AMSI 2013: MEASURE THEORY
Handout 4
Measurable and Borel Functions

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Measurable functions are the generalization of measurable sets: those functions which are well behaved with respect to a measure μ; in particular, a characteristic function χ_A will be measurable iff A is measurable. Subject to some concerns with ∞, measurable functions are those which we will be able to integrate.

The definition of a measurable function can seem somewhat unmotivated, but it can be seen as a generalization of the notion of a continuous function. To that end, we recall the following equivalent characterizations of continuity (adjusted to $\mathbb{R}^*$):

20 **PROPOSITION 13**: Suppose that (X, d) is a metric space and $f : X \to \mathbb{R}^*$. Then the following are equivalent:

(a) For any $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that, if $d(x, y) < \delta$ then $d^*(f(x), f(y)) < \epsilon$.

(b) Whenever $x_j \to x$ in X, then $f(x_j) \to f(x)$ in $\mathbb{R}^*$.

(c) Whenever $U \subseteq \mathbb{R}^*$ is open, then $f^{-1}(U)$ is open in X.

(d) For any $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ and $f^{-1}((a, \infty])$ are open in X.

We make a few simple remarks:

- It is more common to consider proposition 13 for real-valued functions, but the arguments are the same for functions mapping into $\mathbb{R}^*$, or indeed into any metric space. If $f : X \to \mathbb{R}$ is in fact real-valued, then we can also regard f as a function into $\mathbb{R}^*$; it is then easy to check that the continuity of a real-valued function is not affected by the choice of range.
- (a), (b) and (c) are standard equivalences. The point of (d) is that we need only check the inverse images of certain open sets to ensure that f is continuous. Also, (d) more directly motivates the standard definition of measurable functions below.
- If X is topological space then (c) becomes the definition of continuity. In the topological setting, (a) and (b) do not really apply,[1] but the equivalence of (c) and (d) still holds.

With proposition 13 as motivation, we now have

Definition: Suppose μ is measure on X and $f : X \to \mathbb{R}^*$. Then f is *μ-measurable* (or *measurable* if the context is clear) if the set $f^{-1}([-\infty, a))$ is μ-measurable for all $a \in \mathbb{R}$.

Though motivated by the concept of a continuous function, note that the definition does not require X to be a topological space. We also make the following simple observations.

21 REMARKS

(a) It is not part of the definition that $f^{-1}((a, \infty])$ also be measurable, but this will follow as a direct consequence of the measurability of $f^{-1}([-\infty, a))$ for all a. See lemma 14.

(b) $A \subseteq X$ is μ-measurable iff χ_A is μ-measurable.

(c) Suppose that X is a topological space and that μ is a Borel measure on X. Then any continuous $f : X \to \mathbb{R}^*$ is measurable.

(d) Suppose that $f : X \to \mathbb{R}^*$ and $g : X \to \mathbb{R}^*$, and let $N = \{x \in X : f(x) \neq g(x)\}$. If $\mu(N) = 0$ then

$$f \text{ measurable} \iff g \text{ measurable}.$$

In the context of (d), we say that f equals g *almost everywhere*, or *a.e.* for short. This then defines an equivalence relation on functions:

$$f \equiv g \quad \iff \quad f = g \text{ almost everywhere.}$$

Similarly, we talk about a function f being defined almost everywhere. In such a case the notion of f being measurable still makes sense, since the particular method of extending the domain of f to all of X will not affect its measurability.

[1] There is in fact a notion of convergent sequences for general topological spaces. However, this notion does not in general capture the idea of continuity in the desired sense.

