Function

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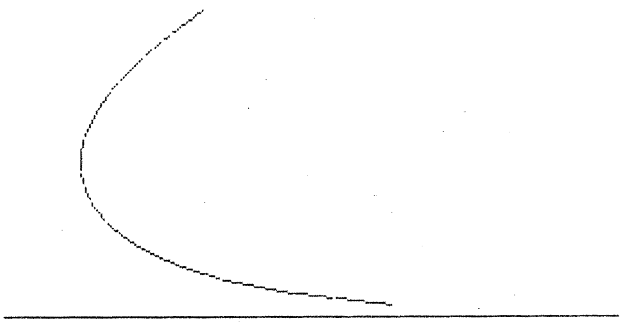

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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* $17 for *bona fide* secondary or tertiary students.

THE FRONT COVER

Our Front Cover for this issue illustrates the feature article "The Bull and the Botanist" (p 70).

The case illustrated is that in which the bull and the botanist run with equal speeds, which is the critical case, although it is shown in the course of the article that the bull never catches the botanist in this case of equal speeds, unless the initial conditions are highly unusual.

In the article, a somewhat unfamiliar set of co-ordinates is used. Here on the cover, the usual conventions are employed instead.

The equation of the curve is

$$x = cy^2 - \ln y,$$

but it should be noted that the conventions under which this simple form is achieved are not those of the article. Here $x = c$ when $y = 1$. Another point of interest is the turning point where the bull reverses its direction and runs from left to right instead of right to left. At this point $\frac{dx}{dy} = 0$, so that

$$y = \sqrt{\frac{1}{2c}}$$

which gives a value

$$x = \frac{1 + \ln(2c)}{2}$$

Readers should contrast this with the different convention adopted in the article.

For a brief account of this curve, but under somewhat different conventions again, see the website

http://www-history.mcs.st-and.ac.uk/history/Curves/Pursuit.html

THE BULL AND THE BOTANIST

Michael A B Deakin, Monash University

I first learned of this problem from one of my Mathematics teachers when I was in High School. Later I discovered that he in his turn had encountered it when it was set as a problem in his Mathematics course at the University of Melbourne. I don't now recall the precise wording, but the gist of it has stayed with me, and I should like to share it with *Function's* readers.

> In a paddock, collecting specimens, is a botanist. Also in the paddock is a bull, who, on seeing the botanist, charges at him. Just when the animal begins its charge, the botanist begins his flight. He heads for the safety of the nearest fence, and runs toward it with a constant speed u. The bull charges in such a way that it is always headed straight for the botanist, proceeding at a constant speed v. What happens?

Here is the situation after a time t has elapsed.

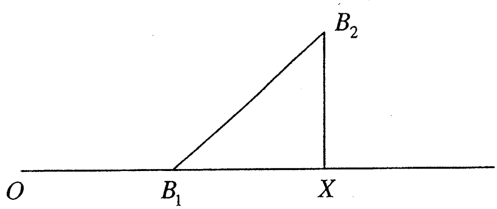

Figure 1

Initially the botanist is at O, which can be taken for the moment as the origin. At time t, he has reached the point B_1, to which we may assign the co-ordinates $(ut, 0)$. The understanding is that the botanist is moving to the right, in the direction OX. The bull is at B_2, whose co-ordinates are (x, y).

The point X is that for which the angle OXB_2 is a right angle. The bull is moving along a path whose tangent B_2B_1 always passes through the botanist. As I have drawn the diagram, $x > ut$, so that the bull is ahead of the botanist as he runs toward the fence (off to the right and out of frame). However, the analysis is general and will equally well apply if the bull is initially behind the botanist.

I will analyse the situation from the point of view of the botanist, who will be concerned with the distance between himself and the bull. This distance $|B_1B_2|$ I will call r, and the angle B_2B_1X I will call θ. (r and θ turn out to provide a more convenient description than do x and y. However, it is possible to use these instead. George Boole, a famous mathematician of the 19th century, included such a discussion in his *Treatise on Differential Equations* on pages 252, 253. See also the cover diagram.) As I have drawn things, the bull is to the botanist's left, and this may be assumed without any loss of generality (apart from the special case, to be discussed briefly later, in which the two characters in our drama are both on the line OX). We may thus restrict discussion to the case $0 \le \theta \le \pi$.

The botanist's velocity along B_1X may be resolved into two components, one along B_2B_1, the other perpendicular to it. Thus the distance between the bull and the botanist is reduced by two effects: the first of these two components and the bull's direct velocity toward the botanist. This provides the first of the two equations below. The second refers to the need the bull has to continuously alter its direction. We thus have:

$$\frac{dr}{dt} = -u\cos\theta - v \qquad (1)$$

$$r\frac{d\theta}{dt} = u\sin\theta\,. \qquad (2)$$

From Equation (2), we see that if initially θ is either 0 or π, then it does not change, but remains at that same value. I leave it to the reader to analyse what happens in these two cases. I will in the rest of this article assume that the situation is otherwise. In all these other cases, θ increases steadily as the action proceeds because the right-hand side of Equation (2) is positive.

If we now divide Equation (1) by Equation (2), we reach the single equation

$$\frac{dr}{d\theta} = \frac{-r(k+\cos\theta)}{\sin\theta} \tag{3}$$

where $k = v/u$, the ratio of the speeds.

This equation may now be solved. The solution is

$$r = \frac{A}{\sin\theta}\left(\frac{1+\cos\theta}{\sin\theta}\right)^k \tag{4}$$

where A is a constant whose value is to be determined from the initial situation. (The reader may check this result by differentiation.)

Now the main point of interest is whether the bull catches the botanist or not. If they do meet up, then r will be zero. The only possibility of this occurring is that $\cos\theta = -1$, i.e. $\theta = \pi$. This means that if the bull does catch the botanist, it will always be from behind. (Remember that we are not discussing the special case in which the bull and the botanist rush headlong toward one another!)

There is a ready corollary of this deduction. From Equation (1), we see that $\frac{dr}{dt} = 0$ when $\cos\theta = -k$. This equation can be satisfied if $k < 1$, that is to say if the botanist can run faster than the bull. But now when $\cos\theta = -k$, the distance between the two is minimised, so that, as θ continues to increase beyond the value achieved at that point, r then increases because the right-hand side of Equation (1) is then positive. It follows that in this case the bull will never catch the botanist, but will always fall behind after a point of closest approach.

(Remember that we are thinking of the bull and the botanist both as moving points, with no size. In real life, things could be rather different!)

