Function

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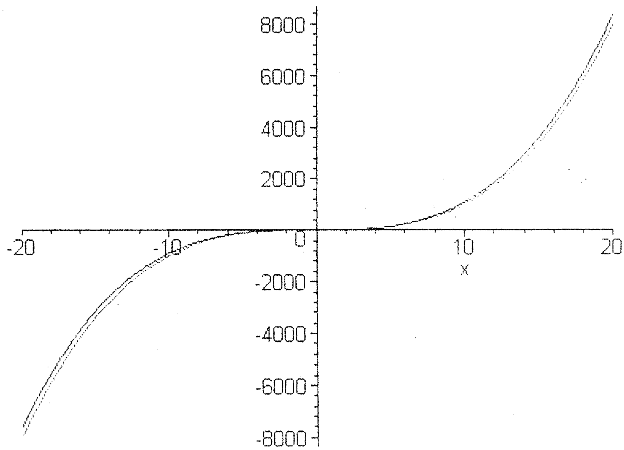

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. It was founded in 1977 by Prof G B Preston, and is addressed principally to students in the upper years of secondary schools, but also more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors, *Function*
School of Mathematical Sciences
PO BOX 28M
Monash University VIC 3800, AUSTRALIA
Fax: +61 3 9905 4403
e-mail: michael.deakin@sci.monash.edu.au

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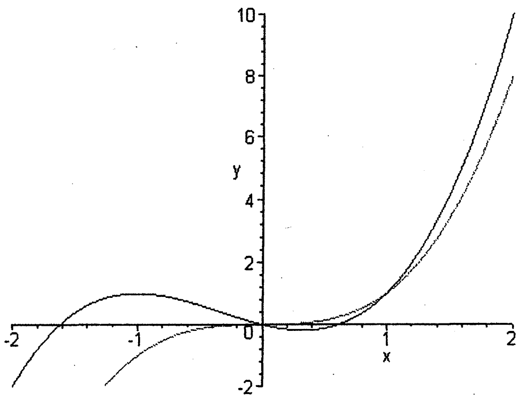
* $17 for *bona fide* secondary or tertiary students.

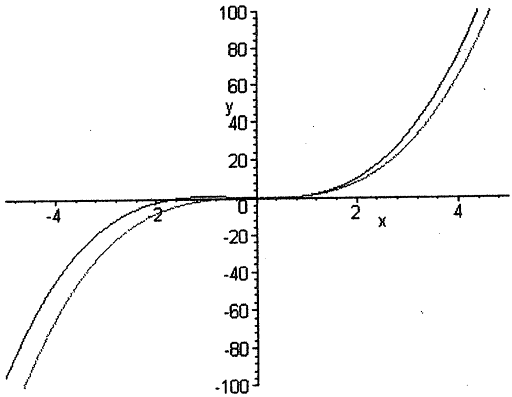
THE FRONT COVER

Our front cover for this issue depicts the two cubics $y = x^3 + x^2 - x$ (black curve) and $y = x^3$ (grey curve) on the same scale and related to the same set of axes. The scale extends from $x = -20$ to $x = +20$, which means that the y-values go from about -8000 to $+8000$. This is a considerable amount of graph: quite a lot more than is commonly displayed. If we take a more typical view of these graphs we see something like this.



On this scale, the two graphs look rather more different from one another. The simpler (grey) graph is symmetric about the origin in the sense that we could imagine its being rotated through 180° about the origin and, as far as we could tell, simply occupying its former position. (This is often referred to as S-symmetry, and it may be expressed by means of the equation $f(-x) = -f(x)$. The other (black) graph does not have this feature and moreover it has a maximum and a minimum instead of the point of horizontal inflection at the origin; it also has three distinct real roots instead of a triple root at the origin.

If we look at the same pair of graphs on a slightly larger scale, we get the picture overleaf. Here the two graphs have become rather more alike, but they are still readily distinguishable.



The cover graph expands the scale even more, and now the graphs have come to look remarkably similar. The features that distinguished the more complicated graph are much less in evidence, as they all relate to small values of x and y.

This material was suggested by an article "Asymptotic Symmetry of Polynomials" by Paul Deiermann and Richard D Mabry in the US journal *Mathematics Magazine* (April 2002), whose analysis however goes well beyond that presented here.

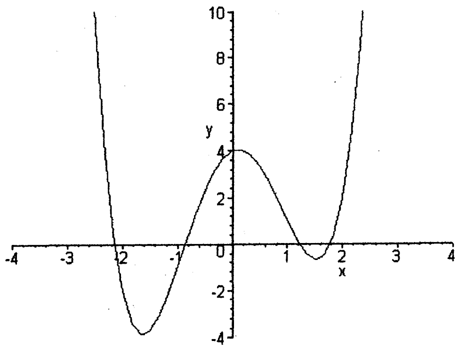
The key to this feature is that, although the extra terms of the more complicated graph actually get larger and tend to infinity as x itself tends to infinity, they do so less rapidly then does the leading term x^3. Another way to say this is that the vertical distance between the two graphs increases as x increases (as can be observed both in the cover diagram and also in those above), the *relative* distance compared with the overall vertical scale actually diminishes.

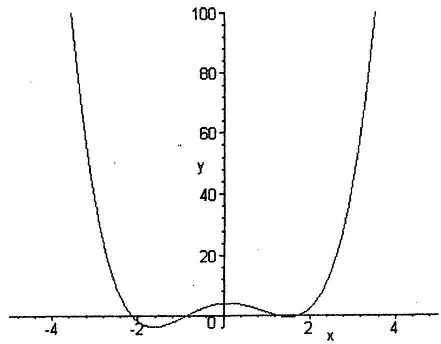
This is what in this context is meant by the term "asymptotic". Readers may only know this word in another different but related meaning. The graph of $y = 1/x$ is asymptotic to zero as $x \to \infty$. The value of y approaches but never attains the value of zero.

