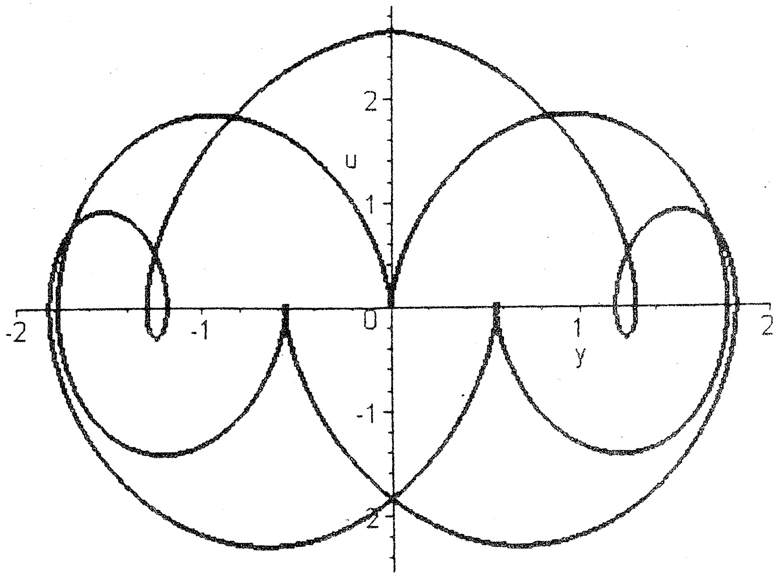
Function

A School Mathematics Journal

Volume 25 Part 3 June 2001



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Reg. by Aust. Post Publ. No. PP338685/0015

Published by Department of Mathematics & Statistics, Monash University

The Front Cover

For our Special Issue to mark 20 years of *Function*, we reran the cover from our very first issue, except that we redid the graphics, using more modern software. The then chief editor produced a better version "in a matter of seconds", whereas 20 years before, the cover graph "had to be produced by computing experts". This prompted us to look at another of *Function's* more memorable covers: the one from *Volume* 3, *Part* 2, which showed a curve with the strange name of "Murphy's Eyeballs".

Perhaps we had better recap. We will say later how the curve arises. For the moment, let us just note that it is very difficult to produce this curve, and probably impossible without a computer. The Murphy after whom it is named is the Murphy of the famous "Murphy's Law": *If anything can go wrong, it will.* The curve was so named by Professor H T Davis, whose computing laboratory at Northwestern University (Illinois) first explored it. As Davis remarked, "[during] the investigation, Murphy's eyes were constantly upon the computation."

And as *we* remarked, "[they] certainly were, for Davis' version contains several inaccuracies." To see what we were on about, compare the two figures below. Davis's is to the left, ours to the right.

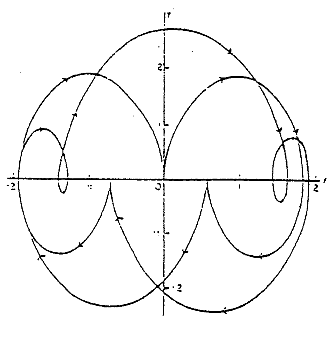

Figure 1. Murphy's Eyeballs as shown by Davis

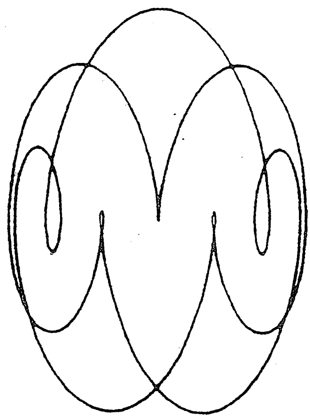

Figure 2. Murphy's Eyeballs as shown on the front cover of *Function*, *Volume* 3, *Part* 2

Ignore the differences of scale, these are merely incidental. Indeed, the new version on the front cover of this issue, uses a scale closer to Davis's than to our 1979 version. What we were more concerned with was the asymmetry of Davis's diagram. It can be shown that the curve must have an axis of symmetry vertically through the origin. Many details of the Davis graph fail to exhibit this feature. Our 1979 version, computed by Sean O'Connor, then a research assistant at Monash, does show the symmetry, and it also resolves a matter that Davis failed to address. Right in the centre of the diagram is a "downward-tending spike". This culminates in a true point: the curve reverses its vertical direction from down to up at the origin. But what of the two "upward-tending spikes" to either side of this?

