Function

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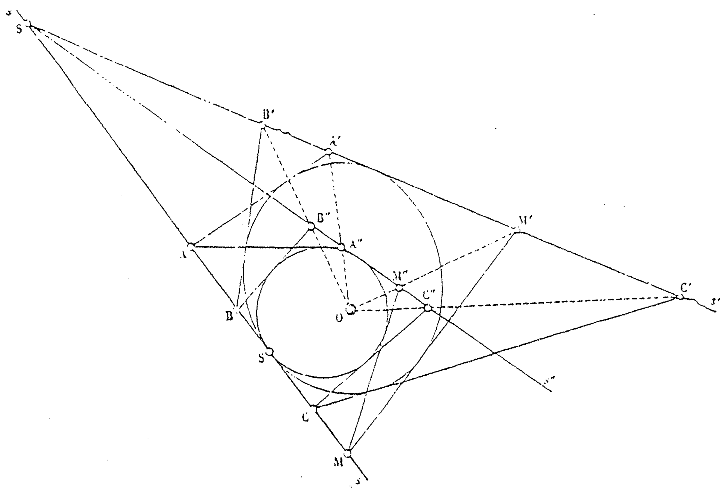

School of Mathematical Sciences – Monash University

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Function is a refereed mathematics journal produced by the School of Mathematical Sciences at Monash University. Founded in 1977 by Prof G B Preston, *Function* is addressed principally to students in the upper years of secondary schools, but more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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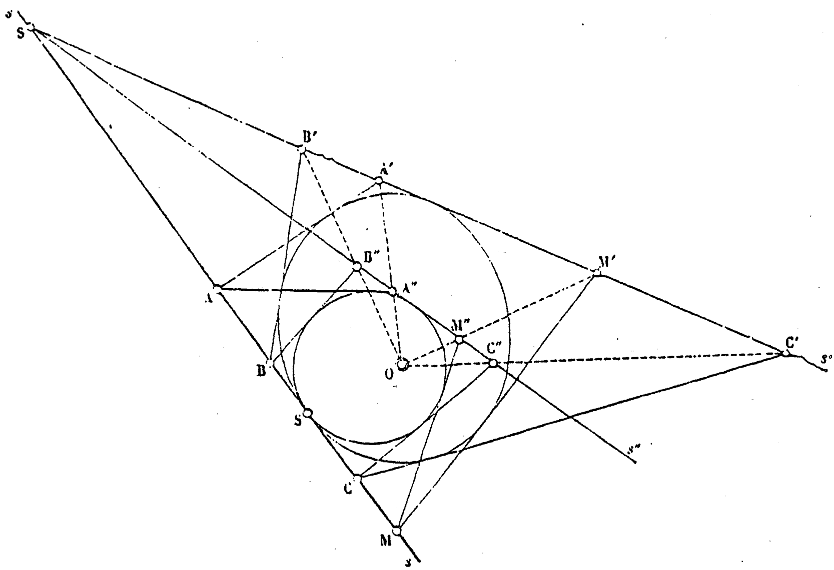
http://www.maths.monash.edu.au/~cristina/function.html

* $14 for *bona fide* secondary or tertiary students.

THE FRONT COVER

The study now known as Projective Geometry began its life with the study of perspective introduced into European art during the Renaissance. See our cover stories for April 1983 and August 1985 (and, for related material, June 1986 and October 1992). The cover diagram for this issue comes from the same tradition. It first appeared in an 1885 textbook, *Elements of Projective Geometry* by Luigi Cremona, very influential in its time and still an excellent introduction to the subject today.

This particular diagram (reproduced below for convenience) shows a circle and its perspective rendering as an ellipse. The circle appears at the lower right, and the perspective rendering is to be set up by means of the images of its tangents under systematic transformation. This transformation involves a fixed point *O*, here taken to be inside the circle (but not at its centre), and a line *s* which in this case is itself taken to be a tangent to the circle, tangent at one of the two points labelled *S*.



Points A, B, C, M are chosen on the line s. Two further lines s', s'' are also drawn, both meeting s in the other point labelled S. The line s'' is taken to be tangent to the given circle. The line s' is to be its perspective rendering and will be a tangent to the ellipse we are seeking to construct.

Through each of the points A, B, C, M draw tangents to the given circle, and let these meet the line s'' in the points A'', B'', C'', M'' respectively. Through the point O draw lines OA'', OB'', OC'', OM'' to meet the line s' in points A', B', C', M' respectively. Now construct the lines AA', BB', CC', MM'. These lines will be tangents to the required ellipse. If we draw enough such tangents, the ellipse will emerge as the envelope of these tangents.

(The notation reflects the understanding that many such lines are to be drawn. Perhaps nowadays, we would write $AA', BB', CC', \ldots\ldots, MM'$, etc, the better to clarify this understanding.)

Several times before we have used curves generated as envelopes as our front cover illustrations. A particularly nice one ushered in our new format in February 1994, but there have been plenty of others. One very elaborate example was the beautifully hand-drawn diagram for June 1979, which like this issue's example displayed an ellipse.

To get a new perspective (pun intended) on the cover diagram, think of it as representing a 3-dimensional situation. Let the lines s, s'' define a plane p'', which will contain as well the circle and its tangents AA'', etc. To fix ideas suppose that all these components have been fabricated (in wire, say).

Now let the lines s, s' define another plane p' at some angle to p''. The lines s', s'' serve to define a third plane, p say. Let l be a line passing through O and perpendicular to the plane p''. Suppose this line to meet the plane p in a point L.

Imagine a light placed at L and casting a shadow, onto the plane p', of the wire model making up the details of the plane p''. Then this is exactly what our cover diagram represents, but it has been shown in a perspective rendering.

The cover for our June 1986 issue may be similarly analysed.

SONNETS, MONKEYS AND EVOLUTION

Michael A B Deakin, Monash University

It seems to have been the French mathematician Émile Borel (1871-1956) who first posed the bizarre example I want to discuss. In a paper on Statistical Mechanics published in 1913, we find (in English translation):

> Imagine that we have organised a million monkeys each to strike at random the keys of a typewriter and that under the watchful eyes of illiterate overseers the simian typists work diligently for 10 hours a day on their million separate typewriters. The illiterate overseers then collate the typed pages and bind them into volumes. At the end of a year, these volumes are found to replicate exactly all the books on every subject and in every language held in the best libraries in the world.

