AMSI 2013: MEASURE THEORY
Handout 8
Product Measures

Marty Ross
martinirossi@gmail.com

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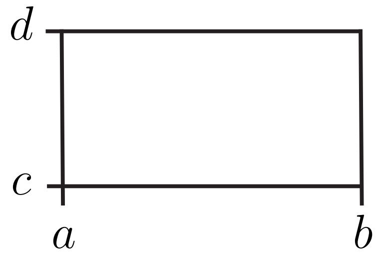
INTRODUCTION

In the theory of Riemann integration, we have the well-known rule (perhaps theorem) for computing a double integral on $P = [a,b] \times [c,d]$ via *iterated integrals*:

(♦) $$\iint\limits_P f(x,y)\,\mathrm{d}A = \int\limits_c^d \int\limits_a^b f(x,y)\mathrm{d}x\mathrm{d}y = \int\limits_a^b \int\limits_c^d f(x,y)\mathrm{d}y\mathrm{d}x\,.$$



As for the convergence theorems, and for differentiating under the integral, the rule can fail for suitably misbehaved functions: see the examples below, following the statement of Theorem 47.

In measure theory, we can think of $\mathrm{d}A$ as integration with respect to 2-dimensional Lebesgue measure $\mathscr{L}^2$, and then one can similarly ask whether the $\mathscr{L}^2$-integral can be evaluated as an iterated integral. In fact, what we do is define a completely new measure, the *product measure* $\mathscr{L}^1 \times \mathscr{L}^1$ on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. More generally, given a measure μ on X and a measure ν on Y, we define the product measure $\mu \times \nu$ on $X \times Y$.

Our goal in this Handout is to show that, for suitably well behaved functions, the product measure satisfies a formula analogous to (♦): this is the *Fubini-Tonelli Theorem* (Theorem 47). Also, in the Lebesgue setting, we prove that the product of two Lebesgue measures is just a higher dimensional Lebsegue measure (Theorem 43).

PRODUCT MEASURES

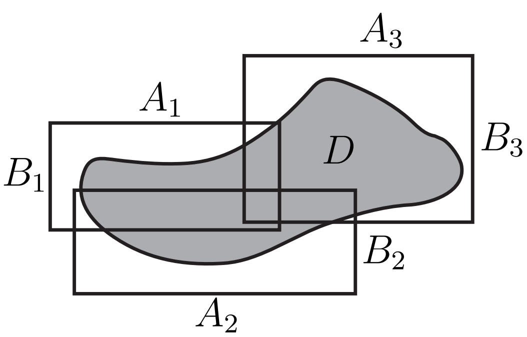
Given a measure μ on X and a measure ν on Y, we want to define a new measure, $\mu\times\nu$ on $X \times Y$. The key property we want, at least for measurable sets, is

$$(\bigstar) \qquad \boxed{\mu\times\nu(A \times B) = \mu(A)\cdot\nu(B)} \qquad A \subseteq X, B \subseteq Y.$$

As for Lebesgue measure, the precise definition of $\mu\times\nu$, is complicated, involving the covering of arbitrary sets by unions of rectangles $A_j \times B_j$.[1] Moreover, unlike the Lebesgue setting, the generality of the sets A and B makes the proving of ($\bigstar$) quite tricky, even assuming that A and B are measurable: this necessitates a subtlety in the definition.

Definition: For μ a measure on X and ν a measure on Y, define $\mu\times\nu : \wp(X \times Y)\rightarrow\mathbb{R}^*$ by

$$\boxed{\mu\times\nu(D) = \inf\left\{\sum_{j=1}^{\infty}\mu(A_j)\nu(B_j) : A_j\subseteq X \ \mu\text{-measurable},\ B_j\subseteq Y \ \nu\text{-measurable}, D\subseteq\bigcup_{j=1}^{\infty} A_j \times B_j\right\}}$$



We shall make a number of remarks, but first we have

PROPOSITION 41: If μ is a measure on X and ν is a measure on Y then $\mu\times\nu$ is a measure on $X \times Y$.

[1] It should be clear that by "rectangle", we simply mean any product set $A \times B$; there is no suggestion that the sides of such a rectangle are intervals, or are in any other way simple sets.

