AMSI 2013: MEASURE THEORY
Handout 6
L^p Spaces

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Suppose μ is a measure on X. We define

$$L^1(\mu) = \{f : f \text{ is } \mu\text{-summable}\}\,,$$

writing L^1, if the context is clear. The superscript 1 suggests, correctly, that there will be collections L^p for other $p \in \mathbb{R}^*$. But let us first concentrate upon the collection of summable functions. We first note that L^1 is closed under addition (by Theorem 21) and scalar multiplication, and thus L^1 is a vector space. Also, if a function f is in L^1 then so is $|f|$ (immediately, from the definition of $\int f$). So, we can define a "norm"

$$\|f\|_1 = \int |f| \,\mathrm{d}\mu\,.$$

We are interested here in $f \in L^1$, but note that $\|f\|_1$ is well-defined, possibly ∞, for any measurable f.

In fact, $\|\cdot\|_1$ is *not* in general a norm. Clearly, $\|cf\|_1 = |c| \cdot \|f\|_1$ for $c \in \mathbb{R}$. And, the triangle inequality follows from the triangle inequality on $\mathbb{R}$, together with monotonicity of the integral:

$$\|f+g\|_1 = \int |f+g| \leqslant \int |f| + |g| = \int |f| + \int |g| = \|f\|_1 + \|g\|_1\,.$$

However, we may not have positivity: for many measures it is possible that $\|f\|_1 = 0$ even if $f \neq 0$ (i.e. even if f is not the zero function). $\|\cdot\|_1$ on L^1 is what is called a *seminorm.*

So, technically, we don't have a norm on L^1, but the seminormness of $\|\cdot\|_1$ is pretty trivial: if $\|f\|_1 = 0$ then f must be 0 almost everywhere. To cope with the lack of positivity, we recall the equivalence relation on measurable functions:

$$f \equiv g \quad \Longleftrightarrow \quad f = g \text{ a.e.}\,.$$

Note that

$$(*) \qquad\qquad f \equiv g \quad \Longrightarrow \quad \|f\|_1 = \|g\|_1\,,$$

and thus we can define $\|\cdot\|_1$ on the set of equivalence classes of L^1. Writing $\hat{f}$ for the equivalence class of $f \in L^1$, we then define

$$\|\hat{f}\|_1 = \|f\|_1\,,$$

with $(*)$ guaranteeing that the definition is independent of the class representative.

It is easy to check that we do now indeed have a norm.[1] Technically, we should define and work with this new space of equivalence classes. However, the general consensus is that life is too short for that. So, we'll stick to L^1, and refer to $\|\cdot\|_1$ as a norm on L^1, keeping the equivalences in the back of our minds.[2]

We can now think of L^1 as a normed space. Our main theorem here, Theorem 31, is that L^1 is a *Banach space*: L^1 is *complete* with respect to the associated metric (in truth, *pseudometric*)

$$d(f,g) = \|f-g\|_1\,.$$

Before proving the completeness of L^1, we first consider the L^p spaces for other p. As well as the norm we have defined above, it is very natural to consider the "inner product"

$$\langle f, g\rangle = \int fg\,\mathrm{d}\mu\,.$$

Modulo the same positivity considerations, this is indeed an inner product, but an inner product on what space? Note that even if f and g are summable, $\langle f, g\rangle$ may be infinite or undefined: for example, let $\mu = \mathscr{L}$ on $(0,1)$ and take $f(x) = g(x) = \frac{1}{\sqrt{x}}$. However, we do have

$$|fg| \leqslant \tfrac{1}{2}f^2 + \tfrac{1}{2}g^2\,.$$

Thus, if both f^2 and g^2 are summable then $\langle f, g\rangle$ will be well-defined and finite. This leads us to define

$$L^2(\mu) = \left\{ f : \int f^2\,\mathrm{d}\mu < \infty \right\}.$$

[1] This is a standard construction, which works for any seminorm.

