AMSI 2013: MEASURE THEORY

Extra Solutions C

37 to 64

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37 Given $\mu(X) < \infty$, we want to show

$$1 \leqslant p < r < \infty \quad \Longrightarrow \quad \left(\fint |f|^p\right)^{\frac{1}{p}} \leqslant \left(\fint |f|^r\right)^{\frac{1}{r}} .$$

To do this, we apply Hölder's Inequality, with $F = |f|^p, G = 1, P = \frac{r}{p}, Q = \frac{P}{P-1} = \frac{r}{r-p}$. This gives

$$\int |FG| \leqslant \|F\|_P \|G\|_Q \quad \Longrightarrow \quad \int |f|^p \leqslant \left(\int |f|^r\right)^{\frac{p}{r}} \cdot \left(\int 1\right)^{\frac{r-p}{r}}$$

$$\Longrightarrow \quad \left(\int |f|^p\right)^{\frac{1}{p}} \leqslant \left(\int |f|^r\right)^{\frac{1}{r}} \cdot (\mu(X))^{\frac{1}{p}-\frac{1}{r}} .$$

Dividing both sides by $(\mu(X))^{\frac{1}{p}}$ gives the desired result.

We assume $f_j \to f$ in L^p. If $p = \infty$ then this obviously implies pointwise convergence a.e.. But for $p < \infty$ it is easy to construct examples for $\{f_j\}$ doesn't converge pointwise anywhere. For example, for $2^n \leqslant j < 2^{n+1}$ let f_j be the characteristic function on $\left[\frac{j-2^n}{2^n}, \frac{j+1-2^n}{2^n}\right]$.

To show there is always a subsequence that converges pointwise a.e., we use the fact that $\{f_j\}$ is Cauchy. This implies that, for any $m \in \mathbb{N}$, we can find an N_m such that

$$j, k \geqslant N_m \quad \Longrightarrow \quad \|f_j - f_k\|_p < \tfrac{1}{2^m} \,.$$

Choosing the N_m inductively, we can also ensure that $\{N_m\}_m$ is a strictly increasing sequence. We show the subsequence $\{f_{N_m}\}$ converges a.e. to f. To do this, first consider

$$g_n = \sum_{m=1}^{n} \left|f_{N_{m+1}} - f_{N_m}\right|$$

Then

$$g_n \nearrow g = \sum_{m=1}^{\infty} \left|f_{N_{m+1}} - f_{N_m}\right| \,.$$

Also, by Minkowski's Inequality,

$$\|g_n\|_p = \left\|\sum_{m=1}^{n} \left|f_{N_{m+1}} - f_{N_m}\right|\right\|_p \leqslant \sum_{m=1}^{n} \left\|f_{N_{m+1}} - f_{N_m}\right\|_p < 1 \,.$$

Thus, by the Monotone Convergence Theorem (Theorem 19),

$$\int g^p = \lim_{n\to\infty} \int g_n^p \leqslant 1 < \infty \,.$$

It follows that $g < \infty$ a.e. That is, for almost every x, the series $\sum \left(f_{N_{m+1}}(x) - f_{N_m}(x)\right)$ of real numbers converges absolutely: this implies the series itself converges. Then

$$f = f_{N_1} + \sum_{m=1}^{\infty} \left(f_{N_{m+1}}(x) - f_{N_m}(x)\right) \,.$$

The mth partial sum is exactly $f_{N_{m+1}}$. That is, $f_{N_m} \to f$ a.e., which is exactly what we wanted to show.

43 X is a topological space, and $\mathcal{F} \subseteq \wp(X)$ contains the closed and open sets, and is closed under countable unions and countable intersections. we want to show that $\mathcal{F} \supseteq \mathcal{B}$. To do this, set

$$\mathcal{G} = \{A \subseteq X : A \in \mathcal{F} \text{ and } \sim A \in \mathcal{F}\}.$$

Clearly $\mathcal{G}$ contains all closed sets (since the complements of the closed sets are the open sets, which are in $\mathcal{F}$). So, if we can show that $\mathcal{G}$ is a σ-algebra then $\mathcal{B} \subseteq \mathcal{G} \subseteq \mathcal{F}$.

