Irrational Thoughts
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Irrational thoughts

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> 'Now as to what pertains to these Surd numbers (which, as it were by way of reproach and calumny, having no merit of their own are also styled Irrational, Irregular, and Inexplicable) they are by many denied to be numbers properly speaking …'
>
> Isaac Barrow (1734)

We begin with the most infamously irrational number:

$$\boxed{\boxed{\sqrt{2}}}$$

This number arises naturally, of course, as the hypotenuse of a right triangle with legs of length 1. Notoriously, $\sqrt{2}$ was found to be irrational by the Pythagoreans in around 500 BC. Their mathematics and philosophy was based upon natural numbers and small number ratios, and thus this discovery would have been very troubling: the Pythagoreans wouldn't be the only ones to react badly to the imposition of the irrational.

At some level, all demonstrations of the irrationality of $\sqrt{2}$ involve a *proof by contradiction*. Suppose that $\sqrt{2}$ is rational, that is that we can write

$$\sqrt{2} = \frac{m}{n} \qquad m, n \text{ integers.} \tag{1}$$

We then show, by some general method, that

$$\sqrt{2} = \frac{m_1}{n_1}$$

where this new fraction is somehow simpler. (For instance m_1 is less than m, or n_1 is less than n, or both). Repeating the procedure,

$$\sqrt{2} = \frac{m_2}{n_2},$$

and then
$$\sqrt{2} = \frac{m_3}{n_3},$$

and so on, each time obtaining a simpler rational expression for $\sqrt{2}$. But clearly, *this can't go on forever*: eventually, the numerator or denominator will be 1, or we'll have ended in some similar absurdity. And that's the contradiction. The assumption that we can write (1), together with our general method for simplifying the fraction, inevitably leads to a contradiction, to an equation that we *know* is false. The only possible conclusion is that (1) is impossible, that $\sqrt{2}$ is in fact irrational.

That's the format of the proof, but we still have to give a method for simplifying the fractions, and it is here that the various proofs differ.

Commonly (and probably what the Pythagoreans did*), one looks at the factors of m and n: it is not hard to show from (1) that m and n are both even, and thus a factor of two can can be cancelled to give a simpler fraction. (Of course this is done, without contradiction, all the time: $\frac{6}{8} = \frac{3}{4}$ for instance. But $\frac{3}{4}$ cannot be simplified further. The contradictory implication of (1) is that we can *always* simplify further). We give here a somewhat less familiar proof; it is in a sense more elementary in that it doesn't rely upon investigating the factors of m or n.

To begin, notice that

$$n < m < 2n. \tag{2}$$

(Both inequalities follow immediately from the fact that $m^2 = 2n^2$.) Now

$$\begin{aligned}
\sqrt{2} &= \frac{\sqrt{2}(\sqrt{2} - 1)}{(\sqrt{2} - 1)} \\
&= \frac{2 - \sqrt{2}}{\sqrt{2} - 1} \\
&= \frac{2 - \frac{m}{n}}{\frac{m}{n} - 1} \qquad \text{(by (1))} \\
&= \frac{2n - m}{m - n} \\
&= \frac{m_1}{n_1}.
\end{aligned}$$

But by (2),

$$\begin{aligned}
2n &< 2m \\
\Rightarrow 2n - m &< m \\
\Rightarrow m_1 &< m.
\end{aligned}$$

Thus the numerator (and similarly, the denominator) of our new fraction is smaller, and we have our contradiction.

Before investigating other irrationals, it is worth pondering for a moment on a fundamental issue we have thus far ignored:

Question

What exactly is an irrational number?

(Notice that $\sqrt{2}$ is geometrically intuitive, but numerically we have only concluded what $\sqrt{2}$ isn't, not what it is.)