Borel used this example several times in his writing. In his book *Probabilities and Life*, the French original of which appeared in 1943, he introduced the story under the name "The Miracle of the Typing Monkeys". Here is this version:

> A typist who knows no language other than French has been kept in solitary confinement with her machine and white paper; she amuses herself by typing haphazardly and, at the end of six months, she is found to have written, without a single error, the complete works of Shakespeare in their English text and the complete works of Goethe in their German text.

Now Borel is concerned to argue for his view that "Phenomena with very small probabilities do not occur", which is basic to his way of relating probabilities to everyday argument.

This account of the typing monkeys continues:

> Such is the sort of event which, though its *impossibility* may not be rationally demonstrable, is, however, so unlikely that no sensible person will hesitate to declare it *actually*

> *impossible.* If someone affirmed having observed such an event we would be sure that he is deceiving us or has himself been the victim of a fraud.

I recall first seeing the example in a popular work by Sir Arthur Eddington (1882-1944), who is remembered as an astronomer and theoretical physicist. In his book *The Nature of the Physical World* (1927), we read:

> If an army of monkeys were strumming on typewriters they *might* write all the books in the British Museum. The chance of their doing so is decidedly more favourable than the chance of the molecules returning to one half of the vessel.

For this and for many related quotes, see the interesting webpage

http://www.research.att.com/~reeds/monkeys.html

Eddington is using the Statistical Mechanics context of Borel's first account to discuss the Second Law of Thermodynamics, which makes the production of order out of disorder practically impossible. He seems to have been fond of the example, and is credited with the following limerick:

> There once was a brainy baboon
> Who always breathed down a bassoon
> For he said "It appears
> That in billions of years
> I shall certainly hit on a tune."

See:

http://www-history.mcs.st-and.ac.uk/~history/Quotations/Eddington.html

In Borel's later version, there are actually no monkeys at all; they have been replaced by a typist. He is here making a more general point and it is important to understand precisely what he is saying. He is arguing for his assertion that events with sufficiently small probabilities do not occur. He is *not* asserting (as some overhasty critics have alleged) that very small probabilities are actually zero. Rather he is concerned with *our response* to accounts of highly unlikely events. To see what is involved, I want to consider more closely the example he quotes.

As Borel has the story, the ask is a very difficult one indeed. So let us make it much easier, and suppose that the typist/monkeys are replaced by a computer whose output is to reproduce *just one of Shakespeare's sonnets*.

Now even the prospect of a computer or someone, anyone, let alone a monkey, happening by mere chance, to type a Shakespearean sonnet strikes us as remote in the extreme. So we would expect to assign a very small probability to this event. Let us begin by estimating how small that probability actually is.

A Shakespearean sonnet comprises fourteen lines of verse, each of ten syllables. Shakespeare himself wrote 154 sonnets. Perhaps the best-known is the eighteenth of the sequence, which begins:

> Shall I compare thee to a summer's day?
> Thou art more lovely and more temperate:
> Rough winds do shake the darling buds of May,
> Etc.

The first line contains 39 characters, including spaces and punctuation marks, the second comes in at 40, and the next comprises 45 characters. All in all, a line contains about 40 characters, so that an entire sonnet will exhibit about 560 characters, and perhaps we should also include in the count the thirteen spaces between the lines.

But let us make the task of producing one somewhat easier and suppose that a sonnet is made up of just 500 characters. What chance is there that a randomly generated set of 500 characters would be recognised as one of Shakespeare's sonnets?

My keyboard has 47 character keys, and the shift key ensures that each of these can generate two different characters. If we add to the list the space-bar and the line-shift, we have a total of 96 different characters. But let us make things easy for our poor computer. Let us suppose that it doesn't need to worry with punctuation, and let us further suppose that it is case-insensitive, and that it also is allowed to ignore the division into lines. So we imagine it with a simplified keyboard with only 27 characters to worry about.

Now Borel wanted the typing to be perfect, but we will be less stringent in our requirements. After all, Shakespeare himself was not exactly particular when it came to spelling, and even published modern editions differ from one to another. So let us allow our computer to make a few misprints. Quite a lot in fact. If up to ten of the characters could be mistyped then we would have over $\binom{500}{10}$ different possibilities. This number is about 2.4×10^{20}. But there are other possibilities also. We might leave out one or more letters or spaces; we might mistakenly insert characters that shouldn't be there. And so on. To err on the side of generosity, let us suppose that for each of Shakespeare's sonnets, there are 10^{25} variations that we are prepared to accept (as misprinted versions). Our use of 500 as the measure of the length may also well mean that we accept as Shakespearean sonnets versions actually missing a line or two, but let us in all these matters err on the side of generosity.

Now set our computer to work. It will print out a random selection of 500 characters, and it has 27^{500} possible ways of doing this. Because there are 154 different sonnets and we are allowing 10^{25} possible ways of generating each one, the chance of producing any one of them is $154 \times 10^{25} \times 27^{-500}$. This works out as less than 10^{-688}, an absolutely minute number!

To get a feel for how small this number really is, suppose our computer to keep generating would-be sonnets at the rate of one a second. The idea is that, like Eddington's baboon, it should eventually succeed. If the computer generates N "sonnets", then with probability $1-10^{-688}$ each will *not* be recognisable as being one of Shakespeare's. The probability that *all* of them fail in this way is $(1-10^{-688})^N$. Suppose we want to reduce this probability of failure to ½. What value of N would we need? This is an easily solved question. We need to satisfy the equation

$$(1-10^{-688})^N = \tfrac{1}{2}.$$

The solution is

$$N = \frac{-\log 2}{\log\left(1-10^{-688}\right)}$$

and it takes some work to estimate a value for this. (Not the sort of thing your average electronic calculator likes to see! Nor did Maple get very far with it either!) However, it's actually quite easy to estimate the value of *N*. We may take the logarithms to any base (check this!) and the best base to use is *e*. The denominator is then reducible to $\ln\left(1-10^{-688}\right)$, where we have now written ln in place of log, in order to emphasise that the logarithms involved are *natural* logarithms.

When x is small, we have $\ln(1-x) \approx -x$, and this approximation becomes very good when x is very small. I won't stop to prove this result, but merely note that if we apply it here, we find, to an excellent approximation, the denominator in the expression for *N* equals -10^{-688}. And so we have

$$N \approx 10^{688} \ln 2 \approx 7 \times 10^{687} .$$

This then is the number of *seconds* it would take for our computer to have a break-even chance of producing something like a real sonnet. If we convert this to years, the result is 2.2×10^{680}. If we use a larger unit and express the result in units equal to the age of the universe (about 1.3×10^{13} years), the result is about 1.7×10^{667} of *these* units. We would be waiting a long, long time!