By construction, $\mathcal{G}$ is closed under complements. To show $\mathcal{G}$ is closed under countable unions, suppose $\{A_j\}$ is a sequence of sets in $\mathcal{G}$: so, each A_j and $\sim A_j$ is in $\mathcal{F}$. Then

$$\begin{cases} \displaystyle\bigcup_{j=1}^{\infty} A_j \in \mathcal{F} & \text{(since } \mathcal{F} \text{ is closed under countable unions)}, \\ \displaystyle\sim \bigcup_{j=1}^{\infty} A_j = \bigcap_{j=1}^{\infty} \sim A_j \in \mathcal{F} & \text{(since } \mathcal{F} \text{ is closed under countable intersections)}. \end{cases}$$

Thus $\mathcal{G}$ is closed under countable unions, as desired.

44

(ii) Let μ be the Anything-Will-Do measure on $X = \{a, b\}$, and let X have the indiscrete topology (so the only open sets are X and $\emptyset$, and thus these are the only Borel sets as well). Then μ is Borel regular, but $A = \{a\}$ is not contained in a Borel set B with $\mu(B \sim A) = 0$. (The only possibility is $B = X$, and that doesn't work).

(iii) Similar to the last example, let $X = \{a, b\}$ be given the indiscrete topology, and let μ be the Anything-Is-Wonderful measure. μ is again Borel regular, and now $A = \{a\}$ is μ-measurable. But there is again no Borel $B \supseteq A$ with $\mu(B \sim A) = 0$.

(46) If μ is Borel regular and $A \subseteq X$ is measurable with $\mu(A) < \infty$ then we want to show $\mu \rfloor A$ is Borel regular. By Theorem 35(b), we can choose a Borel $B \supseteq A$ with $\mu(B \sim A) = 0$. We know by Theorem 35(c) that $\mu \rfloor B$ is Borel regular, so we just have show that $\mu \rfloor B = \mu \rfloor A$. For $C \subseteq X$ we have

$$\mu \rfloor B(C) = \mu(B \cap C) \leqslant \mu(A \cap C) + \mu((B \sim A) \cap C) \leqslant \mu(A \cap C) + \mu(B \sim A) = \mu(A \cap C) = \mu \rfloor A(C) .$$

The other direction is trivial, and so we're done.

(48) X is a locally compact and separable metric space. We want to show that we can write $X = \bigcup_n V_n$, where V_n is open and $\overline{V}_n$ is compact.

Since X is separable, we have a countable dense subset $Y = \{y_1, y_2, \dots\}$. We know that around each y_n there is a compact ball; we just have to be careful to choose these balls to be reasonably large. (For example, taking the interval of radius $\frac{1}{2^n}$ around the n'th rational $q_n \in \mathbb{Q}$ will *not* work in $\mathbb{R}$). So, we set

$$(*) \qquad r_n = \tfrac{1}{2} \min\left(1, \sup\{r : \overline{B}_r(y_n) \text{ is compact}\}\right) .$$

Setting $V_n = B_{r_n}(y_n)$ it is immediate that $\overline{V}_n$ is compact. (Note, this may not be true without the *min* in the definition of r_n). We just have to show that $X = \cup_n V_n$.

Considering $x \in X$, we want to show x is in some V_n. We know that there is an r such that $\overline{B}_r(x)$ is compact. We can also assume that $r \leqslant \frac{3}{2}$ (since closed subsets of a compact set are compact, any smaller closed ball will still be compact). Next, since Y is dense in X, we can find a y_n with $d(x, y_n) \leqslant \frac{r}{3}$. But then $\overline{B}_{\frac{2r}{3}}(y_n) \subseteq \overline{B}_r(x)$, and thus is compact. Then by $(*)$, and since $r \leqslant \frac{3}{2}$,

$$r_n \geqslant \frac{r}{3} .$$

But then $x \in B_{\frac{r}{3}}(y_n) \subseteq B_{r_n}(y_n) = V_n$.