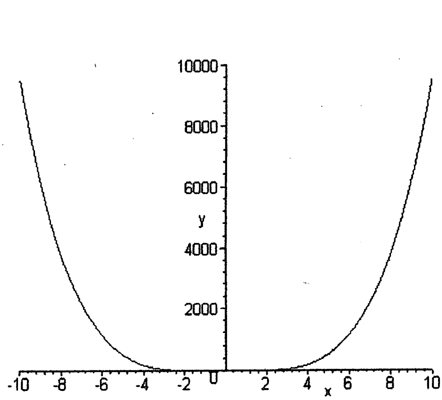
In the case under discussion here, the two curves actually drift apart, but nevertheless they come more and more to resemble one another!

This property applies to all polynomials. However if the degree of the polynomial is even, the asymptotic (simpler) curve exhibits **T**-symmetry ($f(-x) = f(x)$) rather than **S**-symmetry, which applies if the degree is odd.

To illustrate this point, here are some graphs of the quartic polynomial $y = x^4 - 5x^2 + x + 4$.







This last one looks very like the graph of $y = x^4$, you will probably agree, while the first and even the second show a richness of behaviour that fails to be evident on the larger picture.

Response to Unlikely Reports

(Cf. Borel and "The Miracle of the Typing Monkeys", *Function*, April 2002)

Suppose someone tells you that a dog is running down the center of Fifth Avenue [New York]. You might think it unusual, but it's certainly possible, and you would have no reason to doubt the story. If the claim is that it's a lion running down Fifth Avenue, it's still possible, but you would probably want some sort of supporting evidence – perhaps a report of a lion escaping from the Bronx Zoo. But if someone tells you a stegosaurus is running down Fifth Avenue, you would assume that he was mistaken. In some sense it might be "possible" that he's seen a stegosaurus, but it's far more likely that he saw a dog and thought it was a stegosaurus. Indeed, most reasonable people would agree that the possibility that there really could be a stegosaurus running down Fifth Avenue is too small to even bother checking out.

Richard Wilson, quoted by Robert Park
in *Voodoo Science* (OUP, 2000)

CONTINUED FRACTIONS AND INDETERMINATE EQUATIONS

David Shaw, William St , Geelong

The final pages of Hall and Knight's *Higher Algebra* (Macmillan, first published in 1887 and much reprinted since) contain 300 "Miscellaneous Examples", of which Number 111 reads as follows:

> Express $\dfrac{763}{396}$ as a continued fraction; hence find the least positive integral values of x and y which satisfy the equation $396x - 763y = 12$.

This is an example of an indeterminate equation of the first degree. "Of the first degree" because both x and y appear only as first powers, and "indeterminate" because there are two unknowns but only one equation. There is however the further restriction that we are only looking for integral solutions. Such equations are called *Diophantine Equations*.

The aim of this article is to introduce the theory and properties of continued fractions and to show how they may be used to solve such problems. Proofs of the various theorems are omitted in the interest of brevity, but all are readily available either in Hall and Knight's book or else in C D Olds' *Continued Fractions* (Random House, 1963).

An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \ldots}}}$$

is called a continued fraction. Here I will restrict consideration to "simple continued fractions", for which: (1) $b_i = 1$ for all values of i, (2) a_1 is integral, and (3) a_i is positive integral for $i > 1$.

If the sequence of a_i is finite, then the continued fraction is also called "finite". In such a case, we have an expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}$$

The values a_i are called the "partial quotients", or simply "terms" of the continued fraction.

It saves space to write continued fractions more simply, and this convention will be used here. Instead of the bulky expression of the previous paragraph, write instead

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots\frac{1}{a_{n-1} +}\,\frac{1}{a_n}$$

or better still $[a_1, a_2, a_3, \ldots, a_{n-1}, a_n]$.

Any rational number can be expressed as a simple finite continued continued fraction. Any positive rational number $\frac{p}{q}$ may be so expressed by following the steps of the illustrative example $\frac{43}{19}$. Then

$$\frac{43}{19} = 2 + \frac{5}{19} = 2 + \frac{1}{\frac{19}{5}}$$

$$\frac{19}{5} = 3 + \frac{4}{5} = 3 + \frac{1}{\frac{5}{4}}$$

$$\frac{5}{4} = 1 + \frac{1}{4} = 1 + \frac{1}{\frac{4}{1}}$$

$$\frac{4}{1} = 4.$$

Therefore

$$\frac{43}{19} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}} = [2, 3, 1, 4].$$

The working may be arranged as on the left below (where HCF stands for "highest common factor": 1 is the highest common factor of 19 and 43; 3 is the highest common factor of 57 and 129).

$$\begin{array}{llll}
19)\,43\,(2 \leftarrow a_1 & & 57)\,129\,(2 \leftarrow a_1 & \\
\quad 38 & & \quad 114 & \\
\quad \overline{5})\,19\,(3 \leftarrow a_2 & & \quad \overline{15})\,57\,(3 \leftarrow a_2 & \\
\qquad 15 & & \qquad 45 & \\
\qquad \overline{4})\,5\,(1 \leftarrow a_3 & & \qquad \overline{12})\,15\,(1 \leftarrow a_3 & \\
\qquad\quad 4 & & \qquad\quad 12 & \\
\text{HCF} \rightarrow \overline{1})\,4\,(4 \leftarrow a_4 & & \text{HCF} \rightarrow \overline{3})\,12\,(4 \leftarrow a_4 &
\end{array}$$

To the right is the working for $\frac{129}{57}\left(=\frac{43}{19}\right)$. The last divisor is the highest common factor (HCF) of 129 and 57. The process used for obtaining the terms of the continued fraction is identical with that used for obtaining the HCF of two numbers by Euclid's algorithm. (Euclid's algorithm is a method for determining the highest common factor; it is explained in (e.g.) the Hall and Knight book referenced earlier.) The process shows that a continued fraction representation of a rational number must be finite because the remainders (i.e. the divisors) form a decreasing sequence of natural numbers.