It is often said that small probabilities are neutralised by multiple experiments (long times or very many repetitive instances). Thus *my* chance of winning Tattslotto in the next draw may be to all intents negligible, but a lot of people buy tickets, and this means that the chance of *someone* striking it lucky is actually quite high. It all depends on the *balance* between the unlikelihood of the event and the number of trials we can make.

It was once argued that evolution was dependent on such unlikely events (favorable mutations) that we could be practically certain that it could not have occurred. Versions of Borel's argument have been often used by opponents of evolution in order to discredit Darwin and his followers. Such arguments carried a lot of weight before it was realised how long a time was available for the process.

It is a little hard to estimate the small probabilities involved, but though the relevant events are rare they are not impossibly so. Indeed there have been actual instances of favorable mutations occurring in Nature (whereas the typing monkeys stay firmly in the realm of fiction). The most widely quoted case is that of industrial melanism. Moths that came to rest on the bark of trees and used camouflage to avoid their predators developed a black pigmentation in the industrial North of England, where the trees became coated with soot. Now as a result of clean air measures the moths are going back to their original mottled appearance.

This example (and there are others) demonstrates that the appearance of a favorable mutation (though rare) is not *so* rare as to make the basic mechanism of evolution practically impossible. The best estimates show that it is perfectly possible for life to have evolved here on earth, once it became established. In this case, the eons of geological time are long enough to allow events with small probabilities nonetheless to occur, and indeed to occur quite often.

What is still in some dispute is the *origin* of life itself. This involves the spontaneous coming together of the components of a large molecule with the capability of replicating itself. (Once such a molecule exists, the much faster process of replication means that the appearance of a second such molecule is much more rapid. And so on.)

Computation of the relevant probabilities is a matter still very much debated. On some accounts, the probability of life having originated here on earth is so minute that it is a bit like the case of the Shakespearean sonnets. This is argued by one school of thought, which holds that life needs to be carried from place to place in the universe. At the opposite end of the scale are those who argue that life can arise rather easily and so should be a widespread phenomenon, with other cases in other parts of the universe.

One of the more interesting approaches argues still differently. This argument goes that life does not easily arise spontaneously. The chance of life emerging in any one place, this view has it, is extremely small. On this view, and ruling out dissemination through the universe, life other than our own is most unlikely. So how come we are so lucky as to be alive, to have benefited from this monumental piece of luck? Well that is sampling bias; only conscious living creatures can even ask the question!

Acknowledgment. I thank Jim Reeds of A T & T Labs for directing me to his website (noted above) and so filling me in on much of the history and the folklore on the typing monkeys. Readers will find much pleasure and many delights in the quotes he has collected. I myself greatly enjoyed reading them.

THE THREE GUARDIANS

The puzzle discussed here is a hardy perennial and it comes in many versions. It was brought to our attention by Isaac Nativ and Lachlan Harris, whose version was, however, more complicated than that given here.

A traveller is embarked on a quest for treasure and, as one of the tasks his quest requires of him, he needs to identify three strangers whose duty it is to guard the treasure. These guardians can and will answer yes-no questions the traveller puts to them, but one ("the liar") is habitually dishonest, and always tells the opposite of the truth; another ("the truthteller") is scrupulously honest, and always answers truthfully; the third ("the diplomat") answers truthfully or dishonestly quite at random. Our hero must determine which is which.

Here is one way to proceed.

The traveller asks the first guardian: "If I were to ask you whether the second guardian is the diplomat, would you answer 'yes'?"

If the first guardian is the truthteller, then the answer will be 'yes' if the second guardian *is* the diplomat, and 'no' otherwise.

Now consider the case in which the first guardian is the liar. A straight question "Is the second guardian the diplomat?" would elicit the answer 'no' if the second guardian were in fact the diplomat, and 'yes' otherwise. However, asking the question in a more roundabout way makes it a compound question, so that the query "would you answer 'yes'?" will prompt the lie 'yes' in the first eventuality, 'no' in the second.

Thus the truthteller and the liar will both answer 'yes' to the first question if the second guardian *is* the diplomat, 'no' if not.

Finally there is the possibility that the first guardian is the diplomat. In this case, the second guardian is not the diplomat, nor is the third.

So, if the first guardian answers 'yes', then the traveler addresses the next question to the third guardian, who will not be the diplomat. If the first guardian answers 'no', then the traveller addresses the next question to the second guardian, who will not be the diplomat.

So now the traveller can ask the next question of whichever of these two guardians the first answer dictates. The next question is: "If I were to ask you whether the first guardian is the diplomat, would you answer 'yes'?" Now since the respondent to *this* question is *not* the diplomat, the answer will be delivered by either the truthteller or else by the liar, and either way the answer will be "yes" if the first guardian *is* the diplomat, "no" if not.

The traveller now knows whether the first guardian is or is not the diplomat. If the first guardian *is* in fact the diplomat, then the third question is addressed to the same respondent as the second question was. If not, then the same question is addressed to the first guardian. Either way the third question will be answered by a guardian other than the diplomat.

That third question is similar to the first two. "If I were to ask you if you are the truthteller, would you answer 'yes'?". From here on in, the reader can supply the detail.

Three comments, however, are in order.

The first is that, in this simplest of versions, it is in fact possible to replace the second and third questions with others that are less convoluted. However, this cannot be done with other more subtle versions of the puzzle, including that supplied by Isaac Nativ and Lachlan Harris.

The second is that the liar has to be a very sophisticated liar in order to follow all the ramifications of the questions as asked! We may avoid this complication if we allow four questions (again in this simple version of the puzzle).

The third is that the answer given above presupposes that the guardians themselves know which of them is which. The problem would seem to be impossible (with only three questions) if this is not the case.

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LETTERS TO THE EDITOR

Another Challenge to Orthodoxy!