* Ascribing concrete mathematical results to the Pythagoreans is very difficult, and to Pythagoras himself almost impossible. However, it is generally accepted that the Pythagoreans knew of the irrationality of $\sqrt{2}$, and there is agreement that if the Pythagoreans had any sort of argument, it would have been based upon the classification of numbers into even and odd (of which they were certainly aware). B. L. Van der Waerden [1] argues that the Pythagoreans probably did produce such an argument; Walter Burkett, in [2, p. 436] is more sceptical.

A standard response to the above question is

Correct but unhelpful answer

An irrational number is an infinite non-repeating decimal.

It doesn't take much thought to realise that this answer, however correct, is not very illuminating. How does one multiply or divide infinite decimals? How do you even tell what a number's decimal expansion is? (No one knows the complete decimal expansion of $\sqrt{2}$, for instance). This is a genuinely deep question, only satisfactorily answered in the 19th century. (By way of comparison, the complex number $i = \sqrt{-1}$ is often considered to have an air of unreality about it; but $\sqrt{-1}$ is in fact much easier to define than $\sqrt{2}$, and was well understood by about 1800.)

The Pythagoreans were right to be troubled. We won't pursue this matter (later we touch on a more natural method of expressing irrational numbers). Here, we just note that our argument above, as well as the ones below, can be made without explicit reference to irrational numbers. For example,

Alternative statement that $\sqrt{2}$ is irrational

There is no rational number $\frac{m}{n}$ such that $\left(\frac{m}{n}\right)^2 = 2$.

Phrasing the irrationality of $\sqrt{2}$ in this manner, one can go on to prove the statement by rephrasing the calculation above: one simply replaces each occurrence of $\sqrt{2}$ by $\frac{m}{n}$, using the hypothesis $2 = \left(\frac{m}{n}\right)^2$ at the critical stage of the argument.

Having ended our theoretical interlude, we continue the hunt for irrational numbers. Easy targets are other 'algebraic' irrationals: $\sqrt{3}$, $\sqrt{2} + \sqrt{3}$, $\sqrt{n}$, $\sqrt[3]{n}$, etc. Of course not all such numbers *are* irrational: $\sqrt{9}$ for instance. Usually, though, if such a number looks irrational, it is. And, it can usually be proved to be irrational by a variation of a $\sqrt{2}$-proof combined with simple algebraic manipulations. Nonetheless, intuition has its limitations. For example, we have

Question

$$a,\ b \text{ irrational} \overset{?}{\Rightarrow} a^b \text{ irrational?}$$

It is easy to imagine that an irrational number raised to an irrational number must always be irrational, but in fact

Answer

No.

This result has the following simple and intriguing proof. Consider the calculation

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \left(\sqrt{2}\right)^{\sqrt{2}.\sqrt{2}} = \left(\sqrt{2}\right)^2 = 2.$$

Now, either $\sqrt{2}^{\sqrt{2}}$ is rational or it is irrational. In the former case we're clearly done ($a = b = \sqrt{2}$). And, in the latter case we're done by the above calculation $\left(a = \sqrt{2}^{\sqrt{2}},\ b = \sqrt{2}\right)$. So, we have an example, but we just don't know what it is!

In fact, $\sqrt{2}^{\sqrt{2}}$ is irrational, R. Kuz'min proving this in 1930. The recentness of the proof indicates how difficult it can be to prove the irrationality of even an easily defined number: once we go beyond nth roots (and their generalisation, roots of polynomial equations) proving irrationality is almost always tough. (Alternatively, it illustrates how easily one can hide difficult definitions in simple notation: what exactly does it *mean* to raise a number to an irrational power?) Later, we give further illustration by giving a selection of numbers for which the question of irrationality is still unanswered.