At this stage, there are two further points worth noting about the first and last terms. The first point is that

$$a_1 = 0 \text{ if } 0 < \frac{p}{q} < 1.$$

The second is more significant. If the last term, a_n, is greater than 1, we can write

$$\frac{1}{a_n} = \frac{1}{(a_n - 1) + \frac{1}{1}}$$

so that $p/q = [a_1, a_2, \dots, a_{n-1}, a_n] = [a_1, a_2, \dots, a_{n-1}, a_n - 1, 1]$ and if $a_n = 1$, we can add 1 to the previous term a_{n-1}. Therefore a simple finite continued fraction may have either an even or an odd number of terms, whatever we wish. Otherwise the representation is unique.

Now go back to our earlier example,

$$\frac{43}{19} = 2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4}}}$$

and calculate the fractions formed by cutting off the continued fraction, one step at a time, starting from the left (as indicated by the slanting lines drawn across the continued fraction).

$$c_1 = \frac{2}{1} \quad c_2 = \frac{7}{3} \quad c_3 = \frac{9}{4} \quad c_4 = \frac{43}{19}$$

i.e. $c_1 = [2]$ $c_2 = [2, 3]$ $c_3 = [2, 3, 1]$ $c_4 = [2, 3, 1, 4]$.

These fractions are known as the "convergents" of the continued fraction, the last of them being the continued fraction itself.

Now to speak more generally, suppose that

$$\frac{p}{q} = [a_1, a_2, a, \ldots, a_{n-1}, a_n] \ .$$

Then the first convergent c_1 will be equal to $[a_1]$. I.e.

$$c_1 = [a_1] = \frac{a_1}{1} = \frac{p_1}{q_1}$$

Similarly, write

$$c_2 = [a_1, a_2] = a_1 + \frac{1}{a_2} = \frac{a_1 a_2 + 1}{a_2} = \frac{p_2}{q_2}$$

So

$$p_2 = a_2 a_1 + 1, \quad q_2 = a_2 \ .$$

Continuing in this way, we find:

$$c_3 = [a_1, a_2, a_3] = \frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3 + 1}$$
$$= \frac{a_3(a_2a_1 + 1) + a_1}{a_2a_3 + 1}$$
$$= \frac{a_3p_2 + p_1}{a_3q_2 + q_1}$$
$$= \frac{p_3}{q_3}$$

so that

$$p_3 = a_3p_2 + p_1, \quad q_3 = a_3q_2 + q_1.$$

It would seem that the general rule is

$$p_i = a_ip_{i-1} + p_{i-2}, \quad q_i = a_iq_{i-1} + q_{i-2}$$

and this result may be proved, for all values of i after the first two. The problem with those early values is that p_0, q_0, p_{-1}, q_{-1} are none of them defined. However, we can force the fit by adopting the convention that

$$p_0 = 1, q_0 = 0, p_{-1} = 0, q_{-1} = 1.$$

This ensures that the formulas hold for all values of i.

It is left as an exercise to the reader to use these formulas to find the convergents of 43/19 .

Back now to the general case, we have a theorem of great importance. It states that

$$p_iq_{i-1} - p_{i-1}q_i = (-1)^i.$$

This fundamental relation implies that all the convergent fractions are in their lowest terms. Moreover, each is a better approximation to the value of the continued fraction than the one before it. They approach that value in an oscillatory manner: if one is an overestimate, the next will be an underestimate, and *vice versa.*

Now take the special case $i = n$, and write

$$p_n q_{n-1} - p_{n-1} q_n = (-1)^n .$$

And get back to the original problem which concerned the fraction $\frac{763}{396}$ and the equation $396x - 763y = 12$. If we express $\frac{763}{396}$ as a continued fraction, we find

$$\frac{763}{396} = [1, 1, 12, 1, 1, 1, 9] .$$

The convergents are successively $\frac{1}{1}, \frac{2}{1}, \frac{25}{13}, \frac{27}{14}, \frac{52}{27}, \frac{79}{41}, \frac{763}{396}$. If we now look at the second-last convergent $\frac{79}{41}$ and the final (exact) value, we have, by our formula,

$$763 \times 41 - 79 \times 396 = (-1)^7 = -1,$$

because, in this case, $n = 7$.

Therefore

$$79 \times 396 \times 12 - 763 \times 41 \times 12 = 12 .$$

This gives us an immediate solution to the given equation:

$$x = 79 \times 12 = 948, \quad y = 41 \times 12 = 492 .$$

More generally, however, we may write

$$79 \times 396 \times 12 - 763 \times 41 \times 12 = 12 = 396x - 763y,$$

which may be written as

$$396(x - 948) = 763(y - 492),$$

so that

$$\frac{x - 948}{763} = \frac{y - 492}{396} = t \text{ (say).}$$

We may note that t must be an integer because 763 and 396 are relatively prime. The general solution to the given equation is therefore

$$x = 763t + 948, \quad y = 396t + 492.$$

The least positive values of x and y are achieved for $t = -1$, and this gives

$$x = 185, y = 96.$$

This then provides the answer to the question posed by Hall and Knight.

APPROXIMATE FORMULAE

Recent emails to some of the specialist groups in Mathematics have publicised some surprising approximations to complicated expressions. One cluster of results comes from a renewed interest in the life of Benjamin Banneker (1731-1806). Banneker was an amateur, and largely self-taught, mathematician, the son of an American slave, and one of the first African Americans to achieve mathematical prominence. In the 1790s he published a number of almanacs that earned praise from (among others) Thomas Jefferson.

Two of his formulas have been the subject of recent emails. The first is the approximation

$$\frac{5\pi}{9} \approx \sqrt{3}$$

which translates to

$$\pi \approx \frac{9\sqrt{3}}{5}$$

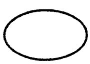
a result accurate to within 1%.

Another is a nice approximation to the altitude of an equilateral triangle. Many readers will have seen in their early brushes with Trigonometry the use of the equilateral triangle of side 1 to reach the trig ratios $\cos 60^{\circ} = \sin 30^{\circ} = 1/2$, $\sin 60^{\circ} = \cos 30^{\circ} = \sqrt{3}/2$.

