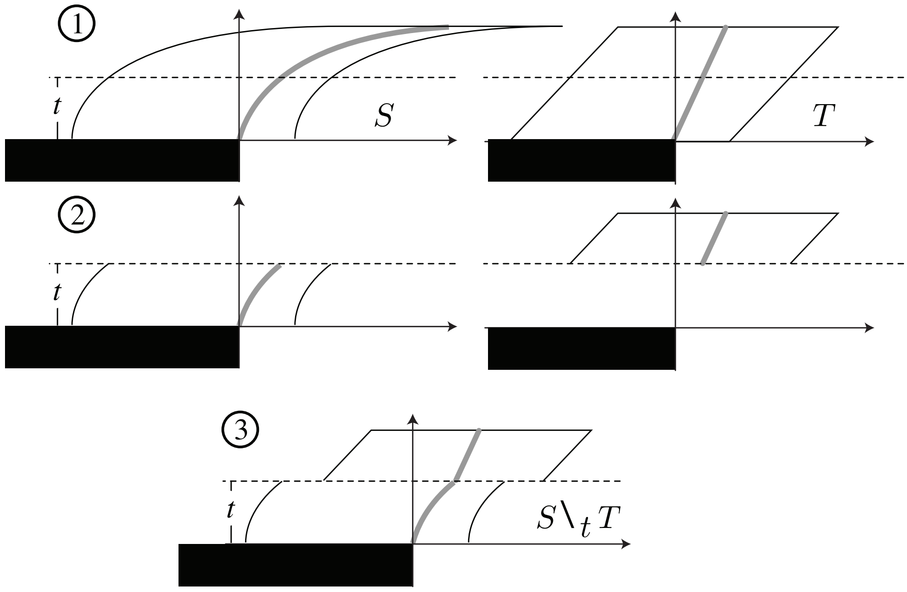

Figure 7: Combining parts of two stacks by aligning the gravity curves.

We conclude that:

Proposition 4.1 (Properties of cut and pasted stacks) *Let S and T be two stacks. If both S and T are stable then so is $S \setminus_t T$.*

$\square$

Here is a nice application of this construction. Suppose that S and T are stable stacks. Then $S_t = S \setminus_t T, t \in [0, 1]$ is a continuous deformation of T into S with all intermediate stacks being stable.

Cutting and pasting is also a useful technique for transforming finite stacks of rectangular blocks. As an example, the stack $har_2 \setminus_{\frac{1}{2}} har_4$ consists of the three blocks pictured in Figure 8, with the lower two overhangs of length $\frac{1}{2}$.

Now replace the top block, by cutting and pasting with har_8 at $t = \frac{3}{4}$, giving $(har_2 \setminus_{\frac{1}{2}} har_4) \setminus_{\frac{3}{4}} har_8$. Continuing this process forever and taking the limit, we arrive at the infinite-block stack

$$HALF = ((((har_2 \setminus_{\frac{1}{2}} har_4) \setminus_{\frac{3}{4}} har_8) \setminus_{\frac{7}{8}} har_{16}) \setminus_{\frac{15}{16}} har_{32}) \cdots$$

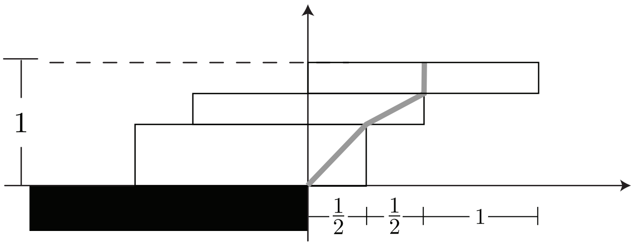

Figure 8: The stack $har_2 \setminus_{\frac{1}{2}} har_4$. The gray curve is the gravity curve.

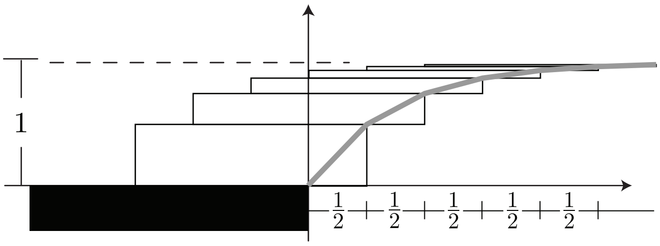

Figure 9: Finite stacks converging to $HALF$. The gray curve is the gravity curve.