We now leave the algebraic world, but we'll delay discussion of Everybody's Favourite Number a while longer. First, we consider another well-known fellow:

$$\boxed{e}$$

In order to discuss the irrationality of e, we need a characterisation of it. However, unlike the situation with $\sqrt{2}$, there is no single obvious choice. The original definition, dating to around 1600, is

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n . \tag{3}$$

This expression arises naturally in finance with the notion of continuously compounded interest. Alternatively, in the study of the calculus, one tends first to introduce the function $f(x) = e^x$ and then $e = e^1 = f(1)$. But of course this approach just shifts the question: what special property determines the base e? In fact, for any base a, there is a constant M such that

$$\frac{d}{dx}\left(a^x\right) = Ma^x.$$

We can then define e to be that (unique) base for which $M = 1$. That is, e is *defined* by the identity

$$\frac{d}{dx}\left(e^x\right) = e^x. \tag{4}$$

Of course, it doesn't matter whether we start with (3) or (4), as either can be proved from the other.

The expression for e we actually want is the infinite series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots . \tag{5}$$

This identity follows readily from either (3) or (4). In the former case one applies the Binomial Formula to the expression $(1 + \frac{1}{n})^n$ (taking the limit as $n \to \infty$ needs some thought, but as all terms in the expansion are positive and increase as n increases, this is not too hard); in the latter case, one uses Taylor's Theorem to expand $f(x) = e^x$ around $x = 0$ to approximate $f(1) = e$, and then one shows that the remainder tends to zero as higher degree polynomials are used in the approximation.

From (5) we can prove

Theorem (Euler, 1737)

e is irrational.

Proof

As for $\sqrt{2}$, the proof is by contradiction. Supposing that e is rational, we have

$$e = \frac{m}{n} \qquad m,\, n \text{ integers.} \tag{6}$$

By (5),

$$e \cdot n! = n! + \frac{n!}{1!} + \frac{n!}{2!} + \dots + \frac{n!}{n!} + \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \dots$$

and thus by (6)

$$\frac{m}{n} \cdot n! - n! - \frac{n!}{1!} - \frac{n!}{2!} - \dots - \frac{n!}{n!} = \frac{n!}{(n+1)!} + \frac{n!}{(n+2)!} + \dots . \tag{7}$$

Now each term on the left hand side is an integer. On the other hand,

$$0 < \text{RHS} = \frac{1}{(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$
$$< \frac{1}{(n+1)} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \dots .$$

This last expression is a geometric series, which sums to

$$\frac{\frac{1}{n+1}}{1 - \frac{1}{n+1}} = \frac{1}{n}.$$

So, though the left hand side of (7) is supposedly an integer, the right hand side is definitely positive but less than one. We have our contradiction.

Though (5) dates to 1665 and Isaac Newton, the above proof was first given by Joseph Fourier in 1815. By contrast, Leonhard Euler's original proof is based on his *continued fraction* expansion,

$$\frac{e-1}{2} = \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \dots}}}}.$$

(The way to read this is, $a_1 = 1$, $a_2 = \frac{1}{1}$, $a_3 = \frac{1}{1+\frac{1}{6}} = \frac{6}{7}$, and so on. Then $\frac{e-1}{2} = \lim a_n$). In fact, every number has a *simple* continued fraction expansion (that is, one with all numerators 1 and all denominators positive integers). For example,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}}.$$

(The expansion for $\sqrt{2}$ is easy to prove, but that for e takes more work). Clearly, a finite continued fraction (i. e. one where eventually all the numerators are zero) is rational. Conversely, Euler proved that *any infinite simple continued fraction is irrational.* In particular $\frac{e-1}{2}$ is irrational, and thus e is as well.

We close our discussion of e by noting that any integer power e^m is also irrational; this is in stark comparison to 2, which of course satisfies the equation $(\sqrt{2})^2 = 2$. The irrationality of e^m is more difficult to prove: Johann Lambert used continued fractions to prove this in 1766, but Fourier's method doesn't apply. (Later we'll indicate a third approach.) As a consequence, log 2 (for example) is irrational, since

$$\log 2 = \frac{m}{n} \Leftrightarrow e^m = 2^n.$$

Thus, since $$e^{\log 2} = 2$$

we have another example of an irrational number to an irrational power being rational.