REMARKS

- Given Proposition 41, we can now refer to $\mu \times \nu$ as the *product measure* on $X \times Y$.
- The condition that the covering rectangles have measurable sides is *not* needed to prove that $\mu \times \nu$ is a measure. The point of the condition is to facilitate the proof of (★) for A and B measurable: see Theorem 42 below.[2] Of course, if μ and ν are Borel regular (or, more generally, regular), then the measurability condition is redundant: given any covering rectangle $A \times B$ we can find $A' \times B' \supseteq A \times B$ with A' and B' measurable, and $\mu(A') = \mu(A)$ and $\nu(B') = \nu(B)$.
- It is not obvious that if μ and ν are Borel measures then so is $\mu \times \nu$. See Theorem 45.

We now show that product measures have the desired product property.

THEOREM 42: Suppose μ is a measure on X and ν is a measure on Y. If $A \subseteq X$ is μ-measurable and $B \subseteq Y$ is ν-measurable then:

(a) $\mu\times\nu(A \times B) = \mu(A) \cdot \nu(B)$;

(b) $A \times B$ is $\mu \times \nu$-measurable.

50 REMARK: Neither (a) nor (b) is in general true for A and B not measurable.

PROOF: To prove (a), we first note that $A \times B$ covers $A \times B$, and therefore it trivially follows that $\mu\times\nu(A \times B) \leqslant \mu(A) \cdot \nu(B)$. To prove the reverse inequality, consider a covering $\{A_j \times B_j\}_{j=1}^{\infty}$ of $A \times B$ by rectangles with measurable sides (i.e. all of the A_j and B_j are measurable). Then

$$\chi_A \cdot \chi_B = \chi_{A\times B} \leqslant \chi_{\left(\bigcup_j A_j \times B_j\right)} \leqslant \sum_{j=1}^{\infty} \chi_{A_j \times B_j} = \sum_{j=1}^{\infty} \chi_{A_j}\chi_{B_j}\,.$$

The μ-measurability of A implies that, for fixed $y \in Y$, the function $\chi_{A\times B}(x,y) = \chi_A(x) \cdot \chi_B(y)$ is a measurable function of x, and similarly for $\chi_{A_j\times B_j}(x,y)$. We can therefore apply the Monotone Convergence Theorem (Theorem 19) to compute

$$\mu(A) \cdot \chi_B(y) = \int \chi_{A\times B}(x,y)\,\mathrm{d}\mu(x) \leqslant \sum_{j=1}^{\infty} \int \chi_{A_j\times B_j}(x,y)\,\mathrm{d}(x) = \sum_{j=1}^{\infty} \mu(A_j)\chi_{B_j}(y)$$

These are now measurable functions of y. So, we can apply the Monotone Convergence Theorem again, to conclude

$$\mu(A) \cdot \nu(B) \leqslant \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j)\,.$$

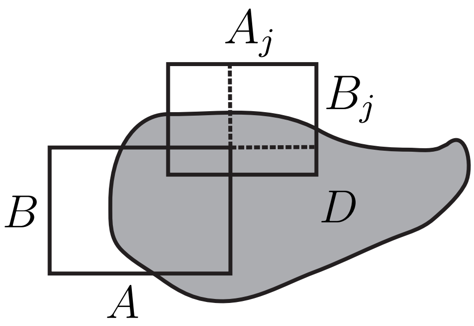
Since this is true for an arbitrary covering of $A\times B$, we conclude $\mu(A)\cdot\nu(B) \leqslant \mu\times\nu(A\times B)$, as desired.

[2] I don't know what happens if $\mu\times\nu$ is defined without the measurability condition, whether the subsequent theorems cease to be true, or just that the proofs become harder. I suspect the former.

For (b), we consider $D \subseteq X \times Y$, and we want to prove that

$$(\blacktriangle) \qquad \mu\times\nu(D) \geqslant \mu\times\nu\,(D \cap (A \times B)) + \mu\times\nu\,(D \sim (A \times B))\ .$$

To this end, consider a covering $\{A_j \times B_j\}$ of D by rectangles with measurable sides. We then use $A \times B$ to cut up each $A_j \times B_j$ into four subrectangles, as pictured.



