# 2, 4, 8, 16, $\pi$

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Starting with the sequence of numbers

$$2, 4, 8, 16, \tag{$\dagger$}$$

we want to say what comes next. The obvious answer is 32. That is, the sequence can be given by the doubling formula

$$2^n\,. \tag{$\dagger\dagger$}$$

So, substituting $n = 1$ gives $2^1 = 2$, substituting $n = 2$ gives $2^2 = 4$, and so on. The new (fifth) number is obtained by substituting $n = 5$, giving $2^5 = 32$.

To justify answers other than 32, we have to come up with some other formula (or precise method) to replace (††). We'll do that in a minute (see the boxed formula on the next page for the formula giving $\pi$), but we first consider a simpler example.

Consider the sequence of numbers

$$4, 1, 2, 13\,. \tag{$*$}$$

What's a formula which fits this? There are others, but one which works is

$$n^3 - 4n^2 + 2n + 5\,. \tag{$**$}$$

That is, substituting $n = 1$ gives $1 - 4 + 2 + 5 = 4$, and so on. This polynomial formula then produces the next (fifth) number to be $125 - 100 + 10 + 5 = 40$.

This example is contrived (we thought of the formula first and then just looked at what numbers it churned out), but the important point is that something like this always works. That is, *whatever sequence of numbers we begin with, we can always find a polynomial formula which works.*

How do we find such a polynomial formula? We can do it by trial and error, but there is also a simple general recipe. (This is called a *Lagrange interpolation polynomial*, and dates to the 18th Century).

We'll now illustrate the recipe for the sequence of numbers

$$(\blacktriangle) \qquad 2, 4, 8, 16, \pi\,.$$

Our answer (there are others) is

$$(\blacktriangle\blacktriangle) \qquad \boxed{\begin{aligned} &\mathbf{2}\,\frac{(n-2)(n-3)(n-4)(n-5)}{(1-2)(1-3)(1-4)(1-5)} \\ +&\mathbf{4}\,\frac{(n-1)(n-3)(n-4)(n-5)}{(2-1)(2-3)(2-4)(2-5)} \\ +&\mathbf{8}\,\frac{(n-1)(n-2)(n-4)(n-5)}{(3-1)(3-2)(3-4)(3-5)} \\ +&\mathbf{16}\,\frac{(n-1)(n-2)(n-3)(n-5)}{(4-1)(4-2)(4-3)(4-5)} \\ +&\boldsymbol{\pi}\,\frac{(n-1)(n-2)(n-3)(n-4)}{(5-1)(5-2)(5-3)(5-4)} \end{aligned}}$$

What happens when we substitute $n = 1$? Well, the first fraction equals 1 (since the numerator and denominator become the same), and all the other fractions equal 0 (since each numerator has a factor $(n-1)$). So, the total result of substituting $n = 1$ is 2.

Similarly, if we substitute $n = 2$ then only the second fraction is non-zero, and the result is 4. And so on. Setting $n = 5$ produces the number $\pi$, just as promised.

What if we wanted the number after 16 to be something else? Say, 3749? So, now we want a formula to fit the sequence

$$(\blacklozenge) \qquad 2, 4, 8, 16, 3749\,.$$

Then, we can use exactly the same formula (▲▲), except we replace $\pi$ by 3749. Ta-da!