[2] It shouldn't be imagined that the representative we choose is always unimportant. For example, the Weird function $W = \chi_{\mathbb{Q}\cap[0,1]}$ from Handout 1 was introduced exactly because of its failure to be Riemann integrable. However, W is equivalent to the zero function, which is trivially Riemann integrable. The moral is, within the confines of measure theory our choice of representative is unimportant; but, as soon as we wish to consider non-measure-theoretic properties, we may need to specify the representative function.

Notice that L^2 is closed under addition, and is thus a vector space, since

$$(f+g)^2 = f^2 + 2fg + g^2 \leqslant 2f^2 + 2g^2 \,.$$

Then $\langle\cdot,\cdot\rangle$ is an inner product on the space L^2, with the associated (semi)norm

$$\|f\|_2 = \sqrt{\langle f,f\rangle} = \left(\int |f|^2\right)^{\frac{1}{2}} \,.$$

Note that $\|f\|_2$ is well-defined for any measurable f. In Theorem 31 we prove that L^2 is complete, and thus that L^2 is a *Hilbert space.*

In accord with L^1 and L^2, we now consider $1 \leqslant p < \infty$. We define

$$L^p(\mu) = \left\{ f : \int |f|^p \,\mathrm{d}\mu < \infty \right\} \qquad 1 \leqslant p < \infty \,,$$

and if $f \in L^p$, we say that f is *p-summable.* Again, L^p is a vector space, since

$$|f+g|^p \leqslant (2\max(|f|,|g|))^p = 2^p \max\left(|f|^p,|g|^p\right) \leqslant 2^p\left(|f|^p + |g|^p\right) \,.$$

We define the associated "seminorm":

$$\|f\|_p = \left(\int |f|^p\right)^{\frac{1}{p}} \qquad f \text{ measurable.}$$

The definition is consistent with the cases $p=1$ and $p=2$, but it is not at all obvious that $\|\cdot\|_p$ defines a seminorm. It is easy to see that $\|cf\|_p = |c|\cdot\|f\|_p$ for $c \in \mathbb{R}$, but the triangle inequality takes significantly more work: see Theorem 30.

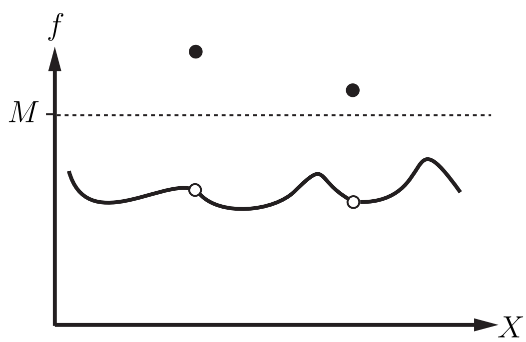
Before getting down to work, there is one more L^p space to define, for $p=\infty$. Here the idea is to capture the space of bounded measurable functions, and to give them a *sup* norm. However, since we want null sets to be irrelevant, the straight-forward definitions are not appropriate. Instead, for $f : X \to \mathbb{R}^*$,we consider those $M \in \mathbb{R}^*$ for which

$$\mu(f^{-1}((M,\infty])) = 0 \,.$$

Of course, $M = \infty$ satisfies this condition, and is easy to check that the set of such M forms a closed interval of $\mathbb{R}^*$. Thus we can define the the *essential supremum* of f to be the smallest such M:

$$(\dagger) \qquad \operatorname{ess\,sup} f = \min\{M \in \mathbb{R}^* : \mu\left(f^{-1}\left((M,\infty]\right)\right) = 0\} \,.$$

Note that the use of open intervals permits us to write *min* rather than *inf* in the definition.