The stack $HALF$ consists of blocks of heights $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$, with all overhangs of length $\frac{1}{2}$. So, $HALF$ has infinite overhang. Also, by applying Proposition 3.1 one can show that the gravity curves of $HALF$ and the nth finite approximation to $HALF$ coincide on the interval $\left[0, \frac{2^n-1}{2^n}\right)$. It follows that $HALF$ is stable and that its gravity curve passes through the top right corners of the blocks, as indicated in Figure 9.

5 Stretching and translating

Another useful method of modifying a stack S is to take the top slab of S, above some height t, and then dilate and translate this slab to form a new stack; see Figure 10. We denote the resulting stack by $\downarrow_t S$.

The stack function for $\downarrow_t S$ is easily determined from the construction, and then the gravity function is easily determined from Proposition 3.1. We summarize this in the following proposition.

Proposition 5.1 (The defining functions of stretched stacks) *Let S be a stack with stack function f and gravity function g, and let $t \in [0, 1)$. Then $\downarrow_t S$ has stack function*

$$f(y(1-t)+t) - g(t)\,,$$

and gravity function

$$g(y(1-t)+t) - g(t)\,.$$

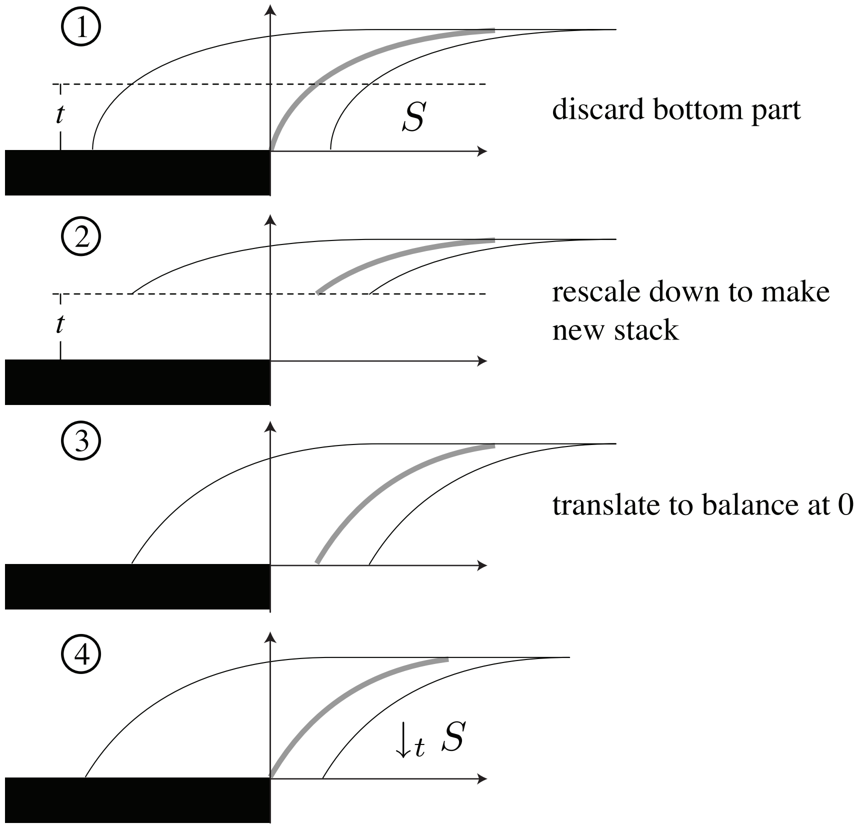

Figure 10: Stretching and translating the top part of a stack into a new stack.

If S is stable then so is $\downarrow_t S$.