This now leaves us with two cases to consider: $k > 1$ and $k = 1$. It will turn out that these cases have to be treated separately. But before we start on this task, it will be best to do a little tidying up. If $x > ut$, as in Figure 1, then the bull must at some stage turn round and proceed toward the right instead of to the left as shown (because as θ increases towards π it will at some stage equal $\pi/2$, so setting up the approach from behind). At this time, the situation is as shown in Figure 2.

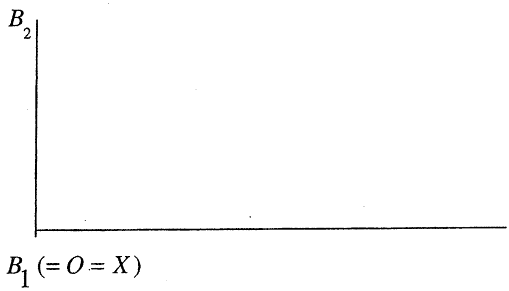

Figure 2

This is a convenient point at which to fix our origin and start our clock. The situation of Figure 1 is thus altered a little, but not in any important sense. Nor are things very different if $x < ut$ initially, as in this case, we may imagine the film being run in reverse, so to speak, again to reach a notional starting point as diagrammed in Figure 2. So our convention shall be that when $t = 0$, $\theta = \pi/2$. The value of r at this time will be called a $(= OB_1)$. If we feed these values into Equation (4), we readily see that $A = a$.

Now, with this out of the way, let us get down to the task of determining the botanist's fate. The case we need most thoroughly to consider is that in which $k > 1$, with the bull being faster than the botanist. Equation (4) tells us the shape of the bull's path, but says nothing about time, and time is what is clearly of the essence here. To introduce time into the discussion, we need to get back to Equations (1) and (2). The overall plan of attack will be to combine Equation (4) with Equation (2), and so to connect θ with t. This indirect attack turns out to work better than a more direct approach via Equation (1).

Even so, things are not easy, and long experience has taught mathematicians to tackle this problem by means of a somewhat elaborate subterfuge. Instead of using θ as the measure of the angle between the bull's path and that of the botanist, we use τ, defined as $\tan(\theta/2)$, which gives us new versions of Equations (2) and (4). These new versions are much simpler than the originals.

We have (after some tricky algebra and trigonometry):

$$r\frac{d\tau}{dt} = u\tau \qquad (5)$$

$$r = A(1+\tau^2)/2\tau^{k+1}. \qquad (6)$$

Equation (5) is the new version of Equation (2), and Equation (6) expresses Equation (4) in the new notation.

If we now substitute from Equation (6) into Equation (5), we find, after some simplification,

$$\frac{A}{2}\left(\frac{1+\tau^2}{\tau^{k+2}}\right)\frac{d\tau}{dt} = u\,. \qquad (7)$$

This equation may be integrated to give

$$\frac{\tau^{1-k}}{1-k} - \frac{\tau^{-1-k}}{1+k} = \frac{2ut}{A} + B. \qquad (8)$$

(Again, check this by differentiation.)

We may now use our data on the initial moment, as defined from Figure 2. At that time, $t = 0$, and $\theta = \pi/2$. This means that $\tau = 1$, so that Equation (8) tells us that

$$B = \frac{1}{1-k} - \frac{1}{1+k} = \frac{2k}{1-k^2}. \qquad (9)$$

We now know enough to determine the fate of the botanist. For the bull to catch him the situation must arise that $\theta = \pi$. This means that $\tau = \infty$, so that $\tau^{1-k} = \tau^{-1-k} = 0$, since $k > 1$. Putting these values into Equations (8) and (9) tells us that at this time

$$t = \frac{ak}{u(k^2 - 1)} \qquad (10)$$

because we found previously that $A = a$.

Thus the botanist escapes if the fence is close enough for him to reach in this time. Its distance must be less than $ak/\left(k^2 - 1\right)$. Otherwise, it is curtains!

We may also tie up another loose end at this point. Earlier it was found that *if* the bull was to catch the botanist, then we needed to satisfy the condition $\theta = \pi$. This is not quite the same as saying that if this condition *is* satisfied, then the bull *will* catch up with the botanist. However Equation (6) makes it clear that if $\theta = \pi$ (and consequently $\tau = \infty$), then r is indeed zero.

Equation (8) demonstrates that we need a separate analysis in the special case $k = 1$. However, Equation (10) shows that the time taken for the bull to reach the botanist tends to infinity as k tends to a limit of 1. This provides strong evidence that the bull will never catch the botanist in this special case.

There is a very simple argument that proves this statement. Look again at Figure 2, and suppose that at some point P on the horizontal axis, the bull and the botanist meet. The botanist will have traveled a distance $|B_1P|$ while the bull will have traveled the clearly longer distance $|B_2P|$; as both travel at the same speed, this is clearly impossible! Thus the pair never meet.

Closer analysis of this case replaces Equation (7) by the simpler equation

$$\frac{A}{2}\left(\frac{1+\tau^2}{\tau^3}\right)\frac{d\tau}{dt}=u. \tag{11}$$

This becomes on integration

$$\frac{-1}{4\tau^2}+\frac{\ln\tau}{2}=\frac{2ut}{A}+B, \tag{12}$$

and this time $B = ¼$.

Equation (6) simplifies to

$$r=\frac{a}{2}\left(1+\frac{1}{\tau^2}\right) \tag{13}$$

from which it follows that as time goes on and $\tau\to\infty$, $r\to a/2$. The bull in the long run thus remains at the constant distance a/2 behind the botanist.

So there we have our complete description of what happens. Problems of this type are quite widespread and there are many variations. The curves generated are known as "pursuit curves". A different pursuit curve turned up on *Function's* cover for October 1998.

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NEWS ITEMS

Vale Bernhard Neumann

The death on October 21st of last year of Bernhard Hermann Neumann was a sad loss for Australian Mathematics. Born on October 15th 1909, he studied at first in his native Germany, earning a doctoral degree in 1931. In 1933, being Jewish, he fled Hitler's Reich for England, where he earned a further doctorate in 1935. Following several academic posts in the UK, he emigrated to Australia in 1959, where he became the first professor and head of the Research School of Mathematical Sciences at ANU.

His first wife, Hanna, was also a mathematician and a Professor of Mathematics at ANU up till her death in 1971. An account of her life and work appeared in *Function* in February 1979. His second wife, Dorothea, was not a mathematician, but shared many of his other interests, notably music. Two of his children have gone on to become professional mathematicians and another is a musician.