This gives us a new covering $\{A'_k \times B'_k\}$, for which

- For each k either $A'_k \times B'_k \subseteq A \times B$ or $A'_k \times B'_k \subseteq \sim (A \times B)$;
- $\displaystyle\sum_{k=1}^{\infty} \mu(A'_k)\nu(B'_k) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j)$.

(Note that the justification of the second claim uses the measurability of A, B, A_j and B_j, but *not* the measurability of $A \times B$ or $A_j \times B_j$: we are not here claiming anything about the product measures of these rectangles.)

We then have

$$\begin{aligned}
\sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) &= \sum_{k=1}^{\infty} \mu(A'_k)\nu(B'_k) \\
&= \sum_{A'_k \times B'_k \subseteq A \times B} \mu(A'_k)\nu(B'_k) \ + \sum_{A'_k \times B'_k \subseteq \sim(A \times B)} \mu(A'_k)\nu(B'_k) \\
&\geqslant \mu\times\nu\,(D \cap (A \times B)) + \mu\times\nu\,(D \sim (A \times B)) \qquad \text{(by definition).}
\end{aligned}$$

Taking the *inf* over all such coverings, we obtain ($\blacktriangle$).

Theorem 42 tells us that rectangles with measurable sides are measurable, and have the desired measure. Some natural results follow from this. To begin,

51 **THEOREM 43**: $\mathscr{L}^{m+n} = \mathscr{L}^m \times \mathscr{L}^n$.

Note that Theorem 43 can be proved without appealing to Proposition 5(b), that $\mathscr{L}^m(P) = v(P)$ for an m-box $P \subseteq \mathbb{R}^n$: consequently, Theorem 42 and Proposition 5(a) give an alternative proof of Proposition 5(b). In fact, a common approach to higher dimensional Lebesgue measure is to directly define

$$\mathscr{L}^m = \mathscr{L}^1 \times \cdots \times \mathscr{L}^1\,,$$

avoiding our m-box definition altogether. In some sense, this makes life easier: certainly, the convergence theorem proof of Theorem 42(a) is (eventually) much simpler than any direct proof of Proposition 5(b). Still, it is natural to define $\mathscr{L}^m$ as we did at that early stage; and, even if more painful, a direct proof of Proposition 5(b) is more transparent.

52 **LEMMA 44**: Suppose X and Y are topological spaces, and suppose $A \subseteq X$ is Borel and $B \subseteq Y$ is Borel. Then $A \times B$ is Borel.

53 **THEOREM 45**: Suppose X and Y are second countable topological spaces, and suppose μ is a measure on X and ν is a measure on Y. Then

(a) If μ and ν are Borel then so is $\mu\times\nu$.

(b) If μ and ν are Borel regular then so is $\mu\times\nu$.

(c) If μ and ν are Radon (in the case that X and Y are locally compact and Hausdorff) then so is $\mu\times\nu$.

REMARKS:

- The hypothesis of second countability guarantees that every open set in $X \times Y$ can be written as a *countable* union of open rectangles, making the proof of (a) straightforward.[3]
- Part (b) follows easily from (a) and Lemma 4, and thus holds whenever (a) does.
- Similarly, for X and Y locally compact and Hausdorff, (c) holds whenever (a) holds.

THE FUBINI-TONELLI THEOREM

We return to the question raised in the introduction, that of integrating a measurable function $f : X \times Y \to \mathbb{R}^*$. In case $f = \chi_{A \times B}$ is the characteristic function of a rectangle with measurable sides, Theorem 42 is exactly the result we want. The next step is to prove the desired result for $f = \chi_S$ for general $\mu\times\nu$-measurable S, which is the content of Lemma 46; after that the Fubini-Tonelli Theorem – Theorem 47 for general f – follows routinely.

Note that, whether a characteristic or general function, it is part of our job is to show that $f(x, y)$ gives rise to measurable functions of the individual variables x and y: this is not merely a matter of definition, and is the reason for the complicated statements of Lemma 46 and Theorem 47. Also, as illustrated by the first counterexample after Theorem 47, the Fubini-Tonelli Theorem is only guaranteed to hold for suitably finite functions. That requirement is encapsulated by the following.