$\square$

A direct calculation using Proposition 5.1 shows that HAR is *self-similar*:

$$\downarrow_t HAR = HAR \quad \text{for any } t \in [0, 1).$$

However, HAR is not the only self-similar stack:

Theorem 5.2 (Classification of self-similar stacks) *Let S be a stack such that $\downarrow_t S = S$ for all $t \in [0, 1)$. Then the stack function f and gravity function g of S are of the form*

$$f(y) = a \log(1 - y) + a + 1$$

and

$$g(y) = a \log(1 - y),$$

for some $a \in \mathbb{R}$.

Notice that $a = 0$ gives the vertical stack, and $a = -1$ gives HAR. Also, $a = 1$ gives HAR reflected in the y-axis. It is clear that the stable stacks are given by $a \in [-1, 1]$, and so $a = \pm 1$ give the extreme stacks.

Proof of Theorem 5.2. Since $\downarrow_t S = S$, the gravity curves of the two stacks have to be identical. So, by Proposition 5.1,

$$g(y(1-t)+t) - g(t) = g(y) \quad \text{for all } y, t \in [0, 1).$$

Rearranging and dividing by $t(1-y)$, we have

$$\frac{g(y+t(1-y)) - g(y)}{t(1-y)} = \frac{g(t)}{t(1-y)}\,.$$

Letting $t \to 0^+$ and applying $g(0) = 0$ gives

$$g'(y) = \frac{g'(0)}{1-y}\,,$$

where $g'(y)$ denotes the right derivative of g. Since $g(0) = 0$, we can conclude that

$$g(y) = a \log(1-y)$$

for some $a \in \mathbb{R}$.[6] Then, from Proposition 3.1,

$$f(y) = 1 - (g(y)(1-y))' = a \log(1-y) + a + 1.$$

□

Notice that the $HALF$ stack introduced in the previous section is self-similar for infinitely many values of t: it is easy to check that

$$\downarrow_{\left(1-\frac{1}{2^n}\right)} HALF = HALF \quad \text{for all } n \in \mathbb{N}.$$

6 Which stacks grow the fastest?

When stacking n identical blocks, with each block supporting at most one block above, it can be shown that the harmonic staircase HAR_n has maximal overhang: see [4, Section II]. However, HAR_n is not unique in this regard, as an adaptation of HAR_n does just as well, as shown in Figure 11.

In this section we will prove similar results for HAR. First, we prove (Theorem 6.1) that HAR has the fastest growing gravity function amongst stable stacks. We then prove (Theorem 6.2) that HAR has one of the fastest growing stack functions, but that it is not unique in this regard.

In what follows, we will consider stacks growing to the right of the table. Of course stacks can also grow to the left, and there are obvious left versions of all our results below.

[6] Even though g may only have a one-sided derivative, the continuity of g still permits us to antidifferentiate g', obtaining the desired expression for g. See, for example, [1, Theorem 7.1].

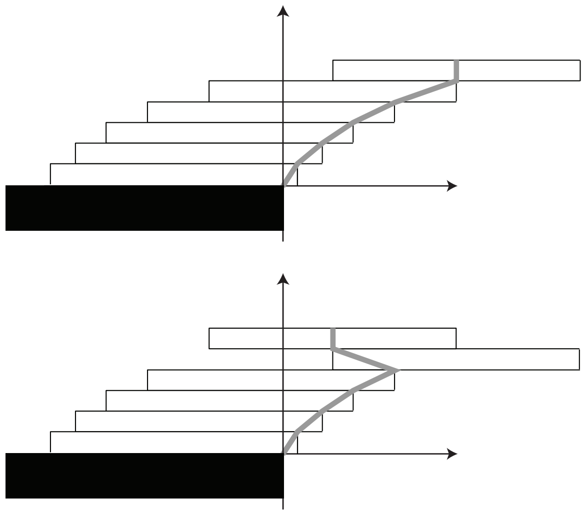

Figure 11: Converting HAR_6 into another stable stack of equal, maximal overhang.

Theorem 6.1 (HAR has the fastest growing gravity function) *Suppose S is a stable stack with gravity function g. Then $har - g$ is a nondecreasing and nonnegative function.*

Note that it is possible that $g = har$ on an initial interval $[0, t]$. However, Theorem 6.1 implies that once har gets in front of g, it remains so.