His research interests lay primarily in the area of Group Theory, especially the theory of infinite groups, but he took a keen interest in the whole of Mathematics. Mike Newman, who wrote his obituary for the Australian Mathematical Society's *Gazette*, commented:

> "Bernhard arrived in Australia with an outstanding reputation for his seminal work on infinite groups and more broadly in algebra. He also published in geometry. More importantly he was a strong supporter of all endeavours in mathematics – he supported people who did mathematics for its own sake, people who applied mathematics and people who taught mathematics. To him it was important to share and spread the joy of doing mathematics."

He was a familiar figure at Mathematics conferences right till the time of his death. He was active in the Mathematics Olympiad movement and many other such activities. His name is attached to the award for the best student presentation at the Australian Mathematical Society's annual conference and (via the Australian Mathematics Trust) to the B H Neumann Awards for Mathematical Enrichment. He was an avid reader of *Function*.

Active to the end, he was still working in the cause of Mathematics at the time of his death. His was a full life and we are the poorer for his leaving us.

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The "Collins Case" Revisited

Lawyers know of many different cases that all go by the name of the "Collins Case". This becomes very clear if these key words are put into a search-engine like Google. However, statisticians pay particular attention to one of these: a 1964 California case, in which a conviction and a subsequent successful appeal both relied on expert statistical evidence. Because of this, there has been a deal of discussion of the case in the statistical literature. *Function* published an account in June 1994. Most recently, an article about it appeared in *The Mathematical Scientist* (November 2002). The author is T Rolf Turner from the University of New Brunswick (Canada). A summary may be found at

http://erdos.math.unb.ca/~rolf/Research/Abstracts/colcase.pdf

The situation was as follows. An old lady was mugged and robbed in an alley in Los Angeles. Although there were no witnesses (other than the victim herself) to the actual crime, a good description was provided to police of a couple observed running away from the scene at the appropriate time. They were a black man with a beard and moustache and a blonde woman with a ponytail. They effected their getaway in a car that was partly yellow in colour. On the basis of this description, the police arrested Janet and Michael Collins, a married couple who fitted this description exactly.

At their trial, a statistician calculated the probabilities as:

A woman's being blonde:	1/3
A man's having a moustache:	1/4
A man's being black and bearded:	1/10
A car's being partly yellow:	1/10
A woman's having a ponytail:	1/10
A couple's being interracial:	1/1000.

These figures were then multiplied together to produce a probability of 1/12,000,000 of finding such a couple. Because of this low probability, it was concluded that Janet and Michael must be the guilty pair.

There are some grounds for querying the calculation. In particular, simple multiplication is only justified with independent events, and these different probabilities are clearly *not* independent (bearded men tend to have moustaches; a black man and a blonde woman *necessarily* constitute an interracial couple; etc). However, the figure of 1/12,000,000 has been accepted by most subsequent commentators, including Turner, so as to concentrate attention on more important points of probability theory. (The appeal, however, had also to consider such matters, as well as the reliability of the witnesses and the accuracy of the statistician's numbers.)

The first point to stress is that, however unlikely such a couple might seem to be, there *was* such a couple. So the relevant question is *not* "How likely is it that a random couple might fit this description?" *but rather* "How likely is it that of all the couples fitting this description, this particular couple is the guilty pair?"

Turner likens the case to the situation of a whole lot of coloured beads in a very large urn, or barrel. Nearly all the beads are white but there is a very small probability that a bead may be red. It is established by careful investigation that the barrel *does indeed* contain a red bead. We now fish around inside the barrel and eventually discover a red bead. What is the probability that *this* bead is the same one as that previously shown to exist?

Turner takes the probability that a bead is red to be p = 1/12,000,000. The number of beads he takes to be 2,000,000, a figure advanced by the defence lawyers when the case went to appeal. (They presumably used this as their estimate of the total number of couples in the relevant area at the relevant time.) Call this number N. So Turner's model of the situation takes a very large barrel containing 2,000,000 beads, each with a probability 1/12,000,000 of its being red. We may say immediately that the number of red beads we *expect* to find in the barrel is 2,000,000/12,000,000 or 1/6.

Thus if we were able to conduct an experiment with many such barrels, we would expect to find no red beads most of the time, one red bead about one-sixth of the time, two red beads rather rarely, three red beads even more rarely, and so on.

This situation, in which N is large, p small and their product Np of moderate size, is one that statisticians model using the Poisson Distribution. (See *Function*, October 1984.) Write $Np = m$. Then the Poisson distribution gives the probabilities of the number of red balls in our barrel as

$$\Pr\{r = n\} = \frac{m^n e^{-m}}{n!}$$

where r is the number of red balls in the barrel. These probabilities add up to 1, as they should, and the expected (mean) value of r is m. Here however the case is a little different, because we already know that there is at least one red bead in the barrel. So we need to adjust the probabilities to:

$$\Pr\{r = 0\} = 0 \qquad\qquad \Pr\{r = n\ (> 0)\} = \frac{e^{-m}}{1 - e^{-m}} \frac{m^n}{n!}$$

(The extra term in the denominator is inserted to ensure that the adjusted probabilities add up to 1.)

If now, $r = 1$, there is one red bead in the barrel and this must be the one we already know about. If $r = 2$, there are two red beads in the barrel and the chance that the one we find is the same as that previously identified is $1/2$. We continue in this way to find that if there are n red beads in the barrel, then the probability that the chance that the bead is the one we found before is $1/n$, and the chance that it is not is $(n - 1)/n$.

So the chance that we have a *different bead* is the sum over all these possibilities:

$$\frac{1}{2}\Pr\{r=2\} + \frac{2}{3}\Pr\{r=3\} + \frac{3}{4}\Pr\{r=4\} + \cdots$$

$$= \frac{e^{-m}}{1-e^{-m}}\left\{\frac{m}{2} + \frac{2m^2}{3\times 2!} + \frac{3m^3}{4\times 3!} + \ldots\right\}$$

This sum may be evaluated, by using for example the EXCEL spreadsheet, or even on a reasonably good hand-held calculator. Only a small number of terms are needed (5 is certainly plenty). The answer for the case $m = 1/6$ is 0.0413.