Definition: Suppose μ is a measure on a set X. Then:

- A set $A \subseteq X$ is *σ-finite* if $A = \bigcup_{j=1}^{\infty} A_j$ where each A_j is measurable with $\mu(A_j) < \infty$.
- A measurable function $f : X \to \mathbb{R}^*$ is *σ-finite* if $\{x : f(x) \neq 0\}$ is σ-finite.

54 We note that

(a) If f is summable then f is σ-finite.

(b) If X is σ-finite then all measurable functions on X are σ-finite.

(c) If X and Y are σ-finite (with respect to μ and ν, respectively), then $\mu\times\nu$ is σ-finite.

[3] I don't know whether (a) remains true without such a hypothesis, but it seems unlikely. Let $\mathcal{F}$ be the σ-algebra generated by the rectangles $A \times B$ with Borel sides, and let $\mathcal{B}$ be the collection of Borel subsets of $X \times Y$. Then $\mathcal{F} \subseteq \mathcal{B}$, by Lemma 44. However, there are topological spaces X and Y for which $\mathcal{F} \subsetneq \mathcal{B}$; presumably, in this situation one can construct Borel μ and ν for which $\mu\times\nu$ is not Borel.

LEMMA 46: Suppose μ is a measure on X and ν is a measure on Y, and suppose $S \subseteq X \times Y$. For $y \in Y$, let

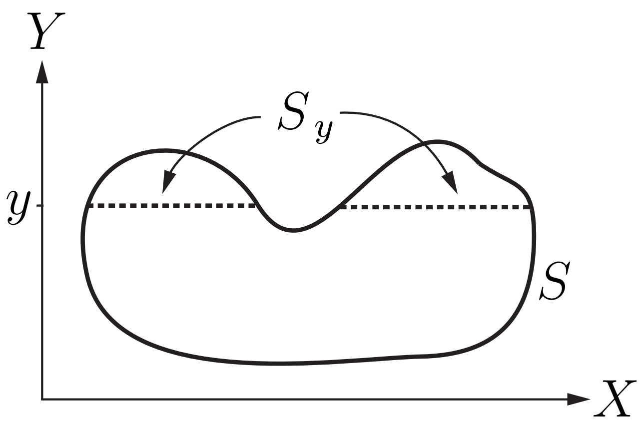
$$S_y = \{x \in X : (x,y) \in S\}\,.$$

If S is σ-finite with respect to $\mu\times\nu$ then:

(i) S_y is a μ-measurable subset of X for ν-a.e. $y \in Y$;

(ii) The function $y \mapsto \mu(S_y)$ is ν-measurable;

(iii) $\mu\times\nu(S) = \int \mu(S_y)\mathrm{d}\nu(y)$.



We prove Lemma 46 at the end of this Handout. We first state, and remark upon:

55 **THEOREM 47 (Fubini-Tonelli Theorem)**: Suppose μ is measure on X and ν is a measure on Y, and suppose that $f : X \times Y \to \mathbb{R}^*$ is $\mu\times\nu$-integrable and σ-finite with respect to $\mu\times\nu$. Then:

(i) The function $x \mapsto f(x,y)$ is μ-integrable for ν-a.e. $y \in Y$;

(ii) The function $y \mapsto \int f(x,y)\mathrm{d}\mu(x)$ is ν-integrable;

(iii)

$$\boxed{\int_{X\times Y} f\mathrm{d}\mu\times\nu = \int_Y \left(\int_X f(x,y)\mathrm{d}\mu(x) \right) \mathrm{d}\nu(y)}$$

REMARKS:

- The Fubini-Tonelli Theorem follows routinely from Lemma 46, writing $f = f^+ - f^-$, and approximating f^+ and f^- by simple functions.
- Interchanging the roles of X and Y, we have

$$\boxed{\int_Y \left(\int_X f(x,y) \mathrm{d}\mu(x) \right) \mathrm{d}\nu(y) = \int_X \left(\int_Y f(x,y) \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x)}$$

- By (54), a summable function is automatically σ-finite, and thus the Fubini-Tonelli Theorem applies to any such function.
- Again by (54), if X and Y are σ-finite then the Fubini-Tonelli Theorem applies to any nonnegative measurable function on $X \times Y$.
- The first example below shows the necessity of the hypothesis of σ-finiteness.
- The second example below shows that even if the two iterated integrals are well-defined, they may not be equal. Thus the hypothesis that f be $\mu\times\nu$-integrable is also necessary.