We now define

$$\begin{cases} \|f\|_\infty = \operatorname{ess\,sup} |f| \,, \\ L^\infty(\mu) = \{f : f \text{ is } \mu\text{-measurable and } \|f\|_\infty < \infty\}\,. \end{cases}$$

37 Note that if $f:\mathbb{R}^n \to \mathbb{R}$ is continuous then, with respect to Lebesgue measure,

$$\|f\|_\infty = \operatorname{ess\,sup} |f| = \sup |f| \qquad (\text{wrt } \mathscr{L}^n).$$

However, with respect to other measures this may not be the case.

It is easy to see that L^∞ is a vector space, and that $\|cf\|_\infty = |c| \cdot \|f\|_\infty$ for $c \in \mathbb{R}$. To prove the triangle inequality, consider $f, g \in L^\infty$ and let $M = \|f\|_\infty$ and $N = \|g\|_\infty$. Then

$$\begin{aligned} & \{x : |f|(x) + |g|(x) > M + N\} \subseteq \{x : |f|(x) > M\} \cup \{x : |g|(x) > N\} \\ \Longrightarrow \quad & \{x : |f(x) + g(x)| > M + N\} \subseteq |f|^{-1}\left((M,\infty]\right) \cup |g|^{-1}\left((N,\infty]\right) \\ \Longrightarrow \quad & \mu\left((|f+g|)^{-1}((M+N,\infty]\right) = 0 \\ \Longrightarrow \quad & \|f+g\|_\infty \leqslant M + N = \|f\|_\infty + \|g\|_\infty \,. \end{aligned}$$

It is also easy to check that

$$f \equiv g \quad \Longrightarrow \quad \|f\|_\infty = \|g\|_\infty \,.$$

Thus $\|\cdot\|_\infty$ is a well-defined seminorm on L^∞, and $\|\cdot\|$ ignores null sets. And, as for L^p for $1 \leqslant p < \infty$, as part of Theorem 31 we shall prove that L^∞ is a Banach space.

Before proving the triangle inequality for L^p, we make a couple of quick observations:

- If $f_j \to f$ uniformly off of a null set, then it easily follows that $\|f_j - f\|_p \to 0$. More generally, with appropriate hypotheses the convergence theorems of Handout 5 can applied to show $\|f_j - f\|_p \to 0$. See the proof of Theorem 31 below.
- Let μ_0 be counting measure on $\mathbb{N}$. Then the spaces $L^p = l^p$ are exactly the well-known sequence spaces.

We now begin the proof of the triangle inequality for general L^p. As part of the proof, we associate L^p and L^q, where

$$\boxed{\frac{1}{p} + \frac{1}{q} = 1}$$

In what follows, we shall always assume that p and q are thus related. Note the particular cases $p = q = 2$, and $p = 1, q = \infty$.

LEMMA 28 (Young's Inequality): For any $a, b \geqslant 0$, and $1 < p, q < \infty$, we have

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}\,,$$

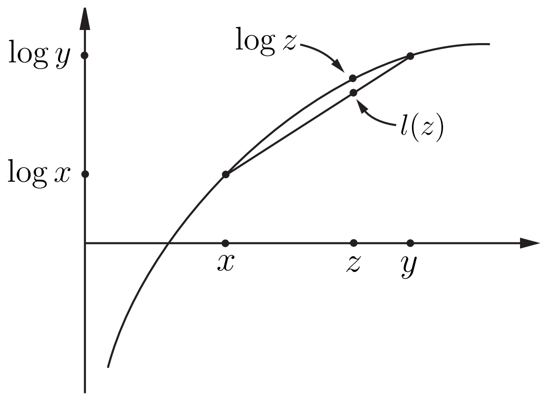
with equality iff $a^p = b^q$.