Now to get back to the Collins case. The calculation is the same, and so we have a probability of 0.0413, or just over 4% that Mr and Mrs Collins were not the guilty couple. We also find the probability that there were two or more such couples to be $1 - \frac{1}{6\left(e^{1/6} - 1\right)} = 0.0810$, i.e. 8.1%. Turner's calculation seems more accurate than any attempted before. In particular it seems more convincing than that supplied as a supplement to the appeal decision (available in the text *Statistics for Lawyers* by M O Finkelstein and B Levin, pp 83-88), which calculated a much higher figure in deciding in favour of Janet and Michael. (They calculated the chance of there being two or more such couples to be 41+%.)

Certainly, Turner's result is not a very high probability, but it is high enough to support a successful appeal, even had it been adduced at the time. The case was a criminal one with the standard of proof that guilt must be established "beyond reasonable doubt". 4% is enough to constitute "reasonable doubt". (Civil cases, however, are decided on the balance of probabilities, and so had the case been a civil one, the appeal should not have succeeded.)

The original confusion presumably came about because the jury were led to believe that the probability of the defendants' innocence was 1/12,000,000. This extremely low probability was seen as not constituting "reasonable doubt". However it was not the correct figure!

Progress on the Twin Primes Conjecture

The "twin primes conjecture" concerns pairs of primes that differ by just 2. (3, 5) is the first such pair, (5, 7) is the next and (11, 13) the next, and so on. The conjecture states that there are infinitely many such pairs. It made a brief appearance in *Function's* History column in April 2001, where it was stated that if it is true (as most mathematicians believe), then such pairs are nonetheless quite rare.

Recently there has been a partial breakthrough. Dan Goldston of the American Institute of Mathematics had a sudden idea while in discussion with Roger Heath-Brown (of Oxford University). He and the Turkish mathematician Cem Yildrin then followed this up and produced new results on the distribution of primes. These increase the case for the conjecture's being true, but fall short of resolving the matter. For a press release on this work, see

http://www.aimath.org/release_goldston.html

This website also has links to more technical accounts of the new results.

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More on the Poincaré Conjecture

In our issue for last August, we reported a claimed proof of the so-called Poincaré conjecture, a deep result in 4-dimensional geometry. As we foreshadowed, the claim proved to be incorrect and the conjecture thus remained unproved. Now however comes another claim, and this time it seems to be standing up.

A Russian mathematician, Grigori Perelman of the Russian Academy of Sciences in St. Petersburg, gave a series of public lectures at the Massachusetts Institute of Technology recently. He presented a number of highly technical results, but these seem to imply the truth of a deep theorem known as Thurston's Geometrization Conjecture.

This in its turn contains the Poincaré Conjecture as a special case. If Perelman's claim passes expert scrutiny therefore, the Poincaré Conjecture will be established. As the proof of the Poincaré Conjecture is one of the Clay Challenge Problems (see *Function*, April 2001), there is a prize of $US1,000,000 for its resolution. For more detail, see the website

http://mathworld.wolfram.com/news/2003-04-15/poincare/

HISTORY OF MATHEMATICS

The First Hurdle (continued)

Michael A B Deakin, Monash University

My last article was cut short by space limitations, and so I need to pick up the story where I left off. The point under discussion was the proof of a proposition known as Euclid I.5, popularly termed the *pons asinorum*. Because Euclid's own proof is very clumsy (unnecessarily so), various attempts were made to improve it. Most of these involved bisecting the isosceles triangle under discussion in one way or another. A favourite was to bisect the apical angle of the triangle. This drew the criticism from Charles Lutwidge Dodgson (aka Lewis Carroll) that it had not previously been proved that this manoeuvre was possible.

The criticism has weight. In Hall & Stevens' *A School Geometry*, the pons asinorum is used in the proof of the theorem that two triangles are congruent if the sides of the one are all equal to the corresponding sides of the other (essentially Euclid I.7). This theorem is used in its turn to show how to bisect an angle. We thus have the following chain of deduction:

Angle bisection $\Rightarrow$ *pons asinorum* $\Rightarrow$ Euclid I.7 $\Rightarrow$ Angle bisection.

The entire argument is circular!

Now read on!

It is, of course, possible to avoid the circularity. Euclid himself does, and so do later writers with different approaches. However, if the Pappus proof discussed in the first part of this article is used, then the entire enterprise becomes much simpler (and the circularity is avoided).

Nowadays the Pappus argument is seen as the best, although it requires some mathematical sophistication to appreciate its force. I first encountered it in a popular Mathematics book *Mathematics in Management*, by Albert Battersby. He speculates that it must have been previously discovered by bright young students, who may well have been marked wrong for using it. As I showed in the first part of this article, at least one such student was marked wrong.

However, Battersby was himself in error in not knowing that Pappus had discovered the argument. Rather he stated that the argument had been discovered by a computer. As he wrote in 1966, when the study of artificial intelligence (AI) was still in its infancy, I was intrigued by this claim.

Eventually, I managed to track down the basis on which it came to be made. Several versions are extant, but the most authoritative comes from (of all places) the magazine *New Yorker* (14/12/1981). There is to be found the record of an interview with Marvin Minsky, one of the pioneers of AI. Minsky claims that it was he who produced the proof by hand simulation of what a suitably programmed machine might do. He showed his proof to a colleague, Nathaniel Rochester, who in turn recruited Herbert Gelernter into the enterprise. It was Gelernter who implemented the program in hardware and so caused a machine to "rediscover" the proof.

Gelernter and Rochester wrote about "problem-solving machines" in the *IBM Journal of Research and* Development (1958) and later Gelernter wrote a paper on his "geometry theorem proving machine", which he presented to a UNESCO meeting in Paris in 1959.

But Minsky made a strange assertion that I have also tried to check out. Minsky later learned that his "hand simulation" was not the first discovery of the proof. He came to believe it to be by Frederick the Great of Prussia. The claim is unsupported by any evidence and strikes me as most unlikely. I have found no evidence that Frederick the Great (King Frederick II of Prussia) had any direct involvement with Mathematics at all. He did support the mathematician Leonhard Euler for a time, but this is not the same as doing Mathematics himself.

I rather think that Minsky must have been talking to his colleague of the early days of AI, Douglas Hofstadter, author of *Gödel, Escher, Bach*, which had just appeared. Frederick does appear in that work, but as a patron, not as a mathematician. Hofstadter gives a correct account of the *pons asinorum*, Gelernter and the Pappus proof. Minsky must have got confused!