Proof of Theorem 6.1: Since $har(0) = g(0) = 0$, we only need to prove that $har - g$ is nondecreasing. To do this, assume by way of contradiction that $har'(t) - g'(t) < 0$ for some $t \in [0, 1)$, where $g'(t)$ refers to the right derivative of g. Let h be the gravity function of $\downarrow_t S$. From Proposition 5.1 and the self-similarity of HAR, it follows that $har = \downarrow_t har < h$ on some interval $(0, s]$.

Now consider the stack $T = (\downarrow_t S) \setminus_s HAR$. By Proposition 4.1 and Proposition 5.1, T is stable. Further, if k is the gravity function and f is the stack function of T, then the stability of T and the choice of s ensures that $har < k \leqslant f$ on $(0, 1)$; see Figure 12. But this is a contradiction: since HAR is balanced at 0, $har < f$ implies that the center of mass of T will be to the right of the origin. $\square$

Having proved the gravity function har of HAR is dominant, what can we say about har as a *stack function* relative to other stack functions? Certainly, HAR need not always be in front of other stacks. For example, the vertical stack begins in front of HAR, before being overtaken.

On the other hand, to have a stable stack S strictly in front of HAR from a certain height on is impossible: if this were the case, then above that height the gravity function of S would also be in front of har, contradicting our previous

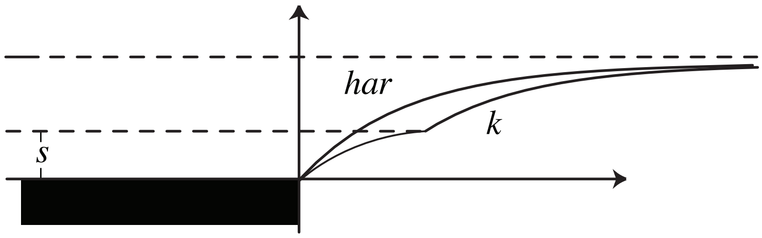

Figure 12: The gravity function k of the stack T is to the right of har.

theorem. However, there do exist stable stacks that effectively compete for the lead:

Theorem 6.2 (The fastest growing stack functions) *There are no stable stacks that stay ahead of HAR for all y near 1. However, there do exist stacks that grow as fast as HAR. That is, there is a stable stack S, with stack function f, such that $f(y) - har(y)$ changes sign infinitely often as y approaches 1.*

Proof of Theorem 6.2: We have already argued that a stable stack cannot stay ahead of HAR for all y near 1. We will now construct a stack S that repeatedly alternates with HAR for the lead: S has stack function f satisfying

$$har(y) < f(y) \quad \text{for values of } y \text{ arbitrarily close to 1.}$$

To see how to construct such a stack S, we first assume that S has the desired lead-changing property, and we use this to derive explicit sufficient conditions for the stack function f. We then show that f can indeed be chosen so that these conditions are satisfied.

We begin by considering the gravity function g of S. Since S is assumed stable, it follows that for any y for which $har(y) < f(y)$, we must also have

$$har(y) - g(y) < 2\,.$$

But, by Theorem 6.1, $har - g$ is nondecreasing. It follows that if S is forever changing the lead with HAR then $har(y) - g(y) < 2$ for *all* y near 1, and thus that $har - g$ converges to some $m \in [0, 2]$ as y goes to 1. We can therefore write

$$g(y) = har(y) - m + \epsilon(y)\,,$$

where $\epsilon \geqslant 0$ is a nonincreasing function, and $\epsilon(y) \to 0$ as $y \to 1$.

Any $m \in (0, 2)$ will suffice for what follows, but for definiteness we take $m = 1$. Then, since S is to be balanced at 0, we must have

$$\epsilon(0) = 1\,.$$

We now assume that $\epsilon(y)$ is differentiable. Then, using Proposition 3.1, we can calculate

$$\begin{aligned} f(y) &= 1 - [g(y)(1-y)]' \\ &= 1 - [(har(y) - 1 + \epsilon(y))(1-y)]' \\ &= -[har(y)(1-y)]' - \epsilon'(y)(1-y) + \epsilon(y)\,. \end{aligned}$$