PROOF: Since the function *log* is concave, the graph of *log* lies above the straight line connecting the straight-line segment l connecting any two points on the graph. Writing

$$z = (1-t)x + ty \qquad 0 \leqslant t \leqslant 1\,,$$

for a point between x and y, we thus have

$$\begin{aligned} & \log(z) \geqslant l(z) \\ \Longrightarrow \quad & \log((1-t)x + ty) \geqslant (1-t)\log x + t \log y\,. \end{aligned}$$



Now set $x = a^p$, $y = b^q$, and $t = \frac{1}{q}$ (implying $1 - t = \frac{1}{p}$). Then

$$\log\left(\tfrac{a^p}{p} + \tfrac{b^q}{q}\right) \geqslant \tfrac{1}{p}\log\left(a^p\right) + \tfrac{1}{q}\log\left(b^q\right) = \log(ab)\,.$$

Taking exponentials, and noting *exp* is increasing, we obtain the desired result. We obtain a strict inequality unless $z = x = y$, that is unless $a^p = b^q$.

THEOREM 29 (Hölder's Inequality): Suppose μ is a measure on X, and suppose that f and g are measurable functions on X. Then

$$(\dagger) \qquad \int |fg| \,\mathrm{d}\mu \leqslant \|f\|_p \cdot \|g\|_q \qquad 1 \leqslant p, q \leqslant \infty.$$

In particular, if $f \in L^p$ and $g \in L^q$ then $fg \in L^1$.

PROOF: We consider $1 < p, q < \infty$ with $0 < \|f\|_p, \|g\|_q < \infty$, the other cases being trivial. Now let

$$\bar{f} = \frac{f}{\|f\|_p} \qquad \bar{g} = \frac{g}{\|g\|_q}\,.$$

Then, by Lemma 28,

$$\int |\bar{f}\bar{g}| \overset{*}{\leqslant} \int \tfrac{1}{p}|\bar{f}|^p + \tfrac{1}{q}|\bar{g}|^q = \frac{1}{p\|f\|_p^p}\int |f|^p + \frac{1}{q\|g\|_q^q}\int |g|^q = \frac{1}{p} + \frac{1}{q} = 1\,.$$

But then clearly $\int |\bar{f}\bar{g}| = \frac{1}{\|f\|_p\|g\|_q}\int |fg|$, and (†) follows.

REMARKS:

- Suppose $1 < p, q < \infty$ and the RHS of (†) is finite. Then Hölder's Inequality is an equality iff the use of Lemma 28 in the proof at $(*)$ is an equality a.e.. That is, we require

  $$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^q}{\|g\|_q^q} \quad \text{a.e.}\,.$$

  Thus, we have equality iff there is a $c \in \mathbb{R}$ such that $|f|^p = c|g|^q$ a.e..

- 38 Suppose $p = 1$, $q = \infty$ and the RHS of (†) is finite. Then Hölder's Inequality is an equality iff $|g| = \|g\|_\infty$ a.e. on $\{x : f(x) \neq 0\}$.

- 39 Suppose $\mu(X) < \infty$. If f is an integrable function, we define the *average* of f:

  $$⨍ f\,\mathrm{d}\mu = \frac{1}{\mu(X)}\int f\,\mathrm{d}\mu\,.$$

  Then a consequence of Hölder's Inequality is

  $$1 \leqslant p < r < \infty \quad \Longrightarrow \quad \left(⨍ |f|^p\right)^{\frac{1}{p}} \leqslant \left(⨍ |f|^r\right)^{\frac{1}{r}}\,.$$

  In particular, if $\mu(X) = 1$ then $\|f\|_p \leqslant \|f\|_r$.

- 40 In general, if $\mu(X) < \infty$ then

  $$(\clubsuit) \qquad \begin{cases} f \in L^r \implies f \in L^p & p \leqslant r\,, \\ \|f\|_\infty = \lim\limits_{p\to\infty} \|f\|_p\,. \end{cases}$$

  However, if $\mu(X) = \infty$ then neither conclusion of $(\clubsuit)$ is true in general.

We can now give the proof of the triangle inequality on L^p, establishing that $\| \cdot \|_p$ is a (semi)norm on L^p, followed by the proof of the completeness of L^p.