(56) **Examples**: Let $X = Y = [0,1]$.

(a) Let $\mu = \mathscr{L}$ and let ν be counting measure. Let $f = \chi_D$, where D is the diagonal:

$$D = \{(x,x) : x \in [0,1]\}\,.$$

Then

$$\int_{[0,1]\times[0,1]} \chi_D \, \mathrm{d}\mathscr{L} \times \nu \neq \int_{[0,1]} \int_{[0,1]} \chi_D \, \mathrm{d}\mathscr{L} \mathrm{d}\nu \neq \int_{[0,1]} \int_{[0,1]} \chi_D \, \mathrm{d}\nu \mathrm{d}\mathscr{L}\,.$$

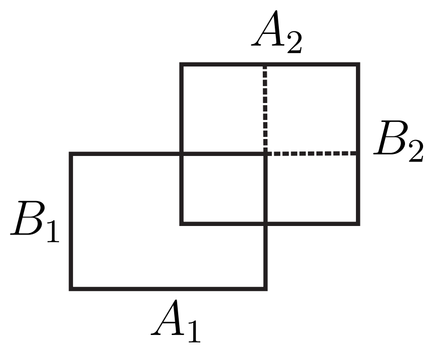
(b) Let $\mu = \nu = \mathscr{L}$, and let $f(x,y) = \frac{x^2-y^2}{(x^2+y^2)^2}$. Then f is σ-finite and

$$\int_{[0,1]} \int_{[0,1]} f(x,y) \, \mathrm{d}\mathscr{L}(x) \mathrm{d}\mathscr{L}(y) \neq \int_{[0,1]} \int_{[0,1]} f(x,y) \, \mathrm{d}\mathscr{L}(y) \mathrm{d}\mathscr{L}(x)\,.$$

PROOF OF LEMMA 46: By the σ-finiteness hypothesis, we can assume $\mu\times\nu(S) < \infty$.

Part 1: We first prove the Lemma in case $S = \bigcup_{j=1}^{\infty} A_j \times B_j$ is the union of rectangles with measurable sides.

Chopping up, as in the proof of Theorem 42, we can assume the rectangles are pairwise disjoint. (First chop $A_2 \times B_2$ with respect to $A_1 \times B_1$, discarding the subrectangle $(A_1 \cap A_2) \times (B_1 \cap B_2)$. Next chop $A_2 \times B_2$ with respect to the disjoint rectangles already obtained, discarding any redundant subrectangles. Continue inductively.)



For $y \in Y$,

$$S_y = \bigcup_{y \in B_j} A_j \,,$$

which is clearly μ-measurable. As well, since the rectangles are pairwise disjoint, this is a disjoint union for each y. Thus.

$$\mu(S_y) = \sum_{y \in B_j} \mu(A_j) = \sum_{j=1}^{\infty} \mu(A_j)\chi_{B_j}(y) \,,$$

which is clearly a measurable function of y. Integrating with respect to y, the Monotone Convergence Theorem, Theorem 42 and countable additivity imply

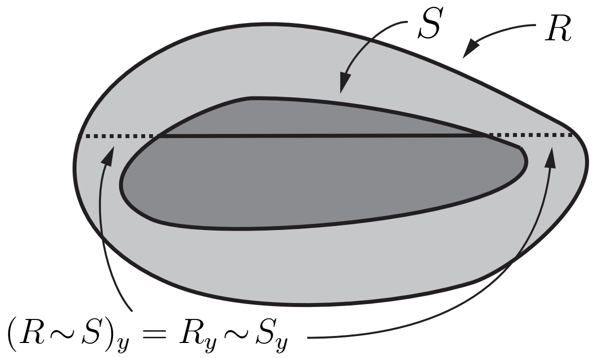
$$\int \mu(S_y)\,\mathrm{d}\nu(y) = \sum_{j=1}^{\infty} \int \mu(A_j)\chi_{B_j}(y)\,\mathrm{d}\nu(y) = \sum_{j=1}^{\infty} \mu(A_j)\nu(B_j) = \sum_{j=1}^{\infty} \mu{\times}\nu(A_j \times B_j) = \mu{\times}\nu(S) \,.$$

This is exactly what we wanted to prove.