(49) For μ a measure on X and ν a measure on Y, we define $\mu \times \nu : \wp(X \times Y) \to \mathbb{R}^*$:

$$\mu \times \nu(D) = \inf\left\{\sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) : A_j \subset X \ \mu\text{-measurable},\ B \subset Y \ \nu\text{-measurable}\right\} \quad D \subset X \times Y.$$

We want to show this is a measure. Only countable subadditivity is nontrivial. So, suppose $\{D_j\}_j$ is a sequence of subsets of $X \times Y$. Fix $\epsilon > 0$, and for each D_j let $\{A_{jk} \times B_{jk}\}_k$ be a covering by rectangles with measurable sides and such that

$$\sum_{k=1}^{\infty} \mu(A_{jk})\nu(B_{jk}) \leqslant \mu \times \nu(D_j) + \frac{\epsilon}{2^j}\,.$$

Then $\{A_{jk} \times B_{jk}\}_{j,k}$ is a covering of $\bigcup_j D_j$, and so

$$\mu \times \nu\left(\bigcup_{j=1}^{\infty} D_j\right) \leqslant \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \mu(A_{jk})\nu(B_{jk}) \leqslant \sum_{j=1}^{\infty} \mu \times \nu(D_j) + \epsilon\,.$$

By the Thrilling ϵ-Lemma, we're done.

51 We want to prove $\mathscr{L}^{m+n} = \mathscr{L}^m \times \mathscr{L}^n$. Fix $D \subseteq \mathbb{R}^{m+n}$. Then

$$\begin{cases} \mathscr{L}^{m+n}(D) = \inf\left\{\sum_{j=1}^{\infty} v(P_j) : D \subseteq \bigcup_{j=1}^{\infty} P_j,\ P_j \text{ an open } (m+n)\text{-box}\right\} \\ \mathscr{L}^{m} \times \mathscr{L}^{n}(D) = \inf\left\{\sum_{j=1}^{\infty} \mathscr{L}^m(A_j)\cdot\mathscr{L}^n(B_j) : D \subseteq \bigcup_{j=1}^{\infty} A_j \times B_j, A_j \subset \mathbb{R}^m,\ B_j \subset \mathbb{R}^n\right\} \end{cases}$$

Note that any open $(m+n)$-box can be thought as an $A_j \times B_j$ with measurable sides. And, it is immediate from Proposition 5 that

$$v(A_j \times B_j) = v(A_j)\cdot v(B_j) = \mathscr{L}^m(A_j)\cdot\mathscr{L}^n(B_j)\,.$$

It follows immediately that $\mathscr{L}^m \times \mathscr{L}^n(D) \leqslant \mathscr{L}^{m+n}(D)$.

We shall prove the reverse inequality for D bounded; the result then follows for general D by continuity of regular measures (Theorem 35(a)). Fixing j, we choose suitable $\eta > 0$ and $\delta > 0$, and we cover A_j and B_j by collections of boxes $\{P_{jk}\}$ and $\{Q_{jl}\}$, such that

$$\left(\sum_{k=1}^{\infty} v(P_{jk})\right)\cdot\left(\sum_{l=1}^{\infty} v(Q_{jl})\right) \leqslant (\mathscr{L}^m(A_j)+\eta)\cdot(\mathscr{L}^n(B_j)+\delta) \leqslant \mathscr{L}^m(A_j)\cdot\mathscr{L}^n(B_j) + \frac{\epsilon}{2^j}\,.$$

(We can find suitable η and δ because D is bounded, and thus A_j and B_j have finite measure). This gives us a covering of D by $(m+n)$-boxes $\{P_{jk} \times Q_{jl}\}_{j,k,l}$, and again, $v(P_{jk})\cdot v(Q_{jl}) = v(P_{jk} \times Q_{jl})$. It follows that

$$\mathscr{L}^{m+n}(D) \leqslant \sum_{j=1}^{\infty} \mathscr{L}^m(A_j)\cdot\mathscr{L}^n(B_j) + \epsilon\,.$$

By the Thrilling ϵ-Lemma, we're done.

(52) If $A \subseteq X$ is Borel and $B \subseteq Y$ is Borel then we want to show $A \times B$ is Borel. It is enough to show $A \times Y$ and $X \times B$ are Borel, since the intersection of these two sets gives the desired set. Define

$$\mathcal{F} = \{C \subseteq X : C \times Y \text{ is Borel}\}.$$

Then $\mathcal{F}$ is easily shown to be a σ-algebra. Also, $\mathcal{F}$ contains any open $V \subseteq X$ (since $V \times Y$ is open, and thus Borel). Thus, by definition of the Borel sets, $\mathcal{F}$ contains all Borel subsets of X; in particular, $A \in \mathcal{F}$, and thus $A \times Y$ is Borel. Similarly if $B \subseteq Y$ is Borel then $X \times B$ is Borel.