THEOREM 30 (Minkowski's Inequality): Suppose μ is a measure on X, and suppose $f, g \in L^p$, $1 \leqslant p \leqslant \infty$. Then

$$(*) \qquad \|f+g\|_p \leqslant \|f\|_p + \|g\|_p \,.$$

PROOF: The cases $p = 1$ and $p = \infty$ have been proved above, so we assume $1 < p < \infty$. Then

$$\|f+g\|_p^p = \int |f+g|^p = \int |f+g||f+g|^{p-1} \leqslant \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1} \,.$$

We will apply Hölder's Inequality on each of these integrals. In preparation, we first calculate

$$\|\, |f+g|^{p-1} \|_q = \left(\int |f+g|^{q(p-1)} \right)^{\frac{1}{q}} = \left(\int |f+g|^p \right)^{1-\frac{1}{p}} = \|f+g\|_p^{p-1} \,,$$

where we have used the equation $\frac{1}{p} + \frac{1}{q} = 1$ twice in the second last step. So, by Hölder's Inequality,

$$\|f+g\|_p^p \leqslant \|f\|_p \cdot \|f+g\|_p^{p-1} + \|g\|_p \cdot \|f+g\|_p^{p-1} \,.$$

If $\|f+g\|_p = 0$ then $(*)$ is trivial. Otherwise, we divide through by $\|f+g\|^{p-1}$ to give the desired result.

THEOREM 31 (Riesz-Fischer Theorem): Suppose μ is a measure on X, and suppose $1 \leqslant p \leqslant \infty$. Then $L^p(\mu)$ is complete with respect to the norm $\| \cdot \|_p$. Thus each L^p is a Banach space, and L^2 is a Hilbert space.

PROOF: 41 The case $p = \infty$ is easier, and is left as an exercise.

We use the standard characterisation, that a normed space is complete if and only if an absolutely convergent sequence is convergent. So, we consider a sequence $\{f_j\}$ in L^p for which $\sum \|f_j\|_p = S < \infty$. Then we want to show $\sum f_j$ converges in L^p to some function f.

By the triangle inequality,

$$\left\| \sum_{j=1}^n |f_j| \right\|_p \leqslant \sum_{j=1}^n \|f_j\|_p \leqslant S < \infty \,.$$

Then, by Monotone Convergence Theorem (Theorem 19),

$$\int \left(\sum_{j=1}^{\infty} |f_j| \right)^p \mathrm{d}\mu = \lim_{n\to\infty} \int \left(\sum_{j=1}^{n} |f_j| \right)^p \mathrm{d}\mu \ \leqslant S^p < \infty \,.$$

It follows that $g = \sum_{j=1}^{\infty} |f_j| \in L^p$, and in particular that g is finite a.e.. Then, since any absolutely convergent series of real numbers is convergent, for a.e. x we can define pointwise

$$f(x) = \sum_{j=1}^{\infty} f_j(x) \,.$$

It is clear that $|f| \leqslant |g|$ pointwise, and so $f \in L^p$. We now show that the series $\sum f_j$ (i.e. the sequence of partial sums) converges to f with respect to the L^p norm. For this, note that

$$\left| f - \sum_{j=1}^{n} f_j \right|^p \leqslant \left(|f| + \sum_{j=1}^{n} |f_j| \right)^p \leqslant 2^p g^p \,.$$

So, by the Dominated Convergence Theorem (Theorem 22),

$$\left\| f - \sum_{j=1}^{n} f_j \right\|_p^p = \int \left| f - \sum_{j=1}^{n} f_j \right|^p \mathrm{d}\mu \ \longrightarrow \ 0 \,.$$

REMARKS:

- It follows that a Cauchy sequence $\{f_j\}$ of functions in L^p converges to some f in L^p.[3]
- If $f_j \to f$ in the L^p norm then a subsequence $\{f_{j'}\}$ will converge pointwise a.e. to f. However, the original sequence $\{f_j\}$ needn't converge pointwise.
- The Riesz-Fischer Theorem suggests an analogy with the completion of the set $\mathbb{Q}$ of rationals to create the set $\mathbb{R}$ of real numbers: given a Borel measure μ on a topological space X, to what extent is the space $L^p(\mu)$ the completion of the space $C(X)$ of continuous functions on X? This is not always the case, even when X is compact and $\mu(X) < \infty$ (implying continuous functions on X are bounded and μ-integrable). However, it is suitably true for "nice" Borel measures. We shall consider this in the next Handout.