Now we have the star of our show:

$$\boxed{\pi}$$

Of course π is naturally defined as the ratio of the circumference of a circle to its diameter. However, unlike the case of $\sqrt{2}$, this *geometric* definition does not immediately transform into *numerical* information. People have been chasing formulas and estimates for π for thousands of years, with varying degrees of success. The following table indicates a very partial history of numerical approximations to π.

When	**Approximation for π**	**Who/Where**
2000 BC	$3\frac{1}{8}$	Mesopotamia
2000 BC	$(\frac{16}{9})^2$	Egypt
1200 BC	3	China
550 BC	3	Old Testament
250 BC	between $3\frac{10}{71}$ and $3\frac{1}{7}$	Archimedes
263 AD	3.14159	Liu Hui
1429	3.14159265358979	Al- Kash i
1706	to 100 decimal places	Machin

1853	to 500 decimal places	Shanks
1897	4	Indiana
1958	to 10000 decimal places	Genuys
1995	to six billion decimal places	Kanada

The 1897 episode wins the prize for π-silliness. An eccentric named Edward Goodwin persuaded the Indiana House of Representatives to pass a bill legislating the value of π (the bill is so bizarrely written it contains geometric claims implying six different values of π). Unfortunately for fans of the absurd, a visiting mathematician enlightened the Indiana Senate before they had a chance to vote the bill into law.

More generally, there is an element of confusion in the table above: we have not indicated whether those who used an approximation to π knew it was an approximation. Certainly, Archimedes knew this, but the situation with some of the early historical values is unclear. In any case, for the context of irrational numbers, we'll leave no room for doubt.

Question

Suppose we know the first eight trillion digits of π. What can that tell us about whether π is rational or not?

Answer

Absolutely nothing.

Fascination with π has given rise to many beautiful formulas. Archimedes' approach, perhaps the most intuitive, was to approximate the unit circle by regular polygons; the *perimeter* of such a polygon then gives an approximation to 2π. He used both inscribed and circumscribed polygons, thus obtaining both lower and upper estimates for π. His method was to double the number of sides repeatedly, (essentially) using half-angle formulas for sine and tangent to express the new perimeters in terms of the old. Starting with a hexagon, he worked up to (at least) a 96-sided polygon, but his method can theoretically be applied to give any desired accuracy. Taking the limit, the inscribed polygons give the expression

$$\pi = \lim_{n\to\infty} 3{\cdot}2^n \sin\left(\frac{\pi}{3{\cdot}2^n}\right).$$

Regular polygons and half-angle formulas were also used by François Viète. In 1579 he calculated the area of these polygons, using a clever algebraic trick to obtain the infinite product

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cdot\cos\frac{\pi}{8}\cdot\cos\frac{\pi}{16}\ldots$$

$$= \sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)}\cdot\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1}{2}}\right)}\right)}\ldots .$$

Given the power of the calculus, one can go to the limit more directly, the area of the unit circle being

$$\pi = 4\int_0^1 \sqrt{1 - x^2}\, dx.$$

An explicit limit can now be obtained by approximating the integral (for example by applying the binomial theorem to expand the integrand $\sqrt{1 - x^2}$ and then integrating term by term). In 1666 Newton used this idea with a slightly different integral to obtain a series beginning

$$\pi = \frac{3\sqrt{3}}{4} + 24\left(\frac{1}{12} - \frac{1}{5\cdot 2^5} - \frac{1}{28\cdot 2^7} - \frac{1}{72\cdot 2^9} - \dots\right).$$

Closely related is the idea of expressing π in terms of inverse trigonometric functions. The earliest of many such results, due to an unknown Indian mathematician from the 15th century, is the famous series expression for arctan 1:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots .$$

Two more identities, too beautiful to overlook, are the infinite product by John Wallis (1655),

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots$$

and the infinite series by Euler (1734),

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{4^2} + \dots .$$

Both are derived from clever analysis of the sine function.