(54)

(a) We want to show that if f is summable then f is σ-finite. Fix j and let $E_n = \{x : |f(x)| \geqslant 1/j\}$. Then E_j is measurable and $\int_{E_n} |f| \geqslant \frac{1}{j}\mu(E_j)$, from which it follows that $\mu(E_j) < \infty$. Thus, $\{x : f(x) \neq 0\} = \cup E_j$ is σ-finite.

(b) Suppose X is σ-finite, so $X = \cup A_j$ with A_j measurable and $\mu(A_j) < \infty$. Suppose f is measurable and let $E = \{x : f(x) \neq 0\}$. Then $E = \cup(E \cap A_j)$ is a countable union of sets of finite measure, and thus f is σ-finite.

(c) Suppose $X = \cup A_j$ and $Y = \cup B_k$ are σ-finite, with all the A_j and B_k measurable, and with each $\mu(A_j) < \infty$ and $\nu(B_k) < \infty$. Then $X \times Y = \cup(A_j \times B_k)$. And, by Theorem 42, each $A_j \times B_k$ is $\mu\times\nu$-measurable with $\mu\times\nu(A_j \times B_k) = \mu(A_j) \cdot \nu(B_k) < \infty$. Thus $X \times Y$ is σ-finite.

(55) We want to prove Theorem 47, the Fubini-Tonelli Theorem.

(i) Suppose $f : X \times Y \to \mathbb{R}^*$ is nonnegative and σ-finite. Using Lemma 20, we can write

$$(\dagger) \qquad f = \sum_{j=1}^{\infty} h_j \chi_{A_j} \qquad h_j \geqslant 0, A_j \ \sigma\text{-finite}.$$

Fix j. Then, by Lemma 46(i) for ν-a.e. $y \in Y$, the slice

$$(A_j)_y = \{x \in X : (x,y) \in A_j\}$$

is μ-measurable. Thus, for ν-a.e. $y \in Y$, the function

$$(*) \qquad x \mapsto \chi_{(A_j)_y}(x) = \chi_{A_j}(x,y)$$

is μ-measurable. Considering all j together, for ν-a.e. $y \in Y$ *every* function given by $(*)$ is μ-measurable. Thus, for ν-a.e. $y \in Y$, the function

$$x \mapsto \sum_{j=1}^{\infty} h_j \chi_{A_j}(x,y) = f(x,y)$$

is μ-measurable. Integrating with the help of Lemma 46 (ii), (iii), and the Monotone Convergence Theorem,

$$\int_Y \left(\int_X f(x,y)\,\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y) = \sum_{j=1}^{\infty} h_j \int_Y \mu((A_j)_y)\nu(y) = \sum_{j=1}^{\infty} h_j \cdot \mu \times \nu(A_j)\,.$$

On the other hand, Lemma 20 applies directly to (†) to give

$$\int_{X \times Y} f \,\mathrm{d}\mu \times \nu = \sum_{j=1}^{\infty} h_j \cdot \mu \times \nu(A_j)\,.$$

This is exactly the result we want for nonnegative f.

(ii) For general σ-finite f, we write $f = f^+ - f^-$, and the desired result follows immediately from the case for nonnegative f.

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(a) We consider $\mathscr{L}$ on $[0,1]$ and μ_0 counting measure on $[0,1]$. We consider $f = \chi_D$ where $D = \{(x,x) : x \in [0,1]\}$. Note that f is measurable, since D is closed and $\mathscr{L} \times \mu_0$ is Borel (by Theorem 45). We then easily calculate

$$\begin{cases} \displaystyle\int_{[0,1]} \left(\int_{[0,1]} \chi_D(x,y)\,\mathrm{d}\mathscr{L}(x) \right) \mathrm{d}\mu_0(y) = \int_{[0,1]} 0 \,\mathrm{d}\mu_0(y) = 0\,, \\ \displaystyle\int_{[0,1]} \left(\int_{[0,1]} \chi_D(x,y)\,\mathrm{d}\mu_0(y) \right) \mathrm{d}\mathscr{L}(x) = \int_{[0,1]} 1 \,\mathrm{d}\mathscr{L}(x) = 1\,. \end{cases}$$

Finally, we can show that

$$(*) \qquad \int \chi_D \, \mathrm{d}\mathscr{L} \times \mu_0 = \mathscr{L} \times \mu_0(D) = \infty.$$

To see this, consider a covering $\{A_j \times B_j\}$ of D by rectangles (by Borel regularity we don't have to worry if the sides are measurable). We can also assume $A_j \subseteq B_j$, since replacing A_j by $A_j \cap B_j$ covers the same points of D. But one of the A_j must have positive Lebesgue measure (since $[0,1] \subseteq \bigcup_j A_j$), and then

$$\mathscr{L}(A_j) > 0 \quad \Longrightarrow \quad \mu_0(B_j) = \infty \quad \Longrightarrow \quad \mathscr{L} \times \mu_0(A_j \times B_j) = \infty.$$

Then $(*)$ follows immediately from the definition of the product measure.