[3] For a direct proof that Cauchy sequences in L^p converge, see §8.4 of *An Introduction to Measure and Integration* by I. Rana (2nd ed., AMS, 2002).

SOLUTIONS

37

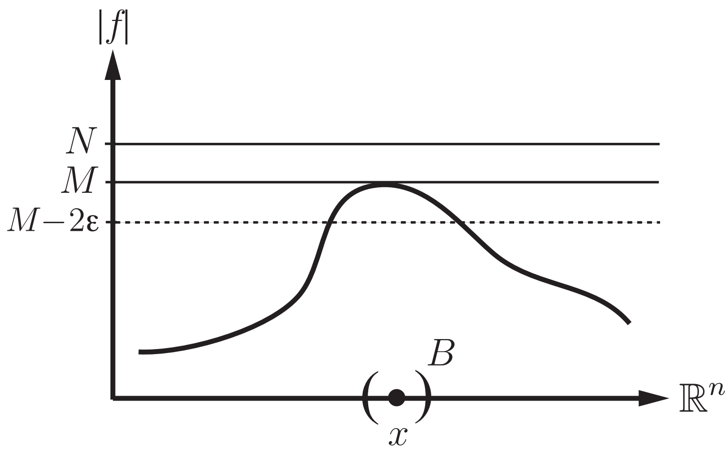
(a) We want to show that if $f : \mathbb{R}^n \to \mathbb{R}$ is continuous then

$$\sup |f| = \operatorname{ess\,sup} |f| \, .$$

with respect to Lebesgue measure. Let $M = \sup |f|$ and $N = \operatorname{ess\,sup} |f|$. Then

$$|f|^{-1}\left((M, \infty]\right) = \emptyset \quad \Longrightarrow \quad \mathscr{L}^n\left(f^{-1}\left((M, \infty]\right)\right) = 0 \quad \Longrightarrow \quad N \leqslant M \, .$$

Next, note that for any $\epsilon > 0$ there is an $x \in \mathbb{R}^n$ with $|f(x)| > M - \epsilon$.



But then, by continuity, $f > M - 2\epsilon$ in a whole open ball B around x. Clearly $\mathscr{L}^n(B) > 0$, and thus

$$\mathscr{L}^n\left(f^{-1}\left((M - 2\epsilon, \infty]\right)\right) > 0 \quad \Longrightarrow \quad N \geqslant M - 2\epsilon \, .$$

Thus $M \leqslant N$, by the Thrilling ϵ-Lemma.

(b) This argument also makes clear how $\operatorname{ess\,sup} |f|$ may be strictly less than $\sup |f|$ for a general measure: the measure just needs to be zero where $|f|$ is large. To take the extreme example, if μ is the zero measure then $\operatorname{ess\,sup} |f| = -\infty$ no matter what f is.