By Theorem 3.2, we also know that $1 - [har(y)(1-y)]' = har(y)$, and so

$$f(y) = har(y) - 1 + \epsilon(y) - \epsilon'(y)(1-y)\,.$$

We want S to be stable, for which we require $f - 2 \leqslant g \leqslant f$. Since $g = har - 1 + \epsilon$, this is equivalent to

$$0 \leqslant -\epsilon'(y)(1-y) \leqslant 2 \quad \text{for all } y.$$

Finally, S being ahead of HAR at height y amounts to $f(y) > har(y)$. So, for the lead-changing property, it suffices to have

$$1 < -\epsilon'(y)(1-y) \quad \text{for values of } y \text{ arbitrarily close to 1.}$$

In summary, it suffices for us to find a nonincreasing and differentiable function $\epsilon : [0,1) \to \mathbb{R}$ such that

$$\begin{cases} \epsilon(0) = 1 \\ \lim_{y \to 1^-} \epsilon(y) = 0 \\ -\epsilon'(y)(1-y) \leqslant 2 \quad \text{for all } y \\ -\epsilon'(y)(1-y) > 1 \quad \text{for values of } y \text{ arbitrarily close to 1.} \end{cases}$$

These conditions will guarantee that the corresponding stack S is stable and balanced at 0, and will repeatedly overtake HAR.

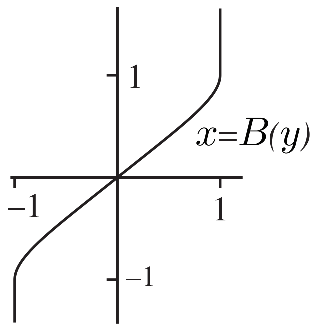

Figure 13: The prototype S-bend B.

To construct an explicit function ϵ satisfying these conditions, we shall take ϵ to have constant segments that are connected by small S-bends of just the right size and slope. To do this, we first define a suitable prototype S-bend; see Figure 13:

$$B(y) = \frac{5}{4}y - \frac{1}{4}y^5, \quad y \in [-1,1]\,.$$

Note that $B(\pm 1) = \pm 1$ and $B'(\pm 1) = 0$. Also, $B' \geqslant 0$ on $[-1, 1]$, with a maximum of $\frac{5}{4}$. We also take $B(y) = 1$ for $y > 1$ and $B(y) = -1$ for $y < -1$.

We now define ϵ by subtracting a sum of suitable linear transformations of B. Specifically, define

$$\epsilon(y) = 1 - \sum_{n=1}^{\infty} B_n(y)\,,$$

where B_n is the S-bend B transformed to rise from 0 to $\frac{1}{2^n}$ on the interval $\left[\frac{2^n-1}{2^n} - \frac{1}{2^{2n}}, \frac{2^n-1}{2^n}\right]$; see Figure 14.

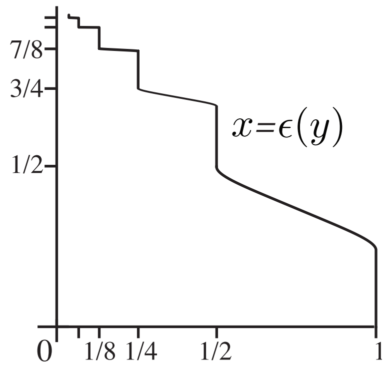

Figure 14: The function ϵ.

Clearly ϵ is nonincreasing, with $\epsilon(0) = 1$ and $\epsilon(y) \to 0$ as $y \to 1$. Also, B_n has a maximum slope of $5 \cdot 2^{n-2}$. And, on the interval where B_n is bending, we have $\frac{1}{2^n} \leqslant 1 - y \leqslant \frac{3}{2^{n+1}}$. It follows that at the point of maximum slope of B_n, we have

$$-\epsilon'(y)(1-y) \geqslant 5 \cdot 2^{n-2} \cdot \frac{1}{2^n} = \frac{5}{4} > 1\,.$$

Furthermore, at every point on the bending interval of B_n, we have

$$-\epsilon'(y)(1-y) \leqslant 5 \cdot 2^{n-2} \cdot \frac{3}{2^{n+1}} = \frac{15}{8} \leqslant 2\,.$$

It follows that ϵ has exactly the properties desired. The resulting stack S is shown shaded in Figure 15, with *har* and the gravity curve of S superimposed. $\square$