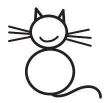
(b) We now consider $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$ with respect to $\mathscr{L}$ on $[0,1]$ in each variable. f is Borel, and thus measurable, since it is continuous except at $(0,1)$; and then f is automatically σ-finite, since $\mathscr{L} \times \mathscr{L}([0,1] \times [0,1]) = 1 < \infty$. Now, by antisymmetry

$$I = \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, \mathrm{d}x \mathrm{d}y = - \int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, \mathrm{d}y \mathrm{d}x \, .$$

So, to show the two integrals are not equal, we just have to show $I \neq 0$. Letting $x = y \tan u$ (for $y > 0$), we have

$$\begin{aligned} \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, \mathrm{d}x &= \int_0^{\arctan(\frac{1}{y})} \frac{y^2 \tan^2 u - 1}{y^4 \sec^4 u} y \sec^2 u \, \mathrm{d}u \\ &= \int_0^{\arctan(\frac{1}{y})} \frac{1}{y}(\sin^2 u - \cos^2 u) = \left[-\frac{1}{y} \sin u \cos u \right]_0^{\arctan(\frac{1}{y})} = \frac{-1}{y^2+1} \, . \end{aligned}$$

Integrating once more, we find $I = -\frac{\pi}{4} \neq 0$.



57 To show that $\mathscr{H}^n_\delta$ is a measure, the only issue is to prove countably subadditivity, and the proof is identical to that for Lebesgue measure. Suppose $A \subseteq \cup_k A_k$ and, for each k, let $\{C_{jk}\}$ be a covering of A_k. Given $\epsilon > 0$, we can choose the C_{jk} so that $\operatorname{diam} C_{jk} \leqslant \delta$ and

$$\sum_{j=1}^{\infty} \omega_n \left(\frac{\operatorname{diam} C_{jk}}{2} \right)^n \leqslant \mathscr{H}^n_\delta(A_k) + \frac{\epsilon}{2^k}\,.$$

Then, since $A \subseteq \cup C_{jk}$,

$$\mathscr{H}^n_\delta(A) \leq \sum_{k=1}^{\infty} \mathscr{H}^n_\delta(A_k) + \epsilon\,.$$

Letting $\epsilon \to 0$ gives the desired result. Next, as $\delta \to 0^+$, $\mathscr{H}^n_\delta$ increase. So, it follows from **4** that $\mathscr{H}^n$ is a measure.

59 For $m > n$ we want to show that

$$\begin{cases} \mathscr{H}^n(A) < \infty \implies \mathscr{H}^m(A) = 0\,, \\ \mathscr{H}^m(A) > 0 \implies \mathscr{H}^n(A) = \infty\,. \end{cases}$$

The critical fact, which follows easily from considering coverings of $A \subseteq \mathbb{R}^n$, is

$$\mathscr{H}^m_\delta(A) \leqslant \frac{\omega_m}{\omega_n} \left(\frac{\delta}{2} \right)^{m-n} \mathscr{H}^n_\delta(A)\,.$$

The desired results now follow by letting $\delta \to 0$.

61 We want to show that if $f : \mathbb{R}^p \to \mathbb{R}^q$ is Lipschitz and if $A \subseteq \mathbb{R}^p$ then

$$\mathscr{H}^n(f(A)) \le (\mathrm{Lip}\, f)^n\, \mathscr{H}^n(A)$$

If $A \subseteq \cup C_j$ then $f(A) \subseteq \cup f(C_j)$. Also $\mathrm{diam}(f(C_j)) \le (\mathrm{Lip}\, f)\, \mathrm{diam}(C_j)$. Thus,

$$\mathscr{H}^n_{(\mathrm{Lip}\, f)\delta}((f(A)) \le \mathscr{H}^n_\delta(A)\,.$$

Letting $\delta \to 0$ gives the result.