Similar to measurability, if X is a topological space we have the notion of $f : X \to \mathbb{R}^*$ being Borel. To be precise, f is a *Borel function* if $f^{-1}([-\infty, a))$ is a Borel subset of X for all $a \in \mathbb{R}$.[2] Note that, whereas a function f need only be defined a.e. to be measurable, for f to be Borel it must be defined everywhere. We then have

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- $A \subseteq X$ is a Borel set iff χ_A is a Borel function.
- If $f : X \to \mathbb{R}^*$ is continuous then f is Borel.
- If $f : X \to \mathbb{R}^*$ is Borel and μ is a Borel measure on X, then f is μ-measurable.
- If $f : \mathbb{R}^* \to \mathbb{R}^*$ is monotonic then f is Borel.
- An important example of a discontinuous Borel function $f : \mathbb{R}^* \to \mathbb{R}^*$ is $f(x) = \frac{1}{x}$. Here, of course we define $f(\pm\infty) = 0$; there is no natural definition for $f(0)$, but however it is defined, the resulting function is Borel.

As the above remarks indicate, there are many similarities between the collections of measurable and Borel functions. This arises from the fact that both $\mathcal{B}$ and $\mathcal{M}_\mu$ are σ-algebras, the key to proving many of the interesting properties. It is illustrated by the double-barrelled nature of the next two results.

23 **LEMMA 14**: Suppose X is a topological space (or a set with a measure μ). Let $f : X \to \mathbb{R}^*$. Then the following are equivalent conditions for f to be Borel (measurable).

(a) $f^{-1}([-\infty, a))$ is Borel (measurable) for all $a \in \mathbb{R}$.

(b) $f^{-1}([-\infty, a])$ is Borel (measurable) for all $a \in \mathbb{R}$.

(c) $f^{-1}(U)$ is Borel (measurable) for all open $U \subseteq \mathbb{R}^*$.

(d) $f^{-1}(B)$ is Borel (measurable) for all Borel $B \subseteq \mathbb{R}^*$.

The point of this lemma is that, when proving a given function f is measurable, we have a choice of the intervals upon which to focus (and there are other choices that we haven't indicated). Then, once we know that f is measurable, we are free to use the fact that many inverse images are measurable.

[2] Many texts will refer to such a function as being Borel measurable. However, since the notion of being Borel is purely topological, it is confusing terminology, and we shall avoid it.

PROPOSITION 15: Suppose X is a topological space (or a set with a measure μ).

(a) If $f\colon X \to \mathbb{R}^*$ is Borel (measurable) and $\phi\colon \mathbb{R}^* \to \mathbb{R}^*$ is Borel, then $\phi \circ f\colon X \to \mathbb{R}^*$ is Borel (measurable).

(b) If $f : X \to \mathbb{R}^*$ is Borel (measurable) and $c \in \mathbb{R}^*$, then $cf, f^2, \frac{1}{f}$ and $|f|$ are Borel (measurable), as long as the new functions are well-defined (well-defined a.e.).

(c) If $f, g\colon X \to \mathbb{R}^*$ are Borel (measurable) then so are $f+g, f \cdot g, \dfrac{f}{g}, \max(f,g)$ and $\min(f,g)$, as long as the new functions are well-defined (well-defined a.e.).

(d) Suppose $\{f_j\colon X \to \mathbb{R}^*\}_{j=1}^{\infty}$ is a sequence of Borel (measurable) functions. Then

- **(i)** $\sup_j f_j$ and $\inf_j f_j$ are Borel (measurable).
- **(ii)** If $f_j \to f$ everywhere (almost everywhere) then f is Borel (measurable).
- **(iii)** $\sum_j f_j$ is Borel (measurable) if it exists everywhere (almost everywhere).
- **(iv)** $\limsup_{j\to\infty} f_j$ and $\liminf_{j\to\infty} f_j$ are Borel (measurable).