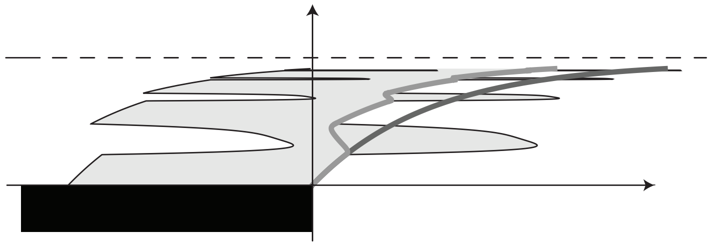

Figure 15: A stable stack that grows as fast as HAR.

7 Afterthought 1: Flipping

Figure 16 illustrates another natural operation for transforming a stack S into a new stack: take the slab above a certain height $t \in [0, 1)$, and reflect that slab about the vertical line through its gravity point. We denote the new stack by $\leftrightarrow_t S$. It is clear that if S is stable then so is $\leftrightarrow_t S$.

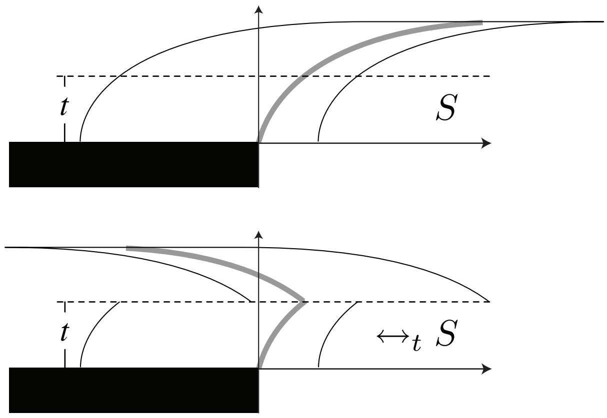

Figure 16: Reflecting the top slab of a stack about an axis through its gravity point.

We now apply this flipping procedure to construct a stable stack that has infinite overhang to both the left and the right. (The following was inspired by similar constructions involving finitely many blocks, described in [2, Chapters 12.5 and 12.7].)

Recall the stack $HALF$ constructed at the end of Section 4, consisting of blocks of heights $\frac{1}{2^n}$, each placed with overhang $\frac{1}{2}$. We now flip this stack above the 1st, 3rd, 6th, and in general the $\frac{(n+1)n}{2}$th block; each pair of flips results in the stack extending $\frac{1}{2}$ further in both directions. Using Proposition 3.1, it is then easy to show that the limiting result of these flips is a stable stack S, with stack function having unbounded oscillation in both directions as $y \to 1$.

Note also that this flipping procedure can be used to construct a stack that continually overtakes the harmonic stack, similar to that constructed at the end of the previous section. For this we modify the harmonic stack by flipping out infinitely many small horizontal slivers that get arbitrarily close to the top of the stack. It is then possible to arrange for these slivers to jut beyond the harmonic stack.

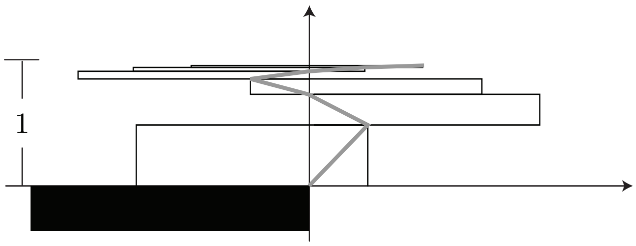

Figure 17: Constructing a stable stack with infinite overhang to both the left and right.

We now momentarily venture into the world of 3-dimensional blocks. We'll create a stack that casts a shadow over the whole xz-plane. (We'll continue to label the vertical direction as y.)

Begin with the oscillating stack S just constructed. Notice that there are infinitely many blocks of S such that a top corner of the block lies above the origin and the gravity curve of S also passes through that corner. Now, thicken the blocks in S to have a thickness d in the z direction, giving a 3D stack $\widehat{S}$. Next, take any fixed irrational number a, and consider the angle $\alpha = a\pi$. Finally, at the height of each of the distinguished corners of S above the origin, successively rotate the top slab of $\widehat{S}$ the angle α around the y-axis.