Banneker pointed out that if the side of the equilateral triangle was initially taken to be 30, then its altitude was very nearly 26. As $\frac{30\sqrt{3}}{2} = 25.98\ldots$, this approximation is very good indeed!

Another series of emails concerned the perimeter of an ellipse. Rather surprisingly there is no simple formula for this. Recall that an ellipse is like a "squashed circle". Its standard equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Here is its shape: 

By convention the longest diameter is called $2a$ and the shortest $2b$. The special case $b = a$ is that of a circle. For this case, the perimeter is of course known; it is $2\pi a$. For a long skinny ellipse, the perimeter approximates $4a$, as the reader may easily verify.

In 1609, Kepler introduced the approximation for the perimeter p:

$$p \approx \pi(a+b)$$

which is reasonably accurate if the ellipse is nearly circular. As Kepler was interested in the elliptical paths of the planets about the sun, and as these are indeed nearly circular, this approximation served him well.

At roughly the same time, the Japanese mathematician Seki (Seki Takamazu) used the approximation

$$p \approx \sqrt{4\pi^2 ab + 16(a-b)^2}$$

which also works well for almost circular ellipses, but as well in the case for which b is very small compared to a. However, there are ratios of a/b for which Kepler's formula is better.

Yet another formula is attributed to Sir Thomas Muir, 1844-1934). It is

$$p \approx 2\pi\left\{\frac{a^{3/2}+b^{3/2}}{2}\right\}^{2/3}$$

There has been a suggestion that this formula was previously used in India, but no details of this claim have been forthcoming.

A 1988 paper by Almkvist and Berndt (*American Mathematical Monthly* **95**, pp 585-608) lists a large number of approximations to p. They attribute to Kepler another formula, this time using a geometric rather than an arithmetic mean:

$$p \approx 2\pi\sqrt{ab}\,.$$

The comments applied to the other Kepler formula apply also to this.

Of the other formulae listed by Almkvist and Berndt, two perhaps deserve special notice. One, by Ekwald (1973), is notable for its simplicity:

$$p \approx 2\pi\left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)^2$$

Another is one of two attributed to the Indian genius Ramanujan (see *Function, Vol* **1**, *Part* **3**). It goes:

$$p \approx \pi\left\{(a+b)+\frac{3(a-b)^2}{10(a+b)+\sqrt{a^2+b^2+14ab}}\right\}$$

This can hardly be called "simple", but it is extremely accurate. It has been used recently in a discussion of the respiratory physiology of insects. (Insects breathe through minute pores in their shells. Throughout the majority of the respiratory cycle, these pores are elliptical in cross-section.)

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LETTERS TO THE EDITOR

Election Prediction Vindicated

Readers of *Function* may be interested to learn that the *Law of Cubic Proportions* (see *Function, Vol* **17**, *Part* **4**, p 120) held exceptionally well for the 2002 election to the lower house of the Victorian parliament.

Perhaps it is worth restating the law here:

In a reasonably homogeneous country with a two-party political system, the ratio of the number of seats won by the two parties is approximately proportional to the cube of the ratio of the total number of votes cast for the two parties.

The law holds whenever the proportion of voters for a party (say Labor) over the total population of seats has an approximately normal distribution, with a standard deviation around 0.11-0.13. This situation has occurred in some elections in the UK, in NZ and in Australian federal elections since 1949.

For the recent Victorian election, there were 88 seats contested. Taking out the two seats won by independents leaves 86. In these 86 seats, the two-party preferred split for Labor : Coalition was 57.8 : 42.2. Applying the law to this case, the number of Labor and Coalition seats would be in the ratio $(57.8/42.2)^3 = 2.569$. Thus we should expect $\frac{2.569}{3.569}\times 86$ seats to go to Labor. This works out to be 61.9 seats, or in round figures 62, with 24 for the Coalition.

That's exactly the state of the parties in the Parliament.

One may add that the proportion of Labor voters ranged from 0.219 (Shepparton) to 0.817 (Thomastown), and the standard deviation was 0. 126.

Ravi Phatarfod
Monash University

Sad News from Wales

I wrote a year ago telling readers of the latest news of my erratic correspondent, Dai Fwls ap Rhyll. Dr Fwls has made a career out of his challenges to accepted orthodoxy in Mathematics and Science. Many of his analyses have been communicated to me and so passed on to *Function*. However, last year I had strong reasons to believe that his health (never the best) had taken a severe turn for the worse. The handwriting was pitifully shaky, the text terse in the extreme. I was however able to deduce that he had found an error of almost 18% in a widely accepted formula.

I now learn that my worst fears have come to pass. He has died. The news reached me only recently, and there were few details. My hope is that the mountain of scrappy manuscripts he left behind will prove to contain treasures of insight and provide further challenge to established orthodoxy.

Here, as a memorial to him, I can only summarise his work over the 25 years of our acquaintance.

I first wrote of his research in *Function* in 1980, the year he won the prestigious Prix le Bon for work on the nature of gravity. This remains his most recognised achievement. Following this he became more and more reclusive, never seeking to publicise his work, but rather sending it only to a small circle of select friends, of which I am proud to have been a member.

Over those years, I have sought to share his discoveries with *Function's* readers. In 1982, 1990 and 1991, he challenged the laws of Arithmetic, and in 1983 and 1999 those of Euclidean Geometry. In 1992, 2000 and 2002, he queried the accuracy of established formulae and numerical methods. In 1996, Probability Theory was his target, and in 1995, 1997 and 2001 he turned his attention to the higher realms of Mathematics. Although it never reached me directly, one of his discoveries turned up on the Internet in 1995. I have no idea who posted it. In 1998, my friend Sue de Nimmes discovered some work that she thought to be his, but this was a mistake on her part. Back in 1986 he wrote directly to *Function* with a speculation about comets, but I don't know how this letter reached the editor; it did not pass through my hands.