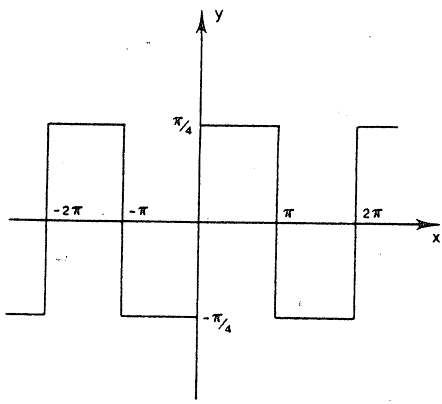
I had almost despaired of hearing from my rather erratic correspondent Dai Fwls ap Rhyll, who tends to contact me about this time with some new challenge or other to mathematical orthodoxy. I had learned indirectly that he is not in the best of health and that he lives an isolated and spartan life in the small Welsh village he has made his home for well over 20 years. I begin to think that he cannot go on forever finding more and more difficulties with the Mathematics taught in our schools and universities.

However, I did hear from him recently. Quite clearly he is in a bad way and I have had to reconstruct his latest research from some very sketchy details written in a pitifully shaky hand. I have made it my practice to send him copies of *Function* almost from the first, and it is to an old cover story that he directed my attention.

"F10/1", he wrote, and even this took me some pains to decipher. His scrawled note went on: "TFC Eqn (*)", which I took to mean "Look up Equation (*) of the Front Cover story". So I did exactly this and found that it said, in *Volume* **10**, *Part* **1**:

$$f(x) = \sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...,$$

and that it referred to a function $f(x)$ whose graph I have copied, and reproduce here.



This would seem to imply that for any x in the interval $0 < x < \pi$, we should have

$$\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ... = \pi/4 .$$

This I think was the conclusion he wanted me to draw, but I did have to interpret some almost illegible hieroglyphics to reach it. Next came another scrawl that, try as I might, I could not interpret as other than a derisive swearword, which I feel it would be out of place to send to *Function.*

Suffice it to say that the good Doctor Fwls evidently did not believe this equation.

Next I read: "100 terms, $x = \pi/200$". It did not take me long to set this up in EXCEL, and lo and behold, I found that the sum came not to $\pi/4$ at all but to an amount 1.179 of this figure. This gave some substance to the next line of Dr Fwls' note which I deciphered as "18% out!"

Thinking that maybe I had not taken enough terms, I had the spreadsheet compute the sum to 200 terms, and used an even smaller value of x ($\pi/400$) to make the convergence even better. The result was the same: an error of 17.9%. As I considered the effect of adding even more terms for even smaller values of x, this result did not vary substantially. If I took 1000 terms and $x = \pi/2000$. I still saw the same result.

I can only think that the textbooks must have it wrong in this matter as in so many others that my correspondent has brought to my attention over the years.

Often I tend to disbelieve him. Could it be, I ask myself, that so many brilliant mathematicians have been wrong about so much and for so long? But once again I am left with the irrefutable evidence of his simple and straightforward demonstration.

It all leaves me most confused.

Kim Dean
Erewhon-upon-Yarra

FROM THE NEWSPAPERS

More on Astrology

The Age (24/1/'02) carried a feature article by David Vaux, a principal research scientist with the Walter and Eliza Hall Institute. Its topic was the continuing attraction of pseudoscience, and one of the aspects of this that he covered was Astrology. Dr Vaux pointed out an additional complication not mentioned in Professor Westfold's discussion in our previous issue.

Things have changed over the last 2000+ years. The fixed pattern of the stars very slowly alters its relation to the earth's orbit around the sun. The effect is referred to as "the precession of the equinoxes", and it means that the (Northern Hemisphere) Spring equinox, the date on which the length of the day there begins to exceed that of the night, alters over the centuries. It used to coincide with the entry of the sun into the constellation Aries. However, the pattern has now altered in such a way that the sun is now in the constellation Pisces at the time of the spring equinox.

The "first point of Aries" remains the origin of the celestial coordinates used by astronomers. It is defined as the point where the celestial equator crosses the ecliptic, just as Professor Westfold explained. However, the sun is no longer coincident with the star in the constellation Aries that gave this origin its name. The heavens have moved on! (Or, rather, the earth's axis has.)

Dr Vaux writes: "The precession of the equinoxes is due to changes in the angle of the axis around which the earth rotates, and performs one complete circuit in 26,000 years. Because astrologers still calculate from Aries instead of the current March equinox location in Pisces, all modern horoscopes are out of phase with the actual stars."

He directed our attention to the website

http://star-www.st-and.ac.uk/~fv/webnotes/chapter16.htm

for a fuller discussion, and there is another account at

http://cse.ssl.berkeley.edu/lessons/indiv/beth/beth_precess.html

In particular, click on the Figures to see more detail. There are also other sites with good discussions.

Yet another recent article on Astrology, but with a different focus, is to be found in *The Age,* 13/2/2002.

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More on Biased Coins

Function's History columns for June and August 2000 contained some inconclusive discussion on biased coins, and the US 1c piece in particular. We now learn of another possible example. In *The Age* (5/1/'02) there is a story on the new eurocoinage. We quote.

"While the notes that began circulating in the 12 members of the eurozone on January 1 are all the same, the coins show national symbols on one side and a map of Europe on the other."

The Belgian version of the one euro coin has a portrait of King Albert on its "heads" side, and if the coin is spun on a flat surface, it comes to rest with this side up in more than 50% of cases. Two professors, Tomasz Gliszczynski and Waclaw Zawadowski, of the Podlaska Academy in Siedlce, together with a group of students, spun the Belgian one euro coin 250 times and found a result of 140 heads. The experiment was repeated in the office of the London newspaper *The Guardian* and the result was 139 heads.

The chance of a coin landing heads 140 or more times out of 250 spins is about 3% if the coin is in fact unbiased, as the head of the Belgian mint claimed it was. This result is sufficiently extreme to be regarded as statistically significant. Barry Blight, a statistician at the London School of Economics, commented that "if the coin were unbiased, the chance of getting a result as extreme as that would be less than 7%". (Blight was considering the case of the coin landing *heads or tails* 140 times in 250 spins; this doubles the likelihood.)

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A BEAUTIFUL PROOF

Our cover story shows a diagram that may be regarded as either a two-dimensional figure or else a perspective rendering of a three-dimensional one. Figure 1 below shows another.