38 Given $f \in L^1$ and $g \in L^\infty$ we want to know when

$$(*) \qquad \int |fg| \, \mathrm{d}\mu = \|f\|_1 \cdot \|g\|_\infty .$$

Fixing $\epsilon > 0$, let

$$A_\epsilon = \{x : |f(x)| \geqslant \epsilon\} \cap \{x : |g(x)| \leqslant \|g\|_\infty - \epsilon\} .$$

If $\mu(A_\epsilon) = \delta > 0$ then clearly

$$\int |fg| = \int\limits_{A_\epsilon} |fg| + \int\limits_{X \sim A_\epsilon} |fg| \leqslant -\delta\epsilon^2 + \int\limits_{A_\epsilon} |f| \cdot \|g\|_\infty + \int\limits_{X \sim A_\epsilon} |f| \cdot \|g\|_\infty = \|f\|_1 \cdot \|g\|_\infty .$$

Thus, for equality in $(*)$, we must have $\mu(A_\epsilon) = 0$ for every ϵ. It follows that for equality it is necessary that $|g| = \|g\|_\infty$ a.e. on $\{x : f(x) \neq 0\}$. Clearly, this condition is also sufficient.

40

(a) For any $p < \infty$,

$$\int |f|^p \leqslant \int \|f\|_\infty^p = \mu(X)\|f\|_\infty^p .$$

Taking p'th roots, and noting $\lim\limits_{p\to\infty} (\mu(X))^{\frac{1}{p}} = 1$, we see $\limsup\limits_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty$. For the reverse inequality, suppose M is such that $|f| \geqslant M$ on a measurable $A \subseteq X$ with $\mu(A) > 0$. Then $\int |f|^p \geqslant \mu(A)M^p$. It follows that $\liminf\limits_{p\to\infty} \|f\|_p \geqslant M$. But for any $\epsilon > 0$, we can find such an M with $M \geqslant \|f\|_\infty - \epsilon$. By the Thrilling ϵ-Lemma, $\liminf\limits_{p\to\infty} \|f\|_p \geqslant \|f\|_\infty$, as desired.

(b) Next, consider $f(x) = \frac{1}{x}$ on $[1, \infty)$, with respect to Lebesgue measure. Then $f \in L^2$ but $f \notin L^1$. As well $\|f\|_\infty = 1$, but $\lim\limits_{p\to\infty} \|f\|_p = 0$.

41 With μ a measure on X, we want to show that $L^\infty(X)$ is complete, Let $\{f_j\}$ be a Cauchy sequence in L^∞. That is, for any $\epsilon > 0$ there is an N such

$$(*) \qquad j, k \geqslant N \quad \Longrightarrow \quad \|f_j - f_k\|_\infty < \epsilon \quad \Longrightarrow \quad |f_j(x) - f_k(x)| < \epsilon \text{ a.e. } x \in X.$$

We first show that $\{f_j(x)\}$ is a Cauchy sequence for a.e. $x \in X$. To do this, for $n \in \mathbb{N}$ let N_n be given by $(*)$ with $\epsilon = \frac{1}{n}$:

$$(n) \qquad j, k \geqslant N_n \quad \Longrightarrow \quad |f_j(x) - f_k(x)| < \frac{1}{n} \text{ a.e. } x \in X.$$

For $j, k \geqslant N$, we then let A_{njk} be the null set of x for which (n) fails. Then

$$A = \bigcup_{\substack{n \in \mathbb{N} \\ j,k \geqslant n}} A_{njk}$$

is a null set, and $\{f_j(x)\}$ is a Cauchy sequence of real numbers for any $x \notin A$.

Thus, for fixed $x \notin A$, $\{f_j(x)\}$ converges to some $y \in \mathbb{R}$, and we define

$$f(x) = y = \lim_{j \to \infty} f_j(x)\,.$$

To show $f_j \to f$ in L^∞, we can let $k \to \infty$ in (n). This gives

$$(n) \qquad j \geqslant N_n \quad \Longrightarrow \quad |f_j(x) - f(x)| \leqslant \frac{1}{n} \text{ a.e. } x \in X.$$

Thus

$$\|f\|_\infty \leqslant \|f - f_1\|_\infty + \|f_1\|_\infty \leqslant 1 + \|f_1\|_\infty < \infty\,.$$

So $f \in L^\infty$, and we similarly have

$$\|f_j - f\|_\infty \to 0\,.$$