To give readers a flavour of his work, let me briefly recount one of his early studies (1982). He set

$$S = 1 - 1 + 1 - 1 + 1 - 1 + \dots ,$$

and determined the value of S in three different ways. He had

$$S = (1-1) + (1-1) + ((1-1) + \dots = 0 + 0 + 0 + \dots = 0,$$
$$S = 1 - (1-1) - (1-1) - (1-1) - \dots = 1 - 0 - 0 - 0 - \dots. = 1,$$

and

$$S = 1 - (1 + 1 - 1 + 1 - 1 + \dots) = 1 - S .$$

From this last equation he deduced that $S = ½$ so that $0 = 1 = 1/2$. The conclusion Dr Fwls drew is that Arithmetic is inconsistent.

Kim Dean, Erewhon-upon-Yarra

COMPUTERS AND COMPUTING

Solving Non-Linear Equations: Part 4, Fixed-Point Iteration

J C Lattanzio, Monash University

The bracketing methods discussed in my last article contrast with a number of other methods known as "open methods". Here I will discuss the so-called "one-point" or "fixed point" method. Other approaches will form the subject of the next two articles.

Begin by rearranging the original equation into an alternative form:

$$x = g(x).$$

A solution of this equation is called a "fixed-point of the function $g(x)$. In most cases, this will be a solution of the original equation.

The procedure begins with a guessed initial approximation x_0. We might obtain this from a sketch-graph or some other information. The calculation proceeds by successive improvements upon this initial estimate. The method is to generate a sequence from the *iterative equation* $x_{i+1} = g(x_i)$, for $i = 0, 1, 2, \ldots$. The sequence of approximations terminates when the relative difference is smaller than some previously specified small quantity. This condition is usually expressed as

$$\varepsilon = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| < \varepsilon^*,$$

where ε^* is the pre-assigned small number.

If the sequence converges to a limit, then this limit will be the solution to the equation.

As an example, consider the solution of the equation $x = e^{-x}$, beginning with an initial guess $x_0 = 1$. Successive approximations are generated from the equations $x_{i+1} = e^{-x_i}$. The table below gives the results of the calculation, with the third column indicating the relative change, or difference. The process is to stop when this becomes smaller than 1%.

This is the same equation as was solved in my previous article, and readers will note that the answer (0.57, to within 1%) is the same as that computed before.

i	x_i	ϵ
0	1.000	——
1	0.368	1.720
2	0.692	0.469
3	0.500	0.383
4	0.606	0.174
5	0.545	0.112
6	0.580	0.059
⋮	⋮	⋮
10	0.568	0.006 < 1%

It is useful to look at a geometric interpretation of fixed-point iteration. Figure 1 (overleaf) shows the first few iterations in the example above. Note that we start at x_0 (=1) and then move up to the graph of $g(x)$. The y-axis now gives us the value of $g(x_0)$. We now need to locate this point on the x-axis, because it is the value of the new estimate x_1. To do this, we "reflect" off the line $y = x$, as shown in the figure. We see that in this case the method gives values that oscillate back and forth from an overestimate to an underestimate and then back, but it is gradually converging to the desired intersection, where $y = x = g(x)$.

Consider now a more general case in which we begin with an equation $f(x) = 0$. There are usually many ways to recast this in the form $x = g(x)$. Unfortunately not all of these give rise to a convergent sequence of values. Suppose that we begin with the equation $x - e^{-x} = 0$. This can clearly be cast in the form $x = e^{-x}$, as used above. But it could equally well be written as $x = -\ln x$. Of these two rewritings, the first (as we have seen) gives rise to convergence. The latter does not.

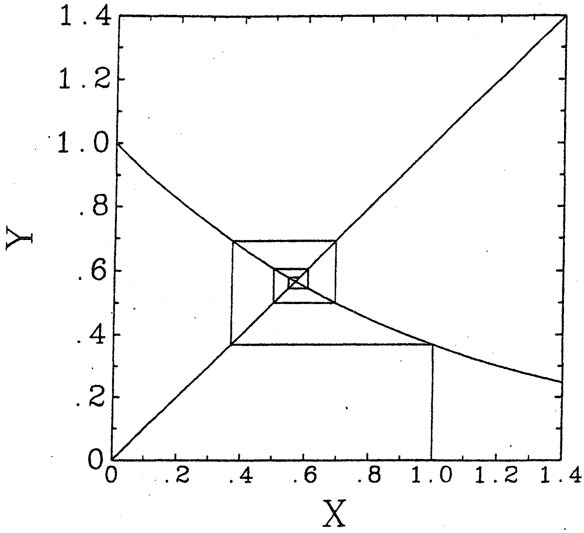


Figure 1

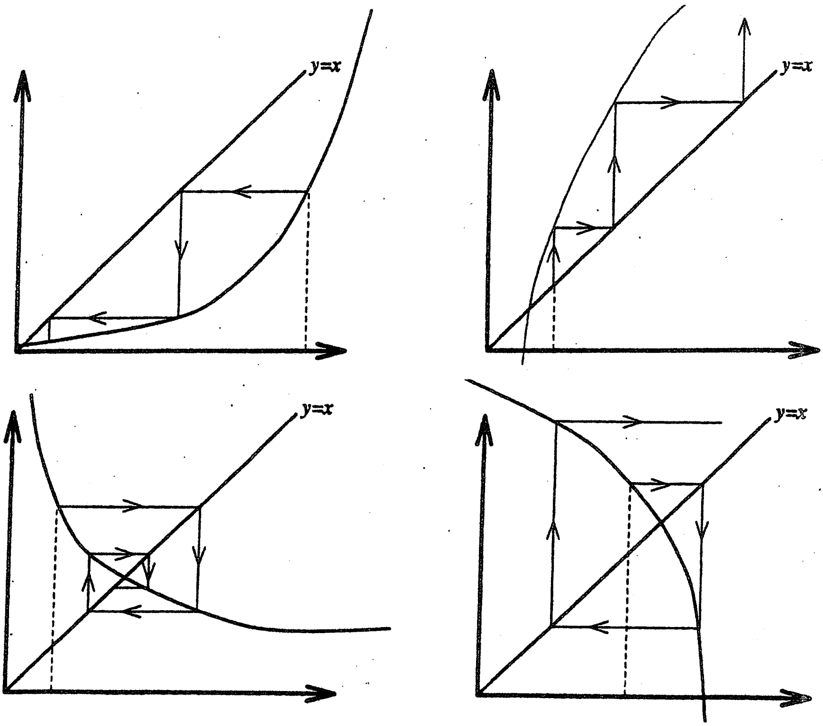
There is a useful test to determine whether the process converges or not. This will now be stated without proof. Let x be the solution and suppose that the process uses estimates in the neighborhood of x. Let $g'(x)$ be the derivative of the function $g(x)$. Then if:

$g'(x) < -1$,	the sequence oscillates but diverges,
$-1 < g'(x) < 0$,	the sequence oscillates but converges,
$0 < g'(x) < 1$,	the sequence converges directly,
$g'(x) > 1$,	the sequence diverges directly.

The second of these behaviours is that of our earlier example, the first is that of the modification in which the logarithm was used. Note that the value of $g'(x)$ in the exponential example was about -0.57, whereas in the logarithmic modification it was about -1.8 in the neighborhood of the root. This is consistent with the criteria given above.

In fact we can go further. The successive relative errors form an approximate geometric sequence with common ratio $g'(x)$.

The graphs below show the complete range of possible behaviour.



Finally, it should be noted that the criterion, if interpreted literally, requires a knowledge of the value we are in fact seeking. However, because we need the inequalities to apply over a *range* of values near the exact value, it suffices to ensure that they hold *near* the root we are seeking.

Here is a final example that demonstrates some of the pitfalls that may arise. Suppose we want to solve the cubic equation

$$x^3 + 4x^2 - 10 = 0.$$

There are various ways in which this could be recast to allow the application of the method:

$$x = x - x^3 - 4x^2 + 10, \qquad x = \left(\frac{10}{x} - 4x\right)^{1/2} \qquad x = \tfrac{1}{2}\left(10 - x^3\right)^{1/2}$$

$$x = \left(\frac{10}{4+x}\right)^{1/2} \qquad x = \frac{2x^3 + 4x^2 + 10}{3x^2 + 8x}$$

You should check that all these rewritings are possible, and then go further. As you explore these different possibilities, you will find that the first two diverge, while the others converge, but at very different rates.

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HISTORY OF MATHEMATICS

The First Hurdle

Michael A B Deakin, Monash University

The school geometry syllabus was for close on 2000 years based on Euclid's *Elements*, a standard textbook of its subject written back in about 300 BC. It was written in Greek, but is now available in English translation. In all, it comes in 13 parts called "books". At the very start of the first book, we encounter Definitions, Axioms and Postulates. These form the basis of the deductive edifice that is to follow. This takes the form of a sequence of "Propositions", or as we now tend to say, Theorems.

The first of these that poses any real difficulty is the fifth in the first book; I.5 is thus its technical name, but it has acquired the nickname of the *Pons Asinorum*, meaning the "Asses' Bridge". Look at the diagram below. It depicts an *isosceles* triangle, one with two equal sides: $AB = AC$.

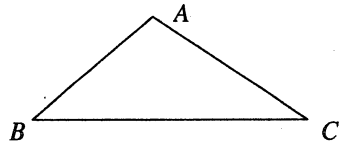


Figure 1

The theorem states that the two "base angles", ABC and ACB, are equal.

Partly because Euclid's proof is somewhat more difficult than it need be, and partly because it produced a diagram rather reminiscent of a cantilevered bridge, it was given the rather derogatory name I have just quoted. "Asses" (weaker students) fell down at this bridge, which in our vernacular we might perhaps rather call a hurdle.

Here is how Euclid proceeded to prove the proposition.

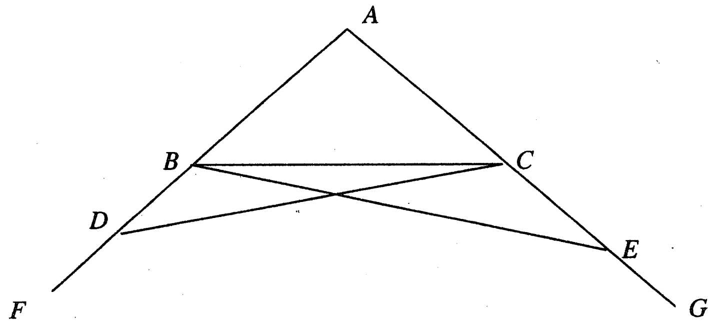


Figure 2

Line *AB* was extended to a point *F* and Line *AC* to a point *G*. Then line segments *BD* and *CE* were cut off from these extensions in such a way that *BD* = *CE*. The triangles *ABE* and *ACD* were now compared. We have:

- *AB* (in the first) = *AC* (in the second), because of the initial data,
- *AE* (in the first) = *AD* (in the second), from the construction, and
- Angle *BAE* (in the first) = Angle *CAD* (in the second), obviously.

Thus, by an earlier proposition (I.4), these two triangles are congruent, i.e. identical except possibly for position and orientation.

We may similarly show that the triangles *BCE* and *CBD* are congruent, since from the first argument we deduce that *BD* = CE and

Angle *BDC* = Angle *CEB*.

From all this it follows that Angle *BCE* = Angle *CBD* and hence that

Angle *ABC* = Angle *ACB*.

This is certainly much more elaborate an argument than anything that precedes it in the *Elements*, and this is probably what gave the theorem its reputation for difficulty. The Eighteenth Century author Tobias Smollett has Peregrine Pickle, the hero of one of his books, experiencing such difficulty: "Peregrine ... began to read Euclid ... but he had scarce advanced beyond the *Pons Asinorum*, when his ardor abated."

As Peregrine advanced (a little way) beyond the *Pons Asinorum*, he would have encountered Proposition I.6 of the *Elements*. That is to say, Proposition 6 of Book I. This proposition is the converse of its predecessor. It takes the equality of the base angles *ABC* and *ACB* as given and seeks to prove that *AB* = *AC*. In this endeavour, it uses a *reductio ad absurdum* argument. This is the first such use in the *Elements*, so that perhaps Peregrine felt that confusions were coming thick and fast.