COMPUTERS AND COMPUTING

Solving Non-Linear Equations: Part 5, The Newton-Raphson Method

J C Lattanzio, Monash University

The Newton-Raphson Method is another "open method", and is the most popular of all methods. It is the method of choice where we are to solve the non-linear equation $f(x)=0$, where $f(x)$ is an easily differentiable function. Start with an approximation x_i to the root we seek, and draw a tangent to the curve $y=f(x)$ at the point $(x_i, f(x_i))$ as shown in the diagram below.

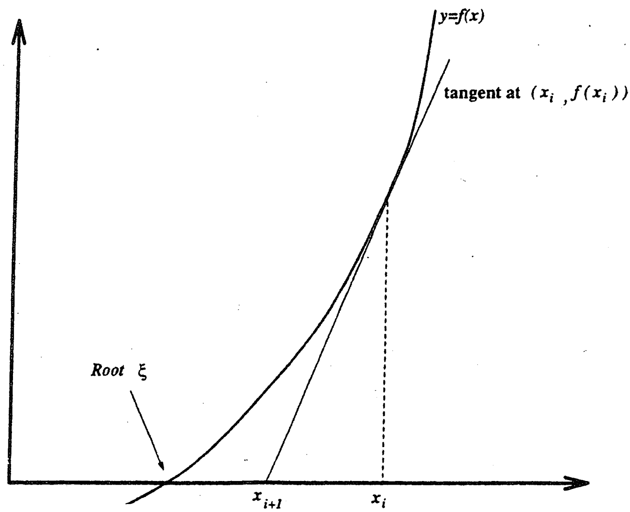

Figure 1 The Newton-Raphson Method

This tangent has the equation

$$y=f(x_i)+(x-x_i)f'(x_i).$$

Where this line intersect the x-axis is often a better approximation to the root we seek, and it is our next approximation x_{i+1}. That is to say

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} .$$

As an example, consider again the example discussed in my previous column in this series: $f(x) = x - e^{-x} = 0$. For this case, we have $f'(x) = 1 + e^{-x}$. So

$$x_{i+1} = x_i - \frac{x_i - e^{-x_i}}{1 + e^{-x_i}} = \frac{x_i + 1}{e^{x_i} + 1}$$

Begin with the approximation $x_0 = 0$. Applying the above method successively generates the following table, where the third column gives the relative error at each step, $\varepsilon_i = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right|$.

i	x_i	ε_i
0	0.000	——
1	0.500	1.00
2	0.566	0.12
3	0.567	0.001

As in the previous column, we are seeking 2-place accuracy and working to 3 places to avoid roundoff errors. Notice that we have achieved accuracy to within 1% in only 3 iterations!

If we analyse the relative error of the Newton-Raphson method, we find that at the (i + 1)th iteration, we have $\varepsilon_{i+1} \approx k\varepsilon_i^2$. This should be compared with the Fixed-Point Iteration method, where the relative errors formed an approximate geometric sequence, that is to say $\varepsilon_{i+1} \approx k\varepsilon_i$.

If we reach a point where the relative errors are small, then the next such in the case of the Newton-Raphson method will be approximately the *square* of this small quantity, and so will be *very small.* It is this result that provides the rapid convergence that often characterises this technique. This is why the Newton-Raphson method is so often the method of choice when it comes to solving a non-linear equation.

This having been said, however, it is as well to be aware that the method does have its pitfalls, and one needs to be forewarned. Although usually showing rapid convergence, it is not guaranteed to converge, and can actually suffer from a variety of problems.

Here I list four of these.

1. If $f'(x_i) = 0$ for some x_i, then the method will clearly fail.
2. If there is a point of inflection near the root, the method may cycle instead of converging.
3. If there are multiple roots, the method may find one not necessarily the nearest to the initial approximation x_0.
4. A near-zero slope can ruin the convergence.

Figure 2 shows an example of the second of these problems. Even if we do not see an *exact* cycle, such problems may make the method impractical. As an exercise, look at the equation $x^3 - x - 3 = 0$, with an initial approximation $x_0 = 0$. You will find that $x_{i+4} \approx x_i$.

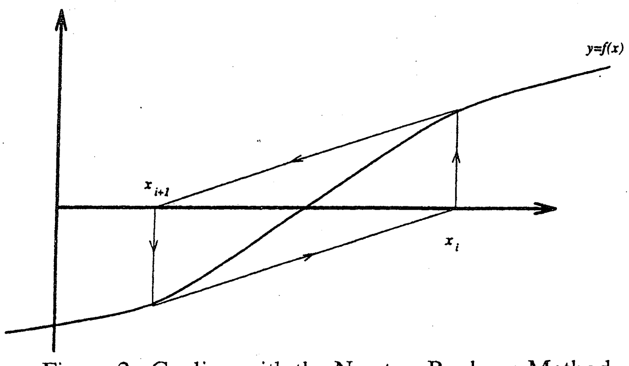

Figure 2. Cycling with the Newton-Raphson Method

Figure 3 shows the third of the problems that may arise. You should continue the geometric constructions to see what happens in the long run.

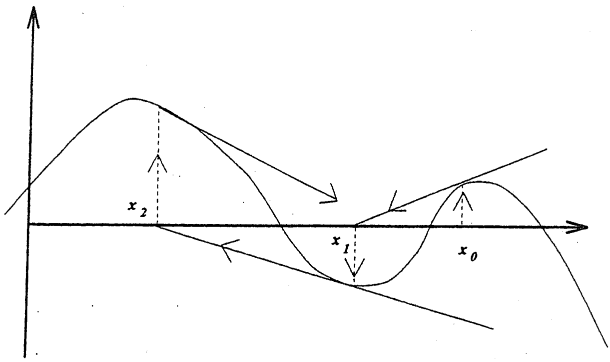

Figure 3. Trying to find a Root

Finally look at Figure 4. This illustrates the fourth of the difficulties, which is actually closely related to the first, although a little more subtle. Again as an exercise continue the process to see what happens in the long run.

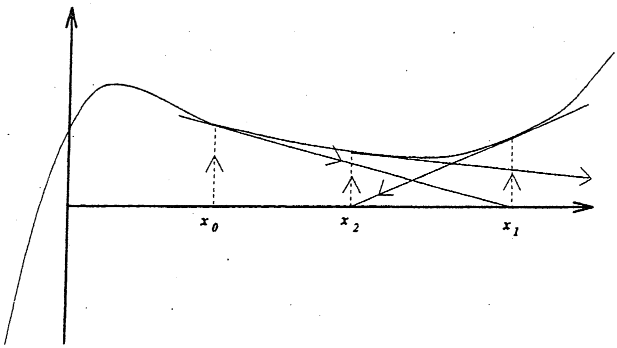

Figure 4. Trying to find a Root

OLYMPIAD NEWS

Hans Lausch, Monash University

The 2003 Australian Mathematical Olympiad

The Australian Mathematical Olympiad (AMO) for 2003 was held in Australian schools on February 11 and 12. On both days, 104 students in years 8 to 12 sat a paper consisting of four problems, for which they were given four hours. These are the two papers. No calculators were allowed and each question was worth 7 points.