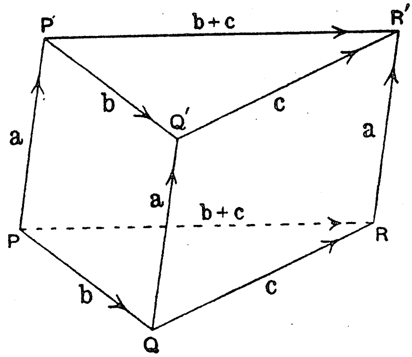

Figure 1

This diagram is taken from C E Weatherburn's *Elementary Vector Analysis*, and its purpose is to illustrate a proof of the vector identity

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c}).$$

However, before we get to the details, a few notes are in order. Weatherburn was the first professor of Mathematics at the University of Western Australia, a post he took up in 1929. By then he had already published his *Elementary Vector Analysis*, which first appeared in 1921 and was reprinted many times before its revision in 1955. In the early years of the twentieth century, the status of Vector Analysis was still controversial (see *Function*, June 1981) and it was Weatherburn's texts that helped to establish it in the place it holds today.

The other point to make is that areas are vector quantities and that the direction of the vector is one normal to the plane of the area, and in the direction given by a right-hand rule. This leads to a very powerful result: *The total vector area of a closed polyhedron is zero.*

We won't stop here to prove this result, although proofs are widely available to any reader who wishes to look into this. Our focus here is the use to which Weatherburn put the result. We quote from his proof.

"Suppose first that the vectors, [i.e. **a**, **b**, **c**] are not coplanar, and consider the triangular prism whose three parallel edges have the length and direction of **a**, and whose parallel ends PQR and $P'Q'R'$ are triangular with [sides **b**, **c** and **b** + **c**]. The sum of the vector areas of this closed polyhedron is zero [see above]. But these areas are represented by the outward normal vectors $\frac{1}{2}\mathbf{c} \times \mathbf{b}$ and $\frac{1}{2}\mathbf{b} \times \mathbf{c}$ for the triangular ends, and $\mathbf{b} \times \mathbf{a}$, $\mathbf{c} \times \mathbf{a}$, $\mathbf{a} \times (\mathbf{b} + \mathbf{c})$ for the other faces. Of these five vectors the first two are equal in length but opposite in direction. Hence the sum of the other three must vanish identically; that is

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a} = \mathbf{0},$$

which is equivalent to

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

"This proves the distributive law for non-coplanar vectors."

Weatherburn then goes on to prove the special case in which the vectors **a**, **b**, **c** are coplanar.

"If the vectors are coplanar, Figure 1 may be regarded as a plane figure. The triangles PQR and $P'Q'R'$ are congruent, and therefore the sum of the areas of the parallelograms $PQP'Q'$, $Q'QRR'$ is equal to that of $PRR'P'$. Hence the relation

$$\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} + \mathbf{c})."$$

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"The good Christian should beware of mathematicians, and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell."

St. Augustine of Hippo (354-430)

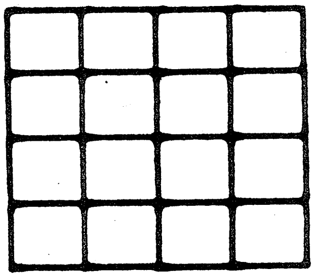
[However, the word translated as "mathematicians" should more correctly be rendered as "astrologers".]

TWENTY YEARS AGO

The Very Best Match Puzzle

In the early 1980s, puzzles based on patterns of matches were printed on the back of matchboxes. They generated a lot of interest at the time, and, courtesy of the Wilkinson Match Company (Bryant and May), *Function* obtained a complete set, together with their solutions. From the mathematical point of view, the most interesting was No 23 in the set, which we reproduce here.

MATCH TRICK No. 23



What is the smallest number of matches you can remove so that no square of any size is left?

This pattern contains 40 matches, and we can count 16 squares of side 1, 9 of side 2, 4 of side 3 and 1 of side 4, for a total of 30 squares altogether. We noted then that there could be simpler versions of the puzzle with:

(a) 4 matches making 1 square,
(b) 12 matches making 5 squares, and
(c) 24 matches making 14 squares.

Or we could have more complicated versions, such as

(d) 60 matches making a total of 55 squares.

We challenged readers to derive the general formula.

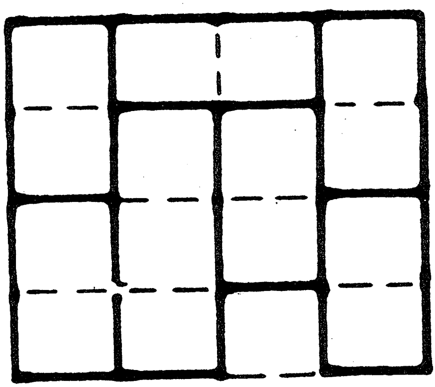
After this, we analysed the problem as follows.

"It is not difficult to show that in Case (a) we need to remove one match to break up the (only) square, in Case (b) we need to remove 3 matches to break up all 5 squares, and in Case (c), a little harder, 6 matches to break up all 14 squares."

Because these three numbers 1, 3, 6 are the first three triangular numbers (see *Function*, *Vol* **25**, *Part* **3**), we might guess that the solution in the case given on the matchbox was the next triangular number (10).

However, this solution is incorrect. Below is the solution provided by the Wilkinson Match Company.

SOLUTION



This shows that the desired result can be achieved by the removal of only 9 matches. Our story continued.

"Even this, though, is not the end of the problem. How do we know there is not an 8-match solution? We could get a computer to check all $\binom{40}{8}$ possibilities for us, but here is a simpler proof due to Derek Holton, [then] one of *Function's* editors.

"Imagine each small square (of side 1) coloured black or white to make up a 4×4 checkerboard. Next note that the largest (4×4 square) must be broken, so at least one of the outer boundary matches must be removed. We may suppose, without loss of generality, that it comes from a white square.

"Now consider the set of 8 small black squares. *No two of these have any match in common.* It follows that to break all these, we must remove 8 matches. Thus, at a very minimum, we need to remove 1 + 8 = 9 matches. That this minimum is achievable is proved by the diagram.

"There seems to be no formula known for the general case. An interesting investigation would be the design of efficient computer techniques for giving the minimum in any particular case."

A few remarks are in order for the general case. Consider an $n\times n$ pattern. This will use $2n(n+1)$ matches, and will contain a total of $\frac{n(n+1)(2n+1)}{6}$ squares. If n is even, we may apply Derek Holton's argument to show that we need to remove at very least $\left(\frac{n}{2}\right)^2+1$ of the matches to break all the squares. Whether this minimum number can always be achieved needs further investigation. Matters are more complicated for odd n.

