AMSI 2013: MEASURE THEORY
Extra Solutions B

20 to 36

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- The fact that $A \subseteq X$ is a Borel set iff χ_A is a Borel function can be proved in exactly the same way as 21(b).

- If $f : X \to \mathbb{R}^*$ is continuous then $f^{-1}([-\infty, a))$ is open, and thus trivially Borel as well. Thus f is Borel.

- If $f : X \to \mathbb{R}^*$ is Borel and μ is a Borel measure on X, then $f^{-1}([-\infty, a))$ is Borel, and therefore also μ-measurable, for any $a \in \mathbb{R}^*$. Thus f is obviously measurable.

- We want to show that if $f : \mathbb{R}^* \to \mathbb{R}^*$ is monotonic then f is Borel. To prove this, suppose $b < c$, $f(b) < a$ and $f(c) < a$. Then any $d \in (b, c)$ will be in the interval with endpoints $f(b)$ and $f(c)$, and so $f(d) < a$ also. It follows that $f^{-1}([-\infty, a))$ is an interval, and is therefore Borel, for any $a \in \mathbb{R}^*$. Thus f is Borel.

- We consider $f : \mathbb{R}^* \to \mathbb{R}^*$ where $f(x) = \frac{1}{x}$, with $f(0) = c$, for some fixed c. We want to show that f is Borel.

  Define $g : \mathbb{R}^* \sim \{0\} \to \mathbb{R}^*$ by $g(x) = \frac{1}{x}$. Then

$$g^{-1}([-\infty, a)) = \begin{cases} \left[\frac{1}{a}, 0\right) & a < 0\,, \\ [-\infty, 0) & a = 0\,, \\ [-\infty, 0) \cup \left(\frac{1}{a}, \infty\right] & a > 0\,. \end{cases}$$

These sets are all obviously Borel. And, for any a, either $f^{-1}([-\infty, a) = g^{-1}([-\infty, a)$ or $f^{-1}([-\infty, a) = g^{-1}([-\infty, a) \cup \{c\}$. Since $\{c\}$ is a closed set, $f^{-1}([-\infty, a)$ is always a Borel set, and thus f is a Borel function.

23 Given $f : X \to \mathbb{R}^*$ Borel or measurable, we want to show the equivalence of :

$$\begin{cases} \text{(a)} \qquad f^{-1}\left([-\infty, a)\right) \text{ is Borel (measurable) for all } a \in \mathbb{R}; \\ \text{(b)} \qquad f^{-1}\left([-\infty, a]\right) \text{ is Borel (measurable) for all } a \in \mathbb{R}; \\ \text{(c)} \qquad f^{-1}(U) \text{ is Borel (measurable) for all open } U \subseteq \mathbb{R}^*; \\ \text{(d)} \qquad f^{-1}(B) \text{ is Borel (measurable) for all Borel } B \subseteq \mathbb{R}^*. \end{cases}$$

We'll focus upon measurability, the arguments for the Borel functions being identical. Trivially (d) implies (c), which implies (a). To see (a) and (b) are equivalent, let

$$\mathcal{M} = \{\mathcal{A} \subseteq \mathbb{R}^* : f^{-1}(A) \text{ is measurable}\}\,.$$

By the properties of f^{-1}, $\mathcal{M}$ is a σ-algebra, whether or not f is measurable. The equivalence of (a) and (b) then follows from

$$\begin{cases} [-\infty, a) = \displaystyle\bigcup_{n=1}^{\infty} [-\infty, a + \tfrac{1}{n}] \in \mathcal{M} \qquad \text{assuming (b)}\,, \\ [-\infty, a] = \displaystyle\bigcap_{n=1}^{\infty} [-\infty, a + \tfrac{1}{n}) \in \mathcal{M} \qquad \text{assuming (a)}\,, \end{cases}$$

The proof that (b) and (a) together imply (c) is the same as for the proof of **20** .

Finally, we show (c) implies (d). Assuming (c), we know $\mathcal{M}$ is a σ-algebra which contains all the open subsets of $\mathbb{R}^*$. But the collection of Borel sets $\mathcal{B}$ on $\mathbb{R}^*$ is the intersection of all such collections, and thus $\mathcal{B} \subseteq \mathcal{M}$. This is exactly the desired conclusion (d).

24 We'll just consider the upper envelope. Let $n \in \mathbb{N}$, and define $f_n : X \to \mathbb{R}^*$

$$f_n(x) = \sup_{y \in U_{\frac{1}{n}}(x)} f(y)\,.$$

Then $f = \lim f_n$, and so we just have to show each f_n is Borel.

Fix n and $a \in \mathbb{R}$, and let

$$A = \{x : f_n(x) > a\} \; .$$

We show that A is an open set. So suppose $x \in A$. Then there is a $y \in U_{\frac{1}{n}}(x)$ with $f(y) > a$. Let $d(x, y) = s < \frac{1}{n}$, and suppose z is such that $d(z, x) < \frac{1}{n} - s$. Then, by the triangle inequality, $d(z, y) < \frac{1}{n}$, and so $z \in A$ also. It follows that A contains an open ball about x. Since x was arbitrary, A is open as desired.