REMARK: We'll prove this proposition in a moment, but we first want to note an asymmetry in lemma 14(d) and proposition 15(a): even in the case where X has a measure, we are considering only B Borel and ϕ Borel. The point is, though X may either be topological or have a measure μ, on the range $\mathbb{R}^*$ we're always employing the topology: for general X, there is no natural measure on $\mathbb{R}^*$ to consider in lemma 14 or proposition 15. *However*, if $X = \mathbb{R}^*$ with Lebesgue measure then the questions become natural. Specifically:

- If $f : \mathbb{R}^* \to \mathbb{R}^*$ is Lebesgue measurable (or even continuous) and $M \subseteq \mathbb{R}^*$ is Lebesgue measurable, does it follow that $f^{-1}(M)$ is Lebesgue measurable?
- If $f\colon \mathbb{R}^* \to \mathbb{R}^*$ is Lebesgue measurable (or even continuous) and $\phi\colon \mathbb{R}^* \to \mathbb{R}^*$ is Lebesgue measurable does it follow that $\phi \circ f$ is Lebesgue measurable?

The general answer in both cases is "No". The easiest way to obtain counterexamples is by employing what is known as the *Cantor ternary function.*[3] This function allows us to define a homeomorphism $f : [0,2] \to [0,1]$ such that $f(D) = C$, where D is a compact set with $\mathscr{L}(D) = 1$, and C is the Cantor set. Now, by ⑱, we know that D contains a non-measurable set E; but setting $F = f(E)$, we then know that $F \subseteq C$ is measurable (since C is null), with $f^{-1}(F) = E$ nonmeasurable. Also, if we set $\phi = \chi_F$, then ϕ is measurable; however, $(\phi \circ f)^{-1}((\frac{1}{2}, \infty]) = E$ is nonmeasurable, implying $f \circ \phi$ is nonmeasurable.

[3]For more details see, for example, §19 of *Measure Theory* by Paul Halmos (Springer, 1974).

PROOF OF PROPOSITION 15: We'll phrase the arguments in terms of measurability, ignoring null sets along the way. (For example, for (d) note that if each f_j is defined except on a null set N_j, then the appropriate limits can be defined except on the null set $\bigcup N_j$). The arguments in the case that the functions are Borel are identical in form.

To prove (a), we note that for $a \in \mathbb{R}$,

$$(\phi \circ f)^{-1}([-\infty, a)) = \{x : \phi(f(x)) < \alpha\} = \{x : f(x) \in \phi^{-1}([-\infty, a))\} = f^{-1}\left(\phi^{-1}([-\infty, a))\right) .$$

Now, ϕ being Borel implies that $\phi^{-1}([-\infty, a))$ is Borel. Then, since f is measurable, it follows from Lemma 14(d) that $f^{-1}(\phi^{-1}([-\infty, a)))$ is measurable. This is exactly what we needed to show.

To prove (b), we now just apply (a) with the obvious choices for the Borel function g.

To prove (c), we first consider $f + g$. Fixing $a \in \mathbb{R}$, we calculate

$$\begin{aligned} (f+g)^{-1}([-\infty, a)) &= \{x : f(x) < a - g(x)\} \\ &= \{x : f(x) < q < a - g(x) \text{ for some } q \in \mathbb{Q}\} \\ &= \bigcup_{q \in \mathbb{Q}} \left(f^{-1}([-\infty, q)) \cap g^{-1}([-\infty, a - q))\right) \end{aligned}$$

Since f and g are measurable, and since $\mathcal{M}_\mu$ is a σ-algebra, it follows that $(f+g)^{-1}([-\infty, a))$ is measurable.

To prove fg is measurable, we first note that if f and g are real-valued then we can write $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$. The measurability of fg then follows. For general f and g, we need to fiddle: we fix $a \in \mathbb{R}$ and we define

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } f(x) \text{ and } g(x) \text{ are finite} \\ a+1 & \text{otherwise} \end{cases} \qquad \hat{g}(x) = \begin{cases} g(x) & \text{if } f(x) \text{ and } g(x) \text{ are finite} \\ 1 & \text{otherwise} \end{cases}$$

It is easy to show that the real-valued functions $\hat{f}$ and $\hat{g}$ are measurable, and so $\hat{f}\hat{g}$ is measurable. We then note that $fg < a$ exactly when either $\hat{f}\hat{g} < a$ or $fg = -\infty$, from which the measurability of $(fg)^{-1}([-\infty, a))$ follows easily.