HISTORY OF MATHEMATICS

Diophantus and his Legacy

Michael A B Deakin

Diophantus is today seen as one of the greatest and most unusual mathematicians of Ancient Greece. Despite the importance that has long attached to his name, very little is known about him. He is usually said to have been born in about AD200, and to have died in about AD284. However, both of these dates are subject to uncertainties of over 100 years. For a fuller account of the evidence on this point, see the discussion at the website

www-history.mcs.st-and.ac.uk/~history/Mathematicians/Diophantus.html

We might wonder at the supposed precision of the 84-year lifespan when there is so much imprecision about everything else. The length of his life is based on a puzzle due to a later writer, Metrodorus, and dating from about 500 AD. It goes (referring to Diophantus)

> " ... his boyhood lasted 1/6th of his life; he married after 1/7th more; his beard grew after 1/12th more, and his son was born 5 years later; the son lived to half his father's age, and the father died 4 years after the son."

We leave the details to the reader, but they have him marrying at the age of 26 and fathering a son who died at the age of 42, four years before Diophantus himself died aged 84. However, all this alleged information may well be completely fictitious!

He is now seen as important for two reasons. The first was that he used abbreviations in his mathematical discussions, and so paved the way for modern algebraic notation. But even more importantly he discussed problems of a type different from those that occupied other mathematicians. He posed problems which we now see as essentially equations, but insisted that the solutions be rational numbers.

This calls for a different approach from the usual, and has led to the development of an entire branch of Mathematics, called Diophantine Analysis in his honour. His principal work is generally called the *Arithmetic*, and we know from his own description of it that it originally comprised 13 "books", of which today only six survive in the original Greek.

It was once thought that these six surviving books were the first six, and it was even hypothesised that they survived as a result of their appearance in a fifth century commentary by Hypatia (whose story was told in this column in February 1992). However in 1973 new evidence came to light that made this theory untenable. An Arab manuscript was found in the Mashhad Shrine Library (Iran), and this proved to contain four more books of the *Arithmetic*. It was clear that these were Books Four to Seven of the original thirteen, but in Arabic translation.

So now we have ten of the books: Books One to Seven and three others of the remaining six. (It is still not clear which of these they are.)

In order to get a feel for what is involved may be gained from Problem 8 of Book Two. This reads (in modern translation):

> To divide a given square number into two squares.
>
> Given square number 16.
>
> x^2 one of the required squares. Therefore $16 - x^2$ must be equal to a square.
>
> Take a square of the form $(mx - 4)^2$, m being any integer and 4 the number which is the square root of 16, e.g. take $(2x - 4)^2$, and equate it to $16 - x^2$.
>
> Therefore $4x^2 - 16x + 16 = 16 - x^2$, or $5x^2 = 16x$, and $x = \frac{16}{5}$.
>
> The required squares are therefore $\frac{256}{25}, \frac{144}{25}$.

This is one of the most famous of Diophantus' derivations. Even though the modern translation uses modern algebraic notation, there are problems of interpretation. At first sight it seems only to find one solution among many and only to apply to the case where the given initial square number is 16. It *is* possible to read Diophantus as merely giving this rather specialist result. However, modern interpretation differs and sees Diophantus as *illustrating* a more general result by means of a general *method.*

We can replace the special number 16 by a more general square, y^2 say, where y is rational. Diophantus, although he took the first steps toward modern algebraic notation, did not get far enough along this road to do this. But now we can trace through the steps given above. Let x^2 be one of the required squares. Therefore $y^2 - x^2$ must be equal to a square. Take it to be a square of the form $(mx - y)^2$, m being rational. Therefore, after some work which I leave to the reader, we discover

$$\frac{y}{x} = \frac{1+m^2}{2m}$$

which is equivalent to the general formula as now given for the solution of this problem.

This problem in Diophantus' *Arithmetic* was the one which prompted the later mathematician Pierre de Fermat to write a famous marginal note in his personal copy, and so to state "Fermat's Last Theorem", which has only recently been proved. (See *Function's* History columns for August 1992 and April 1994.)

The newly discovered material has likewise been a fruitful source of problems. It has been published in three forms. Roshdi Rashed edited an Arabic version, which later provided the basis of a French translation; Jacques Sesiano has produced an English translation and commentary. This text, *Books IV to VII of Diophantus' Arithmetica*, is the most accessible for Australian readers. Rashed and Sesiano are the two acknowledged experts on this new material. Sadly they differ greatly and acrimoniously on a wide range of issues, so it is not possible for those of us who lack their considerable expertise to know quite how things stand.

There are four different lines of specialisation that need to come together in a study of this sort. First, one needs a sound knowledge of Arabic, second, a good acquaintance with the original language (Greek), third, a considerable grasp of the relevant Mathematics, and finally a good historical sense. My own judgement is that Rashed beats Sesiano at

Arabic and Mathematics, but that for the other two the boot is on the other foot.

It is agreed that the books in question are indeed those listed in Sesiano's title, and that they are based on Diophantus' original. However, they have undergone considerable revision, beyond the mere translation from one language to another. Much additional material has been incorporated, but it adds very little of substance. It tends to consist of rather repetitive summaries and detailed checks on the answers. It would seem that this material has been prepared not from Diophantus' original, but rather from a commentary based on it. One theory is that this commentary is Hypatia's.

The translation itself, from Greek to Arabic, is usually attributed to Qustā ibn Lūqā, who lived around or after the middle of the ninth century. Along the way, it may be that either Qustā himself or some other hand included additional material. (It is widely thought that even in the Greek material that has come down to us in its more or less original form, some is not part of what Diophantus actually wrote.)

Sesiano believes that two propositions in particular have been interpolated in this way into the new material. The first is Problem 11 of Book Six, which in our modern notation is

$$x^3 + x^6 = y^2.$$

The original text is here somewhat garbled and Sesiano has rather given himself licence to replace it with an analysis of his own. However, his analysis provides the solution to the problem, even if it may not be what Qustā ibn Lūqā (let alone Diophantus) actually intended.

Sesiano puts $y = nx^3$, where n is such that $n^2 - 1$ is a perfect cube. This gives a subsidiary equation $n^2 - 1 = m^3$. This subsidiary equation is known to have only one solution: $n = \pm 3, m = 2$. (OK, two solutions if you insist, but one is clearly just a simple modification of the other!) Once this substitution is made, it is a simple matter to find the solution $x = \frac{1}{2},\ y = \frac{3}{8}$. Furthermore, this solution is unique (apart from a trivial change of sign and the equally trivial possibility $x = y = 0$).