Part 2: For any $S \subseteq X \times Y$, we show that there is a measurable $R \supseteq S$ for which

$$(*) \qquad \mu\times\nu(S) = \mu\times\nu(R) = \int \mu(R_y)\,\mathrm{d}\nu(y) \,.$$

In particular, R_y is μ-measurable for ν-a.e. y, and the function $y \mapsto \mu(R_y)$ is ν-measurable.



Fix $n \in \mathbb{N}$. By Part 1, together with the definition of $\mu\times\nu$, we can find R_n, a union of rectangles, with

$$(\dagger) \qquad \mu\times\nu(R_n) = \int \mu\left((R_n)_y\right)\, \mathrm{d}\nu(y) \leqslant \mu\times\nu(S) + \frac{1}{n} < \infty\,.$$

Further, we can assume $R_{n+1} \subseteq R_n$. (Given R_n, find any R_{n+1} satisfying (†). Chopping as above, and discarding subrectangles outside of R_n, we can ensure that every rectangle of R_{n+1} lies within some rectangle of R_n).

We let $R = \bigcap_{n=1}^{\infty} R_n$, and the idea is to let $n\to\infty$ in (†). First of all, it is clear that $R \supseteq S$, and that $\mu\times\nu(R) = \mu\times\nu(S)$.

Next, (†) implies for fixed n that $\mu\left((R_n)_y\right) < \infty$ except for a ν-null set. Taking the union over $\mathbb{N}$ of these null sets, we see:

(‡) For ν-a.e. $y \in Y$, we have $\mu\left((R_n)_y\right) < \infty$ for *every* $n \in \mathbb{N}$.

But

$$R_y = \bigcap_{n=1}^{\infty} (R_n)_y\,.$$

By Part 1, each $(R_n)y$ is μ-measurable, and thus so is R_y. Then, by Theorem 8(b) and (‡),

$$\mu(R_y) = \lim_{n\to\infty} \mu\left((R_n)_y\right) \qquad \text{for } \nu\text{-a.e. } y \in Y.$$

In particular, the function $y \mapsto \mu(R_y)$ is a limit of measurable functions, and is thus measurable. As well, each function in this limit is dominated by the function $y \mapsto \mu((R_1)_y)$, which is summable, by (†). Thus, by the Dominated Convergence Theorem (Theorem 22), we can take the limit in (†), giving (∗).

Part 3: Suppose that $\mu\times\nu(S) = 0$. Then Part 2 implies that there is an $R \supseteq S$ with

$$\int \mu(R_y)\,\mathrm{d}\nu(y) = 0$$
$$\implies \quad \mu(R_y) = 0 \quad \text{for } \nu\text{-a.e. } y \in Y$$
$$\implies \quad \mu(S_y) = 0 \quad \text{for } \nu\text{-a.e. } y \in Y$$

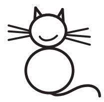
Part 4: Finally, we consider a general measurable S. By Part 2, we can find a measurable $R \supseteq S$ satisfying $(*)$. So, it suffices to show that $\mu(R_y) = \mu(S_y)$ for ν-a.e. $y \in Y$. Note that