Any angle θ is closely approximated by arbitrarily large integer multiples of α. It follows that, no matter how small the thickness d, any point in the direction θ will eventually lie under some block of $\widehat{S}$. It follows that $\widehat{S}$ casts a shadow over the whole xz-plane.

8 Afterthought 2: Balancing the exponential function

In this section, we derive an interesting balancing property of the exponential function. We use this property to reprove the result from Theorem 3.2, that the stack function and gravity function of the harmonic stack coincide.

Proposition 8.1 (Balancing the exponential function) *The region under the graph of the function* e^x *to the left of the point* $x = a$ *balances over a fulcrum at* $x = a - 1$.

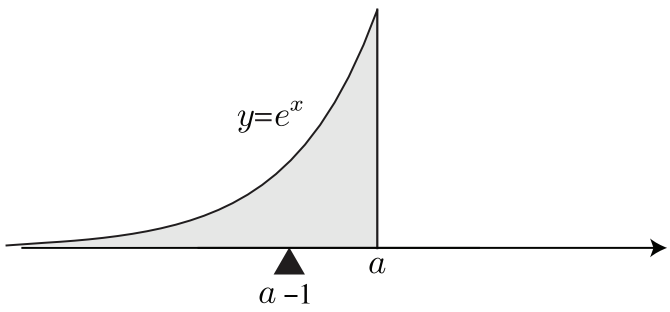

Figure 18: The tail of the region under $y = e^x$ always balances 1 unit to the left of the cut.

Proof. Using the standard formula, we find that the x-coordinate of the center of mass of the tail region is

$$\frac{\int_{-\infty}^{a} xe^x \,\mathrm{d}x}{\int_{-\infty}^{a} e^x \,\mathrm{d}x} = a - 1.$$

□

Figure 19 shows the region between the graph of the exponential function e^x and its horizontal translate e^{x-2}, truncated at a certain height. If the height is less than 1, then this is exactly a top slab of the harmonic stack, rotated 180 degrees about the point $(0, 1/2)$. What we want to prove is that no matter where we cut, the shaded region balances with the fulcrum at a: this establishes again that the gravity function and the stack function of HAR are identical.

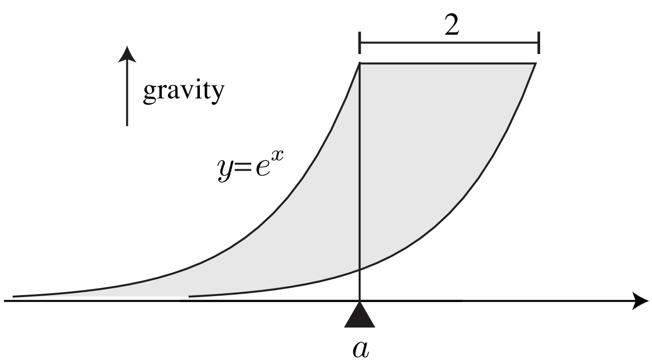

Figure 19: The truncated region between $y = e^x$ and $y = e^{x-2}$ balances at $x = a$.

Consider the region lying between $y = e^x$ and the horizontal translate $y = e^{x-d}$, and to the left of $x = a$; see Figure 20. By the previous proposition this sliver is the difference of two regions that balance over $a - 1$, and hence the sliver also balances over $a - 1$.

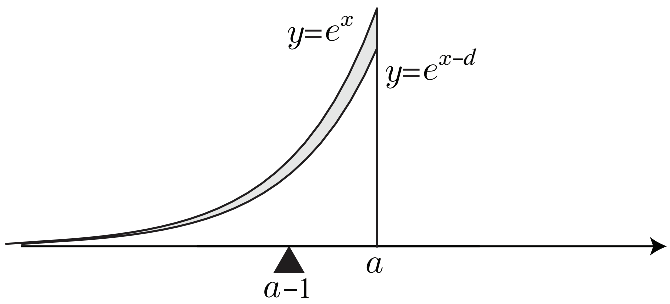

Figure 20: The sliver trapped by $y = e^x$ and its translate balances over a pivot at $a - 1$.