The fact that the subsidiary equation has a unique solution is a result due to Euler, who lived in the eighteenth century. It is now seen as a special case of other more general theorems related to what is known as "Catalan's Conjecture" (Paul Ribenboim has written an entire book devoted to this conjecture).

Sesiano sees the material of this problem as being rather trivial. This is because it does not use the uniqueness result (which Diophantus is unlikely to have known) but merely the observation that $3^2 = 2^3 + 1$, which must have been known from very early times indeed. So Sesiano is inclined to look on this problem as having been interpolated by someone different from Diophantus. Rashed disagrees, but in doing so he tends to attribute the significance of the problem to the uniqueness result.

A similar disagreement concerns a later problem, Number 17 of the same book. We would write this today as

$$x^2 + x^4 + x^8 = y^2.$$

Here is one way to approach this problem.

Put $a = y - x^4$, where because x and y are rational, then so is a. Now substitute $y = x^4 + a$ into the given equation. We then find, after some simplification, that

$$x^2 + x^4 = a^2 + 2ax^4$$

and this equation can clearly be satisfied if we set $a = x = \frac{1}{2}$. We thus easily find the solution $x = \frac{1}{2}$, $y = \frac{9}{16}$, but this analysis does not settle the question of whether there are other possible solutions (again discounting trivial variations and the obvious $x = y = 0$.)

Thus far I have essentially followed Sesiano, but once again, it is Rashed who raises the question of uniqueness. At that time (1985) it was known that there could be at most a finite number of solutions, but whether they reduced to a single one (apart from the trivial extensions) was then still an open problem. However, it was not to stay that way for long. The solution is indeed unique, a result first proved by a young American number theorist, Joseph L Wetherell.

Wetherell kindly sent me a copy of his proof, but its details were complicated, and I lacked the specialist knowledge to follow all it in all its particulars. However it passed more expert scrutiny than mine and appeared in the technical literature. More recently Wetherell has teamed up with a colleague, Victor Flynn, and considerably extended the underlying theory. Flynn and Wetherell revisited the problem from this new perspective and considerably simplified Wetherell's earlier proof.

To this end, they rewrote the equation as

$$Y^2 = X^6 + X^2 + 1,$$

where $X = x$ and $Y = y/x$. This was the equation they then analysed to produce the uniqueness proof, for clearly the new variables will be rational if the old ones are, and vice versa. This rewritten equation was the one they submitted to analysis.

The joint paper by Flynn and Wetherell appeared in the technical journal *Manuscripta Mathematica* in 1999.

Samir Siksek, who summarised it for *Mathematical Reviews*, wrote that he "expects that the method used [in this second analysis] will soon become a standard approach for attacking [such problems]".

So we see that the legacy left by Diophantus still continues to provide challenging problems and to stimulate creative Mathematics.

I do hate sums. There is no greater mistake than to call arithmetic an exact science. There are hidden laws of number which it requires a mind like mine to perceive. For instance, if you add a sum from the bottom up, and then again from the top down, the result is always different.

Mrs. La Touche, 19th C.

MATHS ON SCREEN AND STAGE

Another Mathematical Movie

On the heels of movies like *Stand and Deliver* and *Good Will Hunting*, with their mathematical themes, comes yet another, the much-Oscared *A Beautiful Mind*. It is based on the life of John Nash, and in particular on a biography, with the same name, by Sylvia Nasar.

Nash was born in 1928, and early showed an interest in and aptitude for Mathematics. An account of his life may be found at

http://www-history.mcs.st-and.ac.uk/~history/Mathematicians/Nash.html

He gained a doctorate in 1949 for research on the Theory of Games, although he has also made significant contributions in other areas of Mathematics as well. However it is for the game-theoretic work that his name is most honored. The Theory of Games does apply to games but also much more widely: to decision-making in general, and most especially to economic decisions. *Function* has carried several articles in the area, most recently in April 1998.

Nash has given his name to the Nash equilibrium and to Nash's Theorem. These apply to non-cooperative zero-sum games, that is to say to situations in which n "players" all compete with each of the other $n - 1$ for some payoff, without any collusion between players and where the only way any player can gain more is at the expense of others.

A Nash Equilibrium is a situation in which no player can gain any advantage by means of a unilateral change of strategy. Nash's Theorem ensures the existence of a Nash Equilibrium under the given conditions. This result has been seen to be so significant that Nash shared the 1994 Nobel Prize in Economics in consequence.

(The entrepreneur and inventor Alfred Nobel endowed prizes in Chemistry, Literature, Peace, Physics and Physiology & Medicine. These have been awarded since 1901. More recently, the Bank of Sweden endowed another, recognising advances in Economic Science. This has been awarded since 1969. There is no Nobel Prize in Mathematics as such; *Function* has carried a number of articles on this question, most recently in our last issue.)

However, it is unlikely that a film would have been made of Nash's life were it not for another fact: Nash has had a long battle with schizophrenia. (He now believes himself to be cured.) This certainly makes for an interesting, even inspirational, movie but it seems a pity that Hollywood could not celebrate Mathematics *per se*. A similar situation has arisen with another recently released film, *Iris*, detailing the life of the novelist Iris Murdoch and her decline into dementia. Of this latter film, the critic Philippa Hawker wrote (*Age EG*, 8/2/'02):

> "But her achievements haven't made her the subject of this film, derived from two books by her husband of more than 40 years ... which focused on the Alzheimer's that afflicted the end of her life. Without the fact of her illness, and the intimacy of the memoirs, there wouldn't have been a movie.
>
> "There's something slightly unsettling about this – the notion that Murdoch is now likely to be defined by what undid her, rather than by what she did and wrote."

The case of *A Beautiful Mind* seems similar.

Fermat's Last Tango

Where filmmakers fear to tread, others enter boldly! Recently a musical entitled *Fermat's Last Tango* ran for six weeks at the New York theatre in New York. It is by Joshua Rosenblum and Joanna Sydney Lessner and is based on Andrew Wiles' proof of Fermat's Last Theorem. (For details of this, see *Function*, April 1994.) The reviews have been described as "mixed". The original title was to have been *Proof*, but that one got used by another production (see our previous issue) and so a new name had to be found.