First Day

1. Determine all the triples (p, q, r) of positive integers that satisfy:
   (i) $p(q-r) = q + r$,
   (ii) p, q and r are prime numbers.

2. Determine all functions f that are defined for all real numbers $x \neq 0, 1$ with real numbers as their values, and which satisfy the equation

$$f(x) + \frac{1}{2x} f\left(\frac{1}{1-x}\right) = 1.$$

3. Let ABC be a triangle such that $\angle ACB = 2\angle ABC$, and let D be a point in the interior of ABC satisfying $AD = AC$ and $DB = DC$.
   Prove that $\angle BAC = 3\angle BAD$.

4. Let

$$p(x) = x^{2003} + a_{2002}x^{2002} + a_{2001}x^{2001} + \ldots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_{2002}$ are integers. Let $q(x) = (p(x))^2 - 25$.
Prove that there are not more than 2003 distinct integers m such that $q(m) = 0$.

Second Day

5. After several kilometers of a televised bicycle race along a straight stretch of road on the Nullabor, the favourite Andrew pulled well ahead of the rest of the field closely followed by Brenda and then Chris. For the remainder of the race those three were ahead of the rest and, although they frequently changed places, at no time were all three abreast. During the finish a thunderstorm caused the TV signal to drop out, and when it came back on the race was over. The frustrated viewers only heard that the leading position changed 19 times while the third position changed 17 times and that Brenda came third. Who won the race and why?

6. Let *AD* be a median of the triangle *ABC*. Let the point *E* lie on *AD* (extended if necessary) such that *CE* is perpendicular to *AD*. Suppose that angle *ACE* equals angle *ABC*.

   Prove that either $AB = AC$ or angle *BAC* is a right angle.

7. Let $a_1,\ a_2,\ a_3,\ \ldots$ be a sequence defined by:

   (i) $a_1 = 0$,
   (ii) either $a_{i+1} = a_i + 1$ or $a_{i+1} = -a_i - 1$ for each $i \geq 0$.

   An example is 0, 1, 2, 3, –4, –3, 2, … .
   Prove that $\dfrac{a_1 + a_2 + \ldots + a_n}{n} \geq -\dfrac{1}{2}$ for all positive integers n.

8. Let *S* be any sequence of n letters ($n \geq 1$) not more than 10 of which are different, e.g. MATHEMATICIANS or
   GOOLLLDDMMMMEDALLLLLSYESYES.

Prove that each letter of this sequence can be replaced by a single digit such that

(i) different letters are replaced by different digits,
(ii) the first letter of the sequence is replaced by a digit other than 0,
(iii) the resulting n-digit number is a multiple of 9.

The Fifteenth Asian Pacific Mathematics Olympiad

The Asian Pacific Mathematics Olympiad (APMO) was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the APMO has grown into a major international competition for students from about twenty countries on the Pacific rim as well as from Argentina, Kyrgyzstan, South Africa and Trinidad and Tobago. It was held on March 17/18, with the Australian participants numbering 28.

Here is the contest paper. Four hours were allotted and calculators were not allowed. Each question carried 7 points.

Problem 1. Let a, b, c, d, e, f be real numbers such that the polynomial

$$p(x) = x^8 - 4x^7 + 7x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f$$

factorises into eight linear factors $x - x_i$, with $x_i > 0$ for each $i = 1, 2, \ldots, 8$. Determine all possible values of f.

Problem 2. Suppose $ABCD$ is a square piece of cardboard with a side length a. On a plane are two parallel lines l_1 and l_2 which are also a units apart. The square $ABCD$ is placed on the plane so that the sides AB and AD intersect l_1 at E and F respectively. Also the sides CB and CD intersect l_2 at G and H respectively. Let the perimeters of ΔAEF and ΔCGH be m_1 and m_2 respectively. Prove that no matter how the square was placed, $m_1 + m_2$ remains constant.

Problem 3. Let $k > 14$ be an integer, and let p_k be the largest prime number which is strictly less than k. You may assume that $p_k \geq 3k/4$. Let n be a composite integer. Prove:

(i) if $n = 2p_k$, then n does not divide $(n - k)!$;

(ii) if $n > 2p_k$, then n divides $(n - k)!$.

Problem 4. Let a, b, c be the sides of a triangle, with $a + b + c = 1$, and let $n \geq 2$ be an integer. Show that $\sqrt[n]{a^n + b^n} + \sqrt[n]{b^n + c^n} + \sqrt[n]{c^n + a^n} < 1 + \frac{\sqrt[n]{2}}{2}$.

Problem 5. Given two positive integers m and n, find the smallest positive integer k such that among any k people, either there are $2m$ of them who form m pairs of mutually acquainted people or there are $2n$ of them forming n pairs of mutually unacquainted people.

This year's International Mathematical Olympiad (IMO), to be attended by teams from more than 80 countries, will be held in Tokyo, Japan, in July. Australia has participated in this competition since 1979 and will be represented by six secondary school students. At the 2001 IMO in Washington, DC, USA, an Australian team finished in 25th position, and at the 2002 IMO in Glasgow, Scotland, UK, the Australian team came 26th, while at the 1997 IMO in La Plata, Argentina, our team achieved 9th position. Good luck to our 2003 team!

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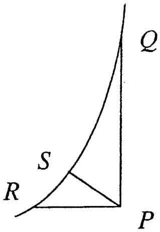
LETTER TO THE EDITOR

A Different Look at Last Issue's Cover

The cover picture on your last issue reminded me of something that has worried me for a long time. When the two curves you showed were compared, you did what most people do and considered the vertical distance between them. I have always thought that our eyes work differently and compare the distance across the curves – at right angles to them so to speak. I tried to look at this in relation to your cover picture.