However we may sympathise also. The proof is unnecessarily complicated. For a start, no use at all is made of the points *F* or *G*. We could quite easily omit all reference to them. Then also we could equally well put the points D and E *inside* the triangle rather than *outside* it. I leave it to readers to check that this leads to a minor simplification in the proof.

But we can achieve a further simplification by allowing for a case intermediate between Euclid's ($AD > AB$, $AE > AC$) and that just mooted ($AD < AB$, $AE > AC$), by allowing D and E to coincide with B, C respectively: $AD = AB$, $AE = AC$. This version of the proof has reached us via a commentary on *The Elements* by the later geometer Proclus, who attributes it to another Greek geometer, Pappus.

Look again at Figure 1. Interpret it to be a picture of two triangles: ABC and ACB. Now compare these two. We have

- AB (in the first) = AC (in the second), because of the initial data,
- AC (in the first) = AB (in the second), similarly, and
- Angle BAC (in the first) = Angle CAB (in the second), obviously.

Thus the two triangles are congruent and, in particular

$$\text{Angle } ABC = \text{Angle } ACB.$$

This proof has become the preferred one among mathematicians, but it does require some mathematical sophistication to appreciate it. Indeed, at least two mathematicians are on record as finding fault with it.

Charles Dodgson (better known as the children's author Lewis Carroll – see *Function*, February 1994) had this criticism in his book *Euclid and His Modern Rivals* (written in the form of a dialogue, and in a rather whimsical style):

Minos: It is proposed to prove I.5 by taking up the isosceles triangle, turning it over, and then laying it down upon itself.

Euclid: Surely that has too much of the Irish bull about it, and reminds one a little too vividly of the man who walked down his own throat, to deserve a place in a strictly philosophical treatise?

Minos: I suppose its defenders would say that it is conceived to leave a trace of itself behind, and that the reversed triangle is laid down upon the trace so left.

This leads us into some rather interesting waters. Euclid I.4, which we have been using in the congruence proofs above is proved by imagining one triangle being picked up and "applied" to the other. It is a nice proof, and I can recall appreciating its force when I first encountered it. Strictly speaking, it is not a proof, because triangles are abstract entities, and can't really be picked up and "applied" like this. Still, it carries conviction.

And we can perhaps say the same of the "physical" version of the Pappus proof as described by Minos in Dodgson's dialogue. But the proof, as I have given it above, does not require the physical interpretation at all. Readers will find a nice discussion of the two (abstract and concrete) forms of the argument in Ian Stewart's *Concepts of Modern Mathematics* (Penguin, 1975).

The other place where the proof is (implicitly) criticised is in the UK journal *Mathematical Gazette* where it was submitted anonymously as an example of a student "howler" in 1937. This did, however, provoke a response the following year from C Dudley Langford, a prominent mathematical educator of the time. One hopes that the anonymous contributor and the editor of the day were both suitably chastened!

Many of the proofs in vogue in the twentieth century in fact tried to circumvent the Euclid proof and its variants altogether. They proceeded by drawing an axis of symmetry *AX* in the triangle of Figure 1. The difficulty is that this may be defined in any one of three ways (later found to be equivalent, but we are not able to assume this at this point in the development of the theory): *AX* could be

- The bisector of Angle *BAC*,
- The line joining *A* to the mid-point *X* of *BC*, or
- The perpendicular drawn from *A* to the line *BC*.

In my own introduction to geometry, we learned the first of these from Hall and Stevens' *A School Geometry*, and it was only after reading Dodgson's criticism that I realised that it involved a circular argument. This I detailed in an earlier paper in *The Mathematical Gazette* (*Vol* **74**, 1990), which also discusses all three variants in some detail.

Continued on p 68

PROBLEMS AND SOLUTIONS

FIRST, A CORRECTION

In Problem 27.1.2, reference was made to a "set of five cards". The word *five* should have read *six.* Thanks to J C Barton and Julius Guest for spotting the error.

SOLUTION TO PROBLEM 26.4.1

The problem began by defining a *convex function* $f(x)$ as one for which

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)$$

[It went on to give two examples that were, in fact wrong. However, two simple functions that *are* convex are x^2 and e^x. Our thanks to Keith Anker for detecting *this* error.] The problem continued by defining a *strongly convex function* as one for which

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)+(x-y)$$

whenever $x > y$. The challenge was to prove that no such functions exist.

Solutions were received from Keith Anker, Šefket Arslangić (Bosnia) and Carlos Victor (Brazil). The arguments were similar; we here follow that of Anker. He put $c = (a + b)/2$, $d = (a + c)/2$, $e = (c + b)/2$. Then if f is strongly convex, we have, by definition

$$f(c) \geq \frac{f(a)+f(b)}{2}+(b-a)$$
$$f(d) \geq \frac{f(a)+f(c)}{2}+\frac{b-a}{2} \geq \frac{3f(a)+f(b)}{2}+(b-a)$$
$$f(e) \geq \frac{f(c)+f(b)}{2}+\frac{b-a}{2} \geq \frac{f(a)+3f(b)}{2}+(b-a).$$

It follows that

$$f(c) \geq \frac{f(d)+f(e)}{2} + \frac{b-a}{2} \geq \frac{f(a)+f(b)}{2} + \frac{3}{2}(b-a).$$

Now if we had started with the inequality

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + \lambda(x-y),$$

instead of the one actually given, then we could show by means of the same argument that

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + \frac{3\lambda}{2}(x-y).$$

From this it follows, by replacing λ by $3\lambda/2$ over and over again that

$$\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right) + \left(\frac{3\lambda}{2}\right)^n (x-y),$$

for all n. Thus it is impossible consistently to assign a finite value to

$$f\left(\frac{x+y}{2}\right)$$

and the definition is revealed as self-contradictory.