Now set the horizontal difference to be $d = \frac{2}{n}$. Then $n + 1$ copies of the sliver fit together seamlessly into the shaded region in Figure 19, with a few curvy triangles missing at the top and an extra sliver sticking out on the right; see Figure 21.

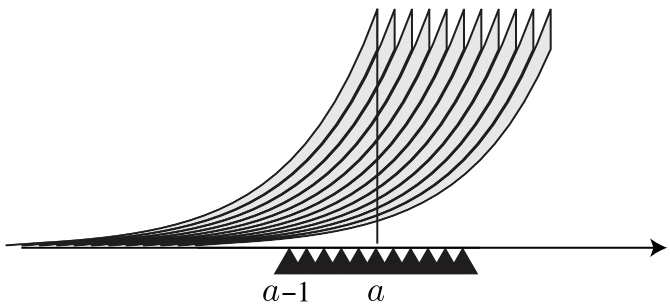

Figure 21: Translates of the sliver combine to approximate the shaded region in Figure 19.

The fulcrums of these $n + 1$ slivers are equally spaced from $a - 1$ to $a + 1$ meaning the fulcrum of the entire region is at the middle point a. Letting $n \to \infty$, the triangular gaps at the top and the extra sliver on the right vanish, proving again that the gravity function and stack function of HAR coincide.

It can also be shown that, within a natural class of functions, the exponential function is characterized by the balancing property established in Theorem 8.1. The proof is a straightforward exercise.

9 Afterthought 3: A candidate but no winner?

Our stacks provide an interesting situation, where the natural candidate for optimality is *not* optimal. Suppose, ignoring what we have learned from Theorem 6.2, that we try to prove HAR is the stack with eventual greatest overhang: that is, for every stable stack function f it is eventually the case that $har \geq f$. We now argue that *if* there is a stack with eventual greatest overhang, *then* it is HAR. This is reminiscent of Jakob Steiner's original and famously flawed approach to the *isoperimetric problem*, that the circle maximizes area amongst closed curves of a given perimeter; see, for example, [3].

Consider any other stable stack S with stack function f and gravity function g. Then, by Theorem 3.2, there exists an interval $[s, t]$ on which $g < f$. Figure 22 shows the stack S and its gravity curve subdivided into a top, middle, and bottom slab, with the middle slab corresponding to the interval $[s, t]$.

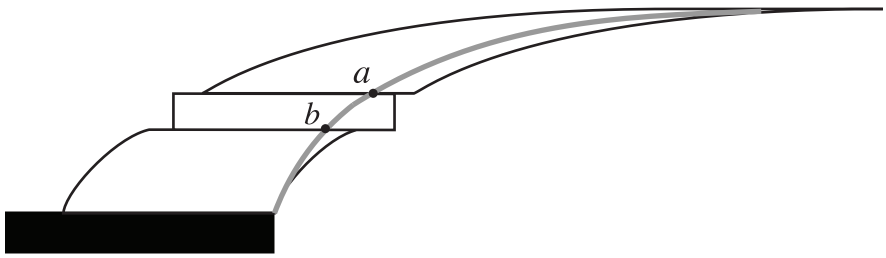

Figure 22: A stable non-harmonic stack together with its gravity curve.

Now, if we slide the top slab slightly to the right, then its gravity point a will stay within the top of the middle slab, ensuring that the top slab will not topple. However, sliding the top slab will move the gravity point of the whole stack to the right, and the stack will not be balanced at 0.

However, we can avoid this by simultaneously sliding the top slab to the right and the middle slab to the left. Clearly, we can do this in such a way that the gravity point b of the combined top and middle slabs stays fixed, and so leaving the gravity curve inside the bottom slab unchanged. This means that the adjusted stack still balances, with the top slab being further to the right than for the original stack: the adjusted stack eventually has greater overhang.

This argument can be applied to any stack other than HAR, and thus establishes that if there is a stack of eventual maximum overhang, then it must be HAR. Note that this does imply that HAR can be regarded as the fastest growing stable stack in a certain sense: HAR is the only stable stack that cannot be completely overtaken by any other stable stack.

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