Let P be a point on the curve $y = x^3$ (this curve not shown on the picture overleaf), and let Q be the point vertically above this on the curve that *is* shown, $y = x^3 + ax^2 + bx + c$. To illustrate, I assume $a > 0$. Let R be the point to the left of P, but on the displayed curve. From P, I drop a perpendicular PS to the same curve. Here is the picture (overleaf):



Then $|PQ| = ax^2 + bx + c \approx ax^2$. R will be the point $(x - h, y)$, where

$$(x-h)^3 + a(x-h)^2 + b(x-h) + c = x^3 = y.$$

So, because we can take x to be large, then $|RP| = h \approx a/3$. Approximate the shape PQR by a triangle. The slope of RQ will be approximately

$$\frac{PQ}{RP} \approx \frac{ax^2}{a/3} = 3x^2.$$

This means that the slope of PS is about $\dfrac{-1}{3x^2}$ which is very small, so that PS is approximately the same as PR, which is h.

So the *visual distance* between the two curves is about h, or $a/3$. This becomes small in relation to x, as we move out from the origin.

I think that it is this phenomenon that explains the apparent similarity of the two curves as we go to larger scales.

Bernard Anderson

Portland College

[This letter provides a new viewpoint on the question that prompted the original article by Deiermann and Mabry. They looked at matters such as the ultimate symmetry about $x = 0$ of (e.g.) $y = (x-1)^4$, whose true axis of symmetry is in fact the line $x = 1$. Readers may care to investigate this example further. Eds]

PROBLEMS AND SOLUTIONS

We begin with the solutions to the problems set last October. Here they are.

Solution to Problem 26.5.1 (submitted by Keith Anker)

The problem read:

Two contestants, *A* and *B*, play the following game. *A*'s initial score is 1, *B*'s is 0. A play consists of the toss of a coin. If it lands heads, 1 is added to *A*'s score (and *B*'s is left unchanged); if it lands tails, 1 is added to *B*'s score (and *A*'s is left unchanged). The game terminates if and when *B*'s score equals *A*'s score. What is the probability that the game will terminate, and what is the expected length of the game?

No solutions were sent in, which perhaps indicates the difficulty of the problem. However, it leads to some very interesting and indeed rather strange Mathematics, and we can use it to give a taste of this aspect of Probability Theory. An equivalent problem is discussed in Chapter III of a standard textbook: William Feller's *An Introduction to Probability Theory and its Applications, Volume I.*

To relate our problem to that discussed by Feller, suppose the game between *A* and *B* is witnessed by two other people, Peter and Paul. Peter and Paul agree to have a side bet on the progress of the game. If the coin lands heads, Paul pays Peter \$1; if it lands tails, Peter pays Paul \$1. If the game ends, then Paul has made exactly \$1, and conversely.

Although all of Feller's discussion is interesting and pertinent, the key passages for our problem are to be found on pp 74 and 75 (of the second edition). There he defines two quantities:

$$u_{2n} = \binom{2n}{n} 2^{-2n} \qquad \text{and} \qquad f_{2n} = \frac{1}{2n} u_{2n-2}.$$

The first of these applies for all non-negative integral n, the second for all positive integral n. The symbol $\binom{2n}{n}$ represents the number of different ways in which n objects may be chosen out of a collection of $2n$ such objects. $\binom{0}{0}$ is understood to be 1. Then the probability that the game ends on toss 1 is f_2 $(=\frac{1}{2})$; the probability that it ends on the third toss is f_4 $(=\frac{1}{8})$, etc. Readers should check these early values for themselves, noting that the game can only end after an odd number of tosses. These results are part of Feller's Theorem 1, which is however couched in terms of Peter and Paul's side bet. The probability that the game ends on toss $2n - 1$ is f_{2n} .

Feller then goes on to show [his Equation (4.9)] that

$$f_2 + f_4 + f_6 + \cdots + f_{2n} = 1 - u_{2n}.$$

Now it may be shown that $u_{2n} \to 0$ as $n \to \infty$. (This may be done in a number of different ways.) Thus we see that the infinite sum $f_2 + f_4 + f_6 + \cdots + f_{2n} + \cdots$ equals 1. This answers the first question: the probability that the game will end (in a finite time) is 1.

The expected length of the game is now

$$1.f_2 + 3.f_4 + 5.f_6 + \cdots + (2n-1)f_{2n} + \cdots$$

and this series may be shown to diverge, again in a number of different ways. Thus the expected length of the game is infinite! The apparent contradiction between this result and the one stated just above is merely one of a large number of paradoxes associated with this game. Feller discusses many more. As he says, "we reach conclusions that play havoc with our intuition".

We can only recommend that readers look more into the many interesting things Feller has to say. The entire chapter is worth careful study.

Solution to Problem 26.5.2 (submitted by Colin Wilson)

The Problem read:

Show that for any triangle, the ratio of the square of its perimeter to the sum of the squares of its sides is greater than 2, but not greater than 3. For what type of triangle is this ratio equal to 3?

Solutions were received from Keith Anker, Sĕfket Arslangić (Bosnia), J C Barton, Julius Guest and the proposer. The details differed somewhat, but Anker and Barton submitted substantially the same elegant solution. Here it is.

Use the standard notation in which the sides of the triangle are a, b and c. The perimeter is then $p = a + b + c$. Now by the triangle inequality

$$a(a+b+c) > a(a+a) = 2a^2,$$

so that $ap > 2a^2$. Similarly $bp > 2b^2$ and $cp > 2c^2$. Add these three inequalities to find that $p^2 > 2(a^2+b^2+c^2)$, which is equivalent to the first of the statements to be proved. But now

$$\begin{aligned}3(a^2+b^2+c^2)-(a+b+c)^2 &= 2(a^2+b^2+c^2-ab-bc-ca)\\ &= (a-b)^2+(b-c)^2+(c-a)^2 \geq 0.\end{aligned}$$

The second result follows. It is also clear that we can only have equality in the final line if $a = b = c$. That is to say, if the triangle is equilateral.

The proposer also noted that if $S = a^2+b^2+c^2$, and if $y = p^2/S$, then y may take any value in the interval $2 < y \leq 3$, and indeed it is possible to find an *isosceles* triangle such that any value of y in this range may be achieved. He did send in proofs of these assertions, but he also noted that the reasoning was somewhat subtle. We are therefore leaving this further aspect of the problem to readers who may wish to follow it up.

Solution to Problem 26.5.3 (submitted by Jim Cleary)

This problem read:

The cells of a 4×4 grid are to be filled with noughts and crosses. There are to be exactly 2 noughts and 2 crosses in each row and exactly 2 noughts and 2 crosses in each column. In how many different ways can this be done?

We here print the solution that Mr Cleary sent to accompany his problem.

Begin with an example of an arrangement satisfying all the constraints:

X	O	O	X
X	X	O	O
O	X	X	O
O	O	X	X

is one such.