SOLUTION TO PROBLEM 26.4.2

Exactly one of the following four statements is false:

(a) Audrey is older than Beatrice
(b) Clement is younger than Beatrice
(c) The sum of the ages of Beatrice and Clement is twice the age of Audrey
(d) Clement is older than Audrey.

Who is the youngest: Audrey, Beatrice or Clement?

Solutions were received from Keith Anker, Šefket Arslangić (Bosnia), Julius Guest and Carlos Victor (Brazil). Here is Arslangić's solution, which is the same as Anker's and Victor's, but slightly different from Guest's.

Statement (b) must be false, because if it were true then *both* of statements (c) and (d) would be false, contrary to data. Thus Clement is older than Beatrice. Statement (a), which we *now* know to be true, tells us that Audrey is older than Beatrice. So Beatrice is the youngest. [As Statement (d) is also now established to be true, Clement is older than Audrey, so Clement is the oldest.]

SOLUTION TO PROBLEM 26.4.3

This problem asked for the value of the integral

$$\int \{5(\sin^3 2x)^{1/2} / \sin^5 x\} dx .$$

Solutions were received from Keith Anker, Šefket Arslangić (Bosnia), J C Barton, Julius Guest (the proposer) and Carlos Victor (Brazil). Here is Guest's.

Introduce the substitution $t = \cot x$, so that $dt = -(1+t^2)dt$. Also

$$\sin x = \frac{1}{(1+t^2)^{1/2}} \quad \text{and} \quad \cos x = \frac{t}{(1+t^2)^{1/2}}$$

so that the integral becomes $-10\sqrt{2}\int t^{3/2} dt = c - 4\sqrt{2}t^{5/2} = c - (2\cot x)^{5/2}$.

SOLUTION TO PROBLEM 26.4.4

This problem asked for the value of the integral

$$\int \{x(1-x^2)\}^{1/3} dx .$$

Solutions were received from Keith Anker, Šefket Arslangić (Bosnia), J C Barton, Julius Guest (the proposer) and Carlos Victor (Brazil).

Barton put $t = x^2$, so that $x = \sqrt{t}$ and $2xdx/dt = 1$. This transforms the integral to $\frac{1}{2}\int t^{-1/3}(1-t)^{1/3} dt$. He now put

$$1 - t = tu^3 \text{ so that } \frac{dt}{du} = \frac{-3u^2 t}{(1+u^3)}$$

The integral is thus further transformed to

$$-\frac{1}{2}\int \frac{3u^3 du}{(1+u^3)^2} = \frac{1}{2}\int \frac{d}{du}(1+u^3)^{-1} u du .$$

Integrating now by parts yields

$$\frac{1}{2}u(1+u^3)^{-1} - \frac{1}{2}\int (1+u^3)^{-1} du .$$

Barton now comments that $\int (1+u^3)^{-1} du$ may be regarded as standard. It reduces to $\frac{1}{3}\ln|1+u| - \frac{1}{6}\ln(1-u+u^2) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2u-1}{\sqrt{3}}\right)$. Finally, putting all this together and restoring the original variable x, we end up with

$$x^2(x^{-2}-1)^{1/3} - \frac{1}{6}\ln\left|1+(x^{-2}-1)^{1/3}\right| + \frac{1}{12}\ln\left\{1-(x^{-2}-1)^{1/2}+(x^{-2}-1)^{2/3}\right\}$$
$$- \frac{1}{2\sqrt{3}}\arctan\left(\frac{2(x^{-2}-1)^{1/3}-1}{\sqrt{3}}\right) + c$$

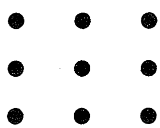
Barton comments "The chief merit of this problem is surely that it turns out to be elementary. The actual calculation is, practically, drudgery."

Now let us move on to the next set of problems.

PROBLEM 27.2.1 (submitted by Peter Grossman)

In a well-known puzzle, you are challenged to draw four line segments passing through nine points arranged in a 3×3 array (as shown below), without lifting your pen off the paper. Most people know that the solution requires some of the line segments to extend beyond the bounds of the array, but a justification for this claim is rarely given.

Prove that no solution is possible in which all of the line segments lie within the bounds of the array.



PROBLEM 27.2.2 (based in part on a problem in *Mathematical Bafflers*, ed Angela Dunn)

The game of periwinkle is the same as noughts and crosses (tic-tac-toe) except that the object is not to place three of your symbols in a row, but to *avoid* doing so. Show that the player moving first can always avoid defeat. Is the second player so lucky?

PROBLEM 27.2.3 (from *Mathematical Bafflers*, ed Angela Dunn)

ABC and *DEF* are two similar triangles, both with integral sides. Two of the sides of *ABC* are equal to two of the sides of *DEF*. The third sides are different from one another and the difference in their lengths is 387. Find the lengths of all the sides of both triangles.

PROBLEM 27.2.4 (from *The Australian Mathematics Teacher*, June 2002)

Hobson is in gaol but is given a chance of escape if he can make the right choice. Three boxes are presented to him, one of which contains the key to his cell, i.e. the means of escape. The three boxes are made of gold, silver and lead, and each carries an inscription. One and only one of the inscriptions is true. They are as follows:

On the gold box:	The key is in this box
On the silver box:	The key is not in this box
On the leaden box:	The key is not in the gold box.

Meanwhile Jobson, in another part of the same gaol, is given a similar opportunity. However in his case, at least one inscription is known to be true and at least one false. In his case they read:

On the gold box:	The key is not in the silver box
On the silver box:	The key is not in this box
On the leaden box:	The key is in this box.

How should Hobson and Jobson decide?

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Continued from p 62

Dodgson's point is that if the angle bisector is to be used, then we must be assured that it can be constructed. This is where Hall and Stevens went wrong.

I will have more to say on this subject in my next column.

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"Among [the Greeks] geometry was held in the highest honour; nothing was more glorious than mathematics. But we [Romans] have limited the usefulness of this art to measuring and calculating."

Marcus Tullius Cicero (106-43 BC)

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