26

(a) We want to show that if $f, g \geqslant 0$ are measurable then $\int f + g \geqslant \int f + \int g$. Let $\phi \leqslant f$ a.e. and $\psi \leqslant g$ a.e. be simple functions. Then $\phi + \psi \leqslant f + g$ a.e. is simple. So, by Lemma 16 and the definition of integral,

$$\int f + g \geqslant \int \phi + \psi = \int \phi + \int \psi \, .$$

Taking the *sup* over all ϕ and ψ, we get the desired result.

(b) If $\{f_j\}$ is a sequence of nonnegative measurable functions, then by (a),

$$\int \sum_{j=1}^{\infty} f_j \geqslant \int \sum_{j=1}^{n} f_j \geqslant \sum_{j=1}^{n} \int f_j \, .$$

Taking the limit in n, we see

$$\int \sum_{j=1}^{\infty} f_j \geqslant \sum_{j=1}^{\infty} \int f_j$$

28 We have

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leqslant \int f^+ + \int f^- \leqslant \int (f^+ + f^-) = \int |f| \, .$$

32

(a) Given $f \geqslant 0$ measurable and $\epsilon > 0$ define

$$\psi(x) = (1+\epsilon)^k \qquad \text{where} \qquad (1+\epsilon)^k < f(x) \leqslant (1+\epsilon)^{k+1}, \quad k \in \mathbb{Z}\,.$$

Then ψ is simple and $f \leqslant \psi \leqslant (1+\epsilon)f$. It follows that $\int \psi \leqslant (1+\epsilon)\int f$. Taking $\epsilon \to 0$, it follows that $\int f$ is the infimum of the integrals of simple functions above f.

(b) Writing $f = f^+ - f^-$ and applying (a) and the definition of the integral, it easily follows that if f is integrable then

$$\begin{aligned}\int f \,\mathrm{d}\mu &= \sup\left\{\int \phi \,\mathrm{d}\mu : \phi \leqslant f \text{ a.e.}, \phi(X) \text{ countable}\right\}\\ &= \inf\left\{\int \psi \,\mathrm{d}\mu : \psi \geqslant f \text{ a.e.}, \psi(X) \text{ countable}\right\}.\end{aligned}$$

34 $F : [a,b] \to \mathbb{R}$ is differentiable on (a,b) and continuous on $[a,b]$, and F' is bounded off of a null set. Then, for any small $h \geqslant 0$, we have

$$\int_a^{b-h} \frac{F(x+h)-F(x)}{h} \mathrm{d}\mathscr{L}(x) = \int_{b-h}^{b} F - \int_a^{a+h} F, .$$

As $h \to 0$, the RHS converges to $F(b) - F(a)$, by the continuity of F. And, the LHS converges to $\int F'$, by the Mean Value Theorem and the Dominated Convergence Theorem.

35 With $F(x,t) = t^3 e^{-t^2 x}$, we set $f(t) = \int_0^\infty F(x,t)\mathrm{d}\mathscr{L}(x)$. Clearly $f(0) = \int_0^\infty 0 = 0$. For $t \neq 0$, we easily integrate to give

$$f(t) = \left[-\frac{t^3}{t^2} e^{-t^2 x}\right]_0^\infty = t\,.$$

Thus $f(t) = t$ for all t and $f'(0) = 1$.

On the other hand,

$$D_2 F(x,t) = \left(3t^2 - 2t^4 x\right) e^{-t^2 x} \quad \Longrightarrow \quad D_2 F(x,0) = 0\,.$$

Thus

$$\int_0^\infty D_2F(x,0)\,\mathrm{d}\mathscr{L} = 0 \neq 1 = f'(0)\,.$$

36 For $I \subseteq \mathbb{R}$ an open interval, we assume $F\colon X \times I \to \mathbb{R}^*$ satisfies

- For each $t \in I$, the function $x \mapsto F(x,t)$ is μ-summable;
- For each $t \in I$, $D_2F(x,t)$ exists for μ-a.e. $x \in X$;
- There is a summable function $M\colon X \to R$ with

$$\sup_{t\in I} |D_2F(x,t)| \leqslant M(x) \quad \text{for } \mu\text{-a.e. } x \in X.$$

Then we want to show $f\colon I \to \mathbb{R}$ defined by

$$f(t) = \int F(x,t)\,\mathrm{d}\mu(x)\,.$$

is differentiable and that

$$f'(t) = \int D_2F(x,t)\,\mathrm{d}\mu(x) \qquad t \in I\,.$$

Fix $t \in I$, and consider $h \neq 0$ small enough that $t + h \in I$. Define

$$f_h(t) = \frac{f(t+h) - f(t)}{h} = \int \frac{F(x,t+h) - F(x,t)}{h}\,\mathrm{d}\mu(x)$$

Now, by the Mean Value Theorem, for every x, t and h there is an $s \in I$ such that

$$\left|\frac{F(x,t+h) - F(x,t)}{h}\right| = |D_2F(x,s)| \leqslant M(x)\,.$$

So, we can apply the Dominated Convergence Theorem to prove

$$f'(t) = \lim_{h\to 0} f_h(t) = \int \lim_{h\to 0} \frac{F(x,t+h) - F(x,t)}{h}\,\mathrm{d}\mu(x) = \int D_2F(x,t)\,\mathrm{d}\mu(x)\,.$$