Next, $\frac{f}{g} = f \cdot \frac{1}{g}$ is measurable. Finally for (c), the measurability of $\max(f, g)$ and $\min(f, g)$ will follow as special cases of (d)(i).

To prove (d)(i), let $u = \sup_j f_j$. Then

$$u^{-1}([-\infty, a]) = \{x : f_j(x) \leqslant a \text{ for all } j \in \mathbb{N}\} = \bigcap_{j=1}^{\infty} f_j^{-1}([-\infty, a]) .$$

By Lemma 14(b), u is measurable. The measurability of $l = \inf_j f_j = -\sup_j(-f_j)$ then follows.

d(ii) will follow immediately from d(iv), and d(iii) will follow immediately from d(ii). Finally, d(iv) follows easily from d(i) and the nature of lim sups and lim infs: applying the Thrilling Inf-Sup Lemma,

$$\limsup_{j\to\infty} f_j = \lim_{j\to\infty} \sup_{n\geqslant j} f_n = \inf_{j\in\mathbb{N}} \sup_{n\geqslant j} f_n\,,$$

and similarly for the lim inf.

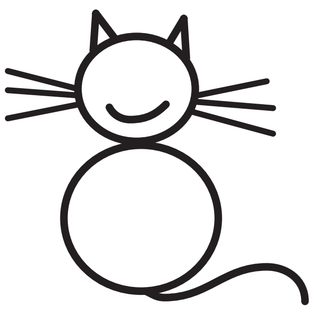
As a simple application of proposition 15, the partial derivatives of a Borel (measurable) $f: \mathbb{R}^n \to \mathbb{R}$ will be Borel (measurable) as long as the derivatives are defined (almost everywhere). Of course, the limit here is defined in terms of a real variable $h \to 0$, but it is obviously sufficient to consider $h = \frac{1}{n}$ for $n \in \mathbb{N}$.

Finally, we want to consider *semicontinuous* functions on a metric space (X, d). To this end, for $x \in X$ and $r > 0$, let $B_r(x)$ be the open ball of radius r around x. Then, for $f: X \to \mathbb{R}^*$ we define the *upper envelope* and *lower envelope* of f:

$$\begin{cases} \overline{f}(x) = \lim\limits_{r\to 0^+} \sup\limits_{y\in B_r(x)} f(y) \\ \underline{f}(x) = \lim\limits_{r\to 0^+} \inf\limits_{y\in B_r(x)} f(y) \end{cases}$$

We say f is *upper semicontinuous* if $f = \overline{f}$, and f is *lower semicontinuous if* $f = \underline{f}$. The intuition is that an upper semicontinuous function can jump up in the limit, but not down, and vice versa for a lower semicontinuous function. With this in mind it is not hard to show that a function f is continuous iff f is both lower semicontinuous and upper semicontinuous. Our particular interest in semicontinuous functions is the following:

24 If X is a metric space and $f: X \to \mathbb{R}^*$, then $\overline{f}$ and $\underline{f}$ are Borel. Thus, upper semicontinuous and lower semicontinuous functions are Borel.



SOLUTIONS

20 (X, d) is a metric space, and $f : X \to \mathbb{R}^*$. We want to show the following are equivalent:

$$\begin{cases} \text{(a)} & \epsilon - \delta \text{ definition of continuity;} \\ \text{(b)} & \text{sequence definition of continuity;} \\ \text{(c)} & f^{-1}(U) \text{ is open for every open } U \subseteq \mathbb{R}^*; \\ \text{(d)} & f^{-1}([-\infty, a)) \text{ and } f^{-1}((a, \infty]) \text{ are open for every } a \in \mathbb{R} \end{cases}$$

$\sim (b) \Longrightarrow \sim (a)$:

Suppose we have a sequence $\{x_n\}$ with $x_n \to x$ but $f(x_n) \not\to f(x)$. Then there is an $\epsilon > 0$ for which $d^*(f(x_n), f(x)) \geqslant \epsilon$ for infinitely many n. This shows that, for that ϵ, there cannot be a $\delta > 0$ such that $d(y, x) < \epsilon \Rightarrow d^*(f(y), f(x))$: no matter how small δ is, there will be one of the bad x_n with $d(x_n, x) < \delta$.