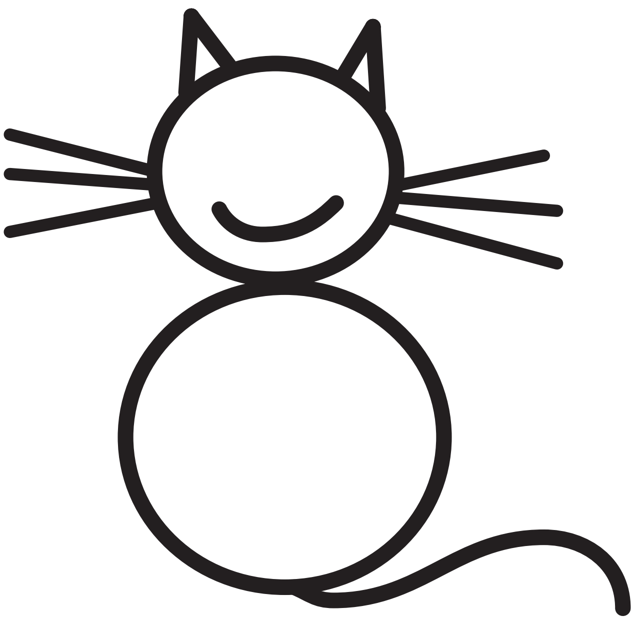
$$R_y = S_y \cup (R{\sim}S)_y \ .$$

Then, since S is measurable and $\mu\times\nu(S) < \infty$,

$$\mu\times\nu(R{\sim}S) = 0$$
$$\implies \quad \mu\left((R{\sim}S)_y\right) = 0 \quad \text{for } \nu\text{-a.e. } y \in Y \qquad \text{(by Part 3)}$$
$$\implies \quad \mu(R_y) = \mu(S_y) \quad \text{for } \nu\text{-a.e. } y \in Y\ .$$

This is exactly what we wanted to prove.





SOLUTIONS

(50) We want to give an example to show that the formula $\mu\times\nu(A\times B)=\mu(A)\cdot\nu(B)$ is not true in general for non-measurable sets. To do this, let $X=\{a\}$ and $Y=\{a,b\}$. Let μ be delta measure at a, and let ν be the Everything-Is-Better measure (Handout 3):

$$\nu(\emptyset)=0 \qquad \nu(\{a\})=\nu(\{b\})=2 \qquad \nu(\{a,b\})=3\,.$$

It is easy to check that ν is in fact a measure. And, it is easy to check that

$$\mu(\{a\})\cdot\nu(\{a\})=2 \qquad \mu\times\nu(\{a\}\times\{a\})=3\,.$$

Also $\{a\}\times\{a\}$ is not $\mu\times\nu$-measurable, since it does not split $\{a\}\times\{a,b\}$ in an additive manner:

$$\mu\times\nu(\{a\}\times\{a,b\})=3\neq 6=\mu\times\nu(\{a\}\times\{a\})+\mu\times\nu(\{a\}\times\{b\})\,.$$

(53) We have X and Y second countable topological spaces, with μ a Borel measure on X and ν a Borel measure on Y.

(a) We want to show $\mu\times\nu$ is Borel. If $V\subseteq X$ and $W\subseteq Y$ are open then $V\times W$ is open, and is measurable by Theorem 42. But such open rectangles form a base for the topology on $X\times Y$, and we can choose a countable base, since we can choose countable bases for X and Y. Thus, every open set in $X\times Y$ is a *countable* union of open rectangles, and is thus measurable.

(b) If μ and ν are Borel regular, we want to show that $\mu\times\nu$ is Borel regular. By Borel regularity of μ and ν, in the definition of $\mu\times\nu$, we can replace any covering by rectangles with a covering by rectangles with Borel sides. Thus

$$\mu\times\nu(D)=\inf\left\{\sum_{j=1}^{\infty}\mu(A_j)\nu(B_j) : A_j\subseteq X \text{ Borel, } B_j\subseteq Y \text{ Borel}\right\} \quad D\subseteq X\times Y.$$

But given such a covering, (52) implies $\bigcup_j A_j\times B_j$ is Borel. We can now argue exactly as for the Borel regularity of Lebesgue measure (Proposition 34).

(c) Given μ and ν are Radon, we now want to prove that $\mu \times \nu$ is Radon. We first show that if $K \subseteq X \times Y$ is compact then $\mu \times \nu(K) < \infty$. Let $\Pi_1 : X \times Y \to X$ and $\Pi_2 : X \times Y \to Y$ be the natural projections. Since these projections are continuous, the sets $K_1 = \Pi_1(K)$ and $K_2 = \Pi_2(K)$ are closed (since X and Y are Hausdorff) and compact. Thus, since μ and ν are Radon, and using Theorem 42,

$$\mu \times \nu(K) \leqslant \mu \times \nu(K_1 \times K_2) = \mu(K_1) \cdot \nu(K_2) < \infty.$$

Next, we want to show that any open set V in $X \times Y$ can be approximated from the inside by compact sets. But using the second countability, V can be written as a countable union of open rectangles R with $\overline{R}$ compact; the approximation result then follows immediately.

Finally, the approximation of A from the outside by open sets follows easily from the definition of the product measure.