Now consider the number of such possibilities. There are ${}^4C_2$ different ways in which 2 objects may be chosen from a set of four. This number is also often represented as $\binom{4}{2}$ and this is the notation we used in our discussion of Problem 26.5.1. In any case, the value is 6. Thus the top row may be filled in 6 different ways. Whichever way we choose, there is still no restriction on the second row, which may also be filled in 6 different ways. Thus there are 36 different ways in which the top two rows may be filled.

If we consider the vertical columns made up from the top two rows, these are of 4 possible types:

$$A = \begin{bmatrix} \mathrm{X} \\ \mathrm{X} \end{bmatrix} \quad B = \begin{bmatrix} \mathrm{O} \\ \mathrm{O} \end{bmatrix} \quad C = \begin{bmatrix} \mathrm{O} \\ \mathrm{X} \end{bmatrix} \quad D = \begin{bmatrix} \mathrm{X} \\ \mathrm{O} \end{bmatrix}$$

We may arrange these in various ways and still be consistent with the constraints imposed. There are 6 arrangements consisting of *A* and *B* only. Call these Type 1. There are a further 6 arrangements consisting of *C* and *D* only. Call these Type 2. Finally there are 4! (=24) possible arrangements using all of *A*, *B*, *C* and *D*. Call these Type 3. Note that the three numbers 6, 6 and 24 add to 36, as they ought to.

Now consider the bottom two rows. If *A* occurs in the top two rows, then *B* must occur in the bottom two, and *vice versa*. Thus there are 6 possible valid arrangements of these forms (involving Type 1). Now consider the possibility that a *C* or a *D* occurs in the top two rows. Then a *C* or a *D* must occur in the bottom two. Thus there are 6×6 (=36) valid arrangements of *these* forms (involving Type 2). Finally if the top two rows contain all four types, then *A* must lie above *B*, *B* must lie above *A*, *C* must lie above *either C or D* and *D* must lie above *either C or D*. As there are 24 possible arrangements of the letters *A*, *B*, *C*, *D*, there are a total of 48 patterns involving Type 3.

Adding up all the possibilities gives a total of 6 + 36 + 48 = 90 possible arrangements.

Solution to Problem 26.5.4 (based on a problem posed by Carl Fischer)

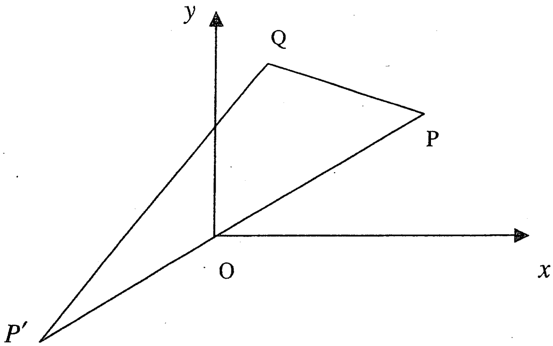
This interesting problem read:

Q is a fixed point inside a rectangle. *P* is any other point inside the same rectangle. It is desired to minimise the mean distance $|PQ|$ over all *P* by appropriate choice of *Q*. Show that *Q* must be the centre of the rectangle.

We received a particularly elegant solution from John Barton.

Let *O* be the centre, and take *O* as the origin of a rectangular co-ordinate system with *x* and *y* axes perpendicular to one another. (It is natural to have these axes parallel to the sides of the rectangle, but this is not strictly necessary.) Let *P* be the point (x, y), and consider also a point P' whose co-ordinates are $(-x, -y)$. Then P, O, P' are collinear. *Q* is another point.

Represent all these points on a diagram (overleaf).



Now note that $|PQ| + |P'Q| \geq |PO| + |P'O|$, by the triangle inequality, with equality if and only if *P* happens to lie on the line *OQ*.

Next suppose that, as we take the mean of $|PQ|$ over all the points *P*, we do so taking the points in pairs, (P, P'). To do this, we must sum over all points *P*, or equivalently over all point pairs (P, P'). Because of the inequality just derived, this sum will be higher for any *Q* than it is for *O* (unless *Q* happens to coincide with *O*), and thus the point *O* minimises this sum and hence also the mean of $|PQ|$.

[This problem is one of a number of related problems, many with practical applications; indeed it came to us from Carl Fischer, a Riverina farmer who was interested in minimising the time involved in liming a paddock. (The saving in time if the lime is placed initially in the centre of the paddock – rather than in a corner – is considerable. The distance to be covered in spreading it can be almost halved!) We may also remark that Barton's solution is much more general than the original problem. Suppose the points *P* form a set *S* with the property that $P \in S \Rightarrow P' \in S$. Then the conclusion applies equally to any such set (finite or infinite). If this symmetry property is not satisfied, however, the problem becomes much more difficult. Even the case of just 3 points *P* (known as the "Steiner Problem") is far from simple. See for example *Function*, October 1987; a related problem was discussed in our June 1985 issue. For yet another variation on the theme, see Problem 27.3.1 below. Eds]

So now on to the next crop of problems.

Problem 27.3.1 (suggested by the considerations just advanced)

It is desired to site the hub Q of a cabling network serving four outlets at A, $(0, a)$; B, $(0, b)$; C, $(c, 0)$ and D, $(-c, 0)$ in such a way as to minimise the total length of cable needed. Find the co-ordinates of Q.

Problem 27.3.2

A cone whose base radius is a and whose base-to-vertex height is h rests with its base on a horizontal surface. It is desired to pick up the cone by grasping it about its curved surface. Under what conditions can this be done?

Problem 27.3.3 (from the 1962 Beijing Mathematical Olympiad, further discussed in *American Mathematical MONTHLY*, Jan 2003, pp 25 …)

A number of students sit in a circle while their teacher gives them candy. Each student initially has an even number of pieces of candy. When the teacher blows a whistle, each student simultaneously gives half of his or her candy to the neighbour on the right. Any student who ends up with an odd number of pieces of candy gets one more piece from the teacher. Show that no matter how many pieces of candy each student has at the beginning, after a finite number of iterations of this process all the students have the same number of pieces of candy.

Problem 27.3.4 (proposed by Dan Buchnick, Israel)

Let ABC be a triangle and let D be a point on AB, E a point on BC, and F a point on CA. Join DE, EF, FD. We have now divided the original triangle into four smaller triangles: ADF, BED, CFE and DEF. Show that of these four triangles, DEF can never have the smallest area.

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