The above are all beautiful identities, but it is not clear that any of them help us determine whether π is irrational. (Certainly, the limit nature of these identities is not enough in itself to conclude anything.) We might hope to mimic our proof of the irrationality of e, but it is tough to come up with a sufficiently neat series for π (even e^m seems to be beyond the reach of such methods). An astonishing series which is tempting but doesn't quite work is

$$\frac{1}{\pi} = \sum_{n=0}^{\infty}\left(\frac{2n!}{n!\,n!}\right)^3 \frac{42n + 5}{2^{12n+4}}.$$

This deep result from the theory of *theta functions*, is due to the amazing Srinivasa Ramanujan (1914).

Another natural approach is to hunt for a simple continued fraction expansion for π. However, though one can compute the terms of the fraction one by one (just as one can compute the decimal expansion of $\sqrt{2}$), the complete simple continued fraction for π is still unknown. There *are* many non-simple fractions for π, beginning with a corollary of Wallis's infinite product, the lovely identity of William Brouncker's (1655):

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \dots}}}.$$

However, such expressions are not necessarily irrational.

In fact the original proof that π is irrational, due to Lambert, is in terms of continued fractions, but in a brilliantly inverted manner. In 1766 he derived the *functional* continued fraction

$$\tan v = \cfrac{1}{\left(\frac{1}{v}\right) - \cfrac{1}{\left(\frac{3}{v}\right) - \cfrac{1}{\left(\frac{5}{v}\right) - \dots}}}.$$

Lambert then proved that if the angle v is rational then the continued fraction must be irrational. But $\tan \frac{\pi}{4} = 1$ is rational, and thus $\frac{\pi}{4}$ (and so π as well) must be irrational.

In principle, Lambert's is a fine proof, but it takes considerable work to justify all the steps*. So we'll give a second proof, a beautiful argument due to Ivan Niven.

Consider the integral $$I = \int_0^1 p(x) \sin \pi x \, dx,$$

where $$p(x) = x^N (1 - x)^N$$

and where the integer N will be chosen (large) later. Noting $p(x) = 0$ at the endpoints, an integration by parts gives

$$I = -\frac{1}{\pi} \int_0^1 p'(x) \cos \pi x \, dx.$$

Now $$p'(x) = Nx^{N-1}(1 - x)^N - Nx^N(1 - x)^{N-1},$$

which is still zero at the endpoints. So, integrating by parts again,

$$I = \frac{1}{\pi^2} \int_0^1 p''(x) \sin \pi x \, dx.$$

We keep integrating, at each stage using the product rule to differentiate $p(x)$. After a few differentiations this will be quite a mess, but a lot of the terms will still be zero at the endpoints: in order for a term to give a non-

* Assigning credit for old theorems can be contentious, as different eras (and different mathematicians) have differing standards of rigour and proof, as well as differing styles of exposition. So, some credit Lambert with proving the irrationality of e because he explicitly considered the convergence of the continued fraction for $\frac{e-1}{2}$: others argue that Euler could have done this without trouble if he had felt it necessary. Similarly, Adrien-Marie Legendre is often assigned some credit for proving the irrationality of π, the claim being that his systematic treatment of continued fractions (1794) is more rigorous than Lambert's analysis of the fraction above: others claim that Lambert's argument, though less elegant, is in fact more rigorous than Legendre's. Claude Brezinski (see [3]) discusses these historical issues in some detail.

zero contribution at an endpoint, either x^N or $(1 - x)^N$ has to be differentiated at least N times, which implies there is a factor of $N!$ in that term. On the other hand, $\deg p = 2N$, so p will be differentiated out completely after $2N$ integrations. Combining these two observations, we must have

$$I = \frac{k_1 N!}{\pi^{N+1}} + \dots + \frac{k_{N+1} N!}{\pi^{2N+1}}, \qquad k_1, \dots, k_{N+1} \text{ integers.}$$