Those interested in learning more of this work should look up

http://www.claymath.org/events/fermatslasttango.htm

and a CD of the work itself may be ordered (for $US18) from

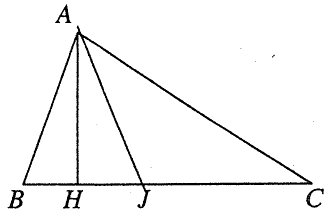
http:www.fermatslasttango.com

PROBLEMS AND SOLUTIONS

More on Problem 25.2.3

This problem (submitted by Claudio Arconcher) generated a large correspondence and some considerable controversy arising from differing interpretations as to quite what it entailed. In our previous issue, an interpretation was supplied, and here a full solution (promised then) is presented, based on this interpretation.

The diagram below will serve to clarify matters. It has no purpose beyond this, as the task set is to *produce* the diagram from some of its features.



Three collinear points *H*, *J* and *M* (the last not shown on the diagram) are given.

Our task is to use the these three points *and nothing else* to construct the triangle *ABC* in such a way that $\angle BAC$ is a right angle, that *J* is the mid-point of *BC*, that *AH* is perpendicular to *BC* and that *AM* bisects $\angle BAC$.

We may assume without any loss of generality that *H* lies to the left of *J*.

Next proceed to a few generalities. Because $\angle BAC$ is a right angle, the point *H* lies *internally* on *BC*, that is to say that under the conventions of the diagram, B lies to the left of H and C to its right. Because J is assumed to lie to the right of H, we easily infer that we need $AB < AC$. All this is compatible with the diagram as shown.

Now consider the position of M. Apply the sine rule to the triangle ABM. This tells us that

$$\frac{BM}{\sin 45^o} = \frac{AM}{\sin B}$$

and likewise from the triangle ACM

$$\frac{CM}{\sin 45^o} = \frac{AM}{\sin C}$$

It follows that $\dfrac{BM}{CM} = \dfrac{\sin C}{\sin B} = \dfrac{AB}{AC}$.

Thus $BM < CM$, and so the point M lies less than halfway along BC. Hence M must lie to the left of J, and if the problem is to have a solution, then the points H, M and J must lie in this order with M between H and J. (It will later emerge that an even more stringent restriction is necessary.)

In order further to analyse the problem, place the origin of a set of co-ordinates at H with the x-axis along the line HJ and positive in this direction. Then B will be the point $(-x, 0)$ and C will be the point $(y, 0)$, where by our assumptions x and y are both positive and $y > x$. A will be the point $(0, z)$ and without loss of generality we may assume $z > 0$. x, y and z are the unknowns, and if we can find them by means of constructible formulae then the problem is solved. The givens are the positions of J and M. Let J be the point $(h, 0)$ and M the point $(k, 0)$. The data are the values of h and k, where by the above analysis $k < h$.

We now derive three equations for the three unknowns x, y, z. In the first place, because J is to be the mid-point of BC,

$$y - x = 2h. \qquad (1)$$

Because AB and AC are perpendicular, we easily discover that

$$xy = z^2. \qquad (2)$$

Finally, because $\tan \angle MAC = 1$, we have, after some simplification,

$$z(y-k) = ky + z^2. \qquad (3)$$

Equation (1) then allows us to eliminate x to leave two equations in the two unknowns y and z. One of these is Equation (3) and the other is

$$z^2 = y(y-2h). \qquad (4)$$

If we now substitute this expression into Equation (3), we reach an expression for z in terms of y. Substitute this now into Equation (4) to find

$$\left(\frac{y(y+k-2h)}{y-k}\right)^2 = y(y-2h). \qquad (5)$$

This equation may be simplified to give (since y is non-zero)

$$(2k-h)y^2 - 2h(2k-h)y + hk^2 = 0. \qquad (6)$$

The roots of this equation are

$$y = h \pm (h-k)\sqrt{\frac{h}{h-2k}}$$

For these roots to be real, we need $h > 2k$, the "more stringent condition" referred to above. That is to say that we must have

$$\frac{HJ}{HM} > 2$$

if the problem is to possess a solution.

In this case, there will be one positive and one negative root. Since we assumed that y was positive, we have

$$y = h + (h-k)\sqrt{\frac{h}{h-2k}}$$

The negative root is in fact $-x$, as can be checked from Equation (1). So

$$x = -h + (h-k)\sqrt{\frac{h}{h-2k}}$$

We can complete the solution by reference to Equation (2), from which we find, after some simplification,

$$z = k\sqrt{\frac{h}{h-2k}}$$

Now to the construction. Here is one possibility among several. A length $k\sqrt{\frac{h}{h-2k}}$ may be constructed, using ruler and compass alone, by the methods outlined by D F Charles in *Function*, *Vol* **23**, *Part* **4**. The perpendicular through H to the originally given line is readily constructed, so the point A is determined. There are now several ways to complete the full construction, but these we leave to the reader.

Further solutions are being held over. It has been brought to our attention by two of our overseas correspondents that there was not enough time between the publication of a problem and its solution to allow them to consider the problems fully and yet meet the demands of the mail. Accordingly our new policy shall be that the solution to each problem will appear three issues later than the problem itself (rather than two, which has been our practice for some time).

In accordance with this new policy, the solutions to the problems set in *Volume* **25**, *Part* **5** will be held over till the next issue.

We proceed immediately to four new problems.

PROBLEM 26.2.1 (the second of the three "Professor Cherry" problems, this one from p 44 of Todhunter's *Algebra*; see the note accompanying Problem 26.1.1)

Show that

$$\{(a-b)^2+(b-c)^2+(c-a)^2\}^2 = 2\{(a-b)^4+(b-c)^4+(c-a)^4\}.$$

PROBLEM 26.2.2 (adapted from *Revista de Matematică din Timişoara*, Romania)

If x, y, z, w are real numbers, show that

$$x^4+y^4+z^4+w^4 \geq 4xyzw$$

and list all the cases for which equality holds.

PROBLEM 26.2.3 (from the same source)

Show that, if a and b are real numbers and n a natural number, then

$$\left(\frac{a+b}{2}\right)^n \leq \frac{a^n+b^n}{2}$$

Hence solve the equation

$$\sqrt[n]{x-1}+\sqrt[n]{3-x}-\sqrt[n]{x-2}=2,$$

where n is a positive integer.

PROBLEM 26.2.3 (in part from *Mathematical Bafflers*, edited by Angela Dunn)

Let N be the product of four consecutive positive integers. Prove:

(1) N is divisible by 24
(2) N is not a perfect square.

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