$\sim (c) \Longrightarrow \sim (b)$:

Suppose there is an open U such that $f^{-1}(U)$ is not open. Then some $x \in f^{-1}(U)$ is *not* contained in an open ball $B \subseteq f^{-1}(U)$. So, there is a sequence $\{x_n\} \subseteq \sim f^{-1}(U)$ with $x_n \to x$. But then $\{f(x_n)\} \subseteq \sim U$: since $f(x) \in U$ is open, there is an open ball $B_\epsilon(f(x)) \subseteq U$, and thus it is impossible for $f(x_n) \to f(x)$.

$(c) \Longrightarrow (a)$:

Fix $x \in X$ and $\epsilon > 0$. Then $U = \{y : d^*(y, f(x)) < \epsilon\}$ is open. So, by hypothesis, $f^{-1}(U)$ is open. Obviously $x \in f^{-1}(U)$, and so there is a $\delta > 0$ such that $B_\delta(x) \subseteq f^{-1}(U)$. That is,

$$d(x, z) < \delta \Longrightarrow z \in B_\delta(x) \Longrightarrow z \in f^{-1}(U) \Longrightarrow f(z) \in U \Longrightarrow d^*(f(z), f(x)) < \epsilon\,.$$

$(c) \Longrightarrow (d)$: Trivial.

$(d) \Longrightarrow (c)$:

For any $a, b \in \mathbb{R}$, $f^{-1}((a, b)) = f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty])$ is open, by hypothesis. Now any open $U \subseteq \mathbb{R}^*$ can be written $U = \cup_\alpha I_\alpha$ as a union of open intervals in $\mathbb{R}^*$. Then $f^{-1}(U) = f^{-1}(\cup_\alpha I_\alpha) = \cup_\alpha f^{-1}(I_\alpha)$ is open.

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(i) If $f : X \to \mathbb{R}^*$ is measurable, then for any $a \in \mathbb{R}^*$, $f^{-1}([a,\infty]) = \sim f^{-1}([-\infty,a))$ is measurable. But then $f^{-1}((a,\infty]) = f^{-1}(\cup_{n\in\mathbb{N}}[a+\frac{1}{n},\infty]) = \cup_{n\in\mathbb{N}} f^{-1}([a+\frac{1}{n},\infty])$ is also measurable.

(ii) If $A \subseteq X$ then

$$\chi_A^{-1}([-\infty,a)) = \begin{cases} \emptyset & a \leqslant 0\,, \\ \sim A & 0 < a \leqslant 1\,, \\ \mathbb{R} & a \geqslant 1\,. \end{cases}$$

It follows immediately that χ_A is measurable iff A is measurable.

(iii) X is a topological space, μ is a Borel measure, and $f : X \to \mathbb{R}^*$ is continuous. Then $f^{-1}([-\infty,a))$ is open, and thus Borel, and thus μ-measurable, for every $a \in \mathbb{R}$. Thus f is measurable.

(iv) For $f, g : X \to \mathbb{R}^*$, let

$$N = \{x \in X : f(x) \neq g(x)\}\,.$$

Then, for any $a \in \mathbb{R}$

$$f^{-1}([-\infty,a)) = g^{-1}([-\infty,a)) \cup \{x : f(x) < a \text{ and } g(x) \geqslant a\} \sim \{x : f(x) \geqslant a \text{ and } g(x) < a\}\,.$$

The last two sets are subsets of N, and thus if N is null then these sets are also null and thus measurable. Thus if $g^{-1}([-\infty,a))$ is measurable then so is $f^{-1}([-\infty,a))$. By symmetry, the converse conclusion also holds.