There's no contradiction in that, but now suppose π is rational,

$$\pi = \frac{m}{n}.$$

Multiplying both sides by $m^{2N+1}/N!$,

$$\frac{m^{2N+1}}{N!} I = k, \qquad k \text{ an integer.} \tag{8}$$

And *now* we have a contradiction. For $0 < I < 1$ (since each term in the integral is between 0 and 1). And, if N is chosen large enough,

$$0 < \frac{m^{2N+1}}{N!} < 1.$$

Thus the left side of (8) is between 0 and 1, whereas the right side is supposedly an integer.

The above is a variation of an argument by Charles Hermite, and other numbers can be proved irrational by similar means. In particular, integer powers e^m of e can thus be proved irrational. As for π, taking a little more care, the proof above actually shows π^2 is irrational; further, with a similar argument one can show that π is not the solution of any quadratic equation with rational coefficients. (The latter is a stronger statement: for example, $(1 + \sqrt{2})^2$ is irrational but $1 + \sqrt{2}$ is a root of the equation $x^2 - 2x - 1 = 0$. Here, when asking whether numbers are the solutions of polynomial equations, we are edging into the much more difficult question of the *transcendental* nature of numbers). However, though higher powers of π are indeed irrational, this is significantly more difficult to prove.

We close by introducing a few numbers which are presumed to be irrational but for which this has yet to be proved.

There are zillions of ways to combine numbers artificially, and essentially all the outcomes of all of these combinations are not known to be irrational. For example, the character of

$$\pi + e \qquad \pi e \qquad \pi^e$$

is unknown. One might be tempted to throw e^π in with this lot, but in 1929 Alexandr Gel'fond showed this number to be irrational ($e^\pi = (-1)^{-i}$, which is natural enough in the world of complex numbers). Also, though $\pi + e$ and πe are not known to be irrational, it is easy to show *at least one* of them must be irrational. To see this, consider the quadratic equation

$$x^2 - (\pi + e)x + \pi e = 0,$$

whose solutions are π and e. Now we know that π is not the solution of any such equation with rational coefficients, thus one of $\pi + e$ or $\pi + e$ must be irrational.

More interesting are values of the *Riemann zeta function*,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots, \qquad s = 2, 3, 4, \dots .$$

We have seen a special case of this above, Euler's result that

$$\zeta(2) = \frac{\pi^2}{6}.$$

As well, Euler showed $\zeta(4) = \dfrac{\pi^4}{90}.$

and in general $\zeta(2n) = a_n\pi^{2n}$, a_n rational.

(The a_n can be written in terms of the so-called *Bernoulli numbers*). As a consequence, all the $\zeta(2n)$ are known to be irrational.

The values $\zeta(2n + 1)$ are much more mysterious. For a long time, no one had any idea how to approach these numbers. It was a complete shock when, in 1978, an unknown mathematician named Roger Apéry proved that $\zeta(3)$ is irrational. The irrationality of $\zeta(5)$ and the values beyond are still unproved.

To explain our last example, first note that one cannot define $\zeta(1)$ except to be ∞, since the *Harmonic Series*,

$$1 + \frac{1}{2} + \frac{1}{3} + \dots$$

does not converge. However, if we subtract $\log k$ in the correct way, we *do* get a finite quantity γ, known as Euler's constant:

$$\gamma = \lim_{k \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k} - \log k\right).$$

As with all the numbers we have considered, it makes absolutely no practical difference whether γ is rational or not. But mathematicians want to know, simply for the sake of knowing. Euler's constant is the grand prize for current hunters of the irrational.

References

1. B. L. Van der Waerden, *Science Awakening* (1st edn), P. Noordhoff (1961).
2. Walter Burkett, *Love and science in ancient Pythagoreanism*, Harvard University Press (1972).
3. C. Brezinski, *History of continued fractions and Padé approximants*, Springer (1991).

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