Sean resolved this matter also, and his diagram reflects this. But now the truth can be told. The computed curve from Sean's program was in fact doctored by *Function's* artist, who took half the curve and reproduced it in mirror image to give the symmetry, and she also used little dabs of whiteout to supply the small loops that Sean could prove to exist inside the "upward-tending spikes", but which didn't show up in the computer-drawn image. So we cheated a little to produce what we still think is "the first correct picture of Murphy's Eyeballs ever published".

How would modern computer software perform on this demanding task? To answer this question, we need first to look at the mathematical basis of the curve. It is quite an involved story.

If a simple pendulum oscillates in a plane, then the deviation y of its string from the vertical, measured in radians, is given by the equation

$$y'' + \sin y = 0\,, \tag{1}$$

where the prime indicates differentiation with respect to time, whose units are so chosen as to produce the simplest possible form of the equation. When the amplitude of the oscillation is small, we may approximate $\sin y$ by y itself, and so reach the equation

$$y'' + y = 0\,. \tag{2}$$

This equation is the standard form of the equation of simple harmonic motion, which is widely applicable in a variety of physical contexts, the simple pendulum being one of them. In this case, however, it is an approximation, not the exact equation of motion. (Many school and university texts of Physics tend to gloss over this point!)

A better approximation to Equation (1) is

$$y'' + y - \frac{y^3}{6} = 0, \tag{3}$$

an equation known as the Duffing Equation after the author of a 1918 monograph on the subject. Although this equation is itself only an approximation to the basic Equation (1), it has been found to possess so many interesting properties in its own right, that it has come to be studied independently of its beginnings as an approximation to Equation (1). [It has even given rise to a generalised version

$$y'' + y + ry^3 = 0, \tag{4}$$

which is also known as the Duffing Equation.]

The principal complication of the Duffing Equation (3) [or (4)] is that the period of oscillation is dependent on the amplitude of that oscillation. If a pendulum is moved to one side through some angle and then released from rest, and if that initial displacement is small, then the time taken for each successive oscillation will not depend appreciably on the magnitude of that initial displacement. Equation (2) has a basic solution $y = A \cos t$, where A is the initial displacement. The period of oscillation is 2π, whatever the value of A. In the case of Equation (3), however, this simple property breaks down.

Davis determined that for this equation, if the initial displacement was $\pi/3$, then the period was $2\pi/\omega$, where, by means of some absolutely heroic computations, he found the value of ω to be 0.928 451.

So far, we have been concerned with the motion of a "free oscillator"; the pendulum, once released, is left to its own devices. But Davis was also interested in the study of "forced oscillations", where the pendulum is subject to a periodic applied force. In the course of his investigation, he was led to the special case of a "forced Duffing oscillator" with the equation of motion

$$y'' + y - \frac{y^3}{6} = 3\sin(3\omega t) \tag{5}$$

This is the equation that gives rise to Murphy's Eyeballs. The curve arises if we plot y against y', which is here called u. To get the plot, we used the MAPLE package and first used the series of instructions:

```
> with(DEtools):
> eq1:=diff(y(t),t)=u(t);
> eq2:=diff(u(t),t)=-y(t)+(y(t)^3)/6+3*sin(3*0.928451*t);
> ini:=y(0)=0,u(0)=0;
>DEplot({eq1,eq2},[y(t),u(t)],0..15.64, [[ini]],stepsize=.01,
linecolour=black);
```

This produced the figure at left on p 69, and you will readily see that it is actually *worse* than Davis's version. At least *his* picture of Murphy's Eyeballs showed a closed curve that started and finished at the origin. The problems arise however from the same cause as affected Davis's picture. As the computation proceeds, the small errors introduced by the numerical solution (which is never exact, just a very good approximation) compound, and throw the overall result out of kilter.

But we were able to use the machine to do what the artist had to do by hand back in 1979. In fact, it's quite easy. We merely rewrote the final line of code in symmetric form as

```
>DEplot({eq1,eq2},[y(t),u(t)],-7.82..7.82,
[[ini]],stepsize=.01,
linecolour=black);
```

and so produced the picture on the cover.

To resolve the question of the "upward spikes", we let the program run as a "magnifying glass" and replaced the final line yet again by

```
>DEplot({eq1,eq2},[y(t),u(t)],3.1..3.5, [[ini]],stepsize=.01,
linecolour=black);
```

and produced the figure at right on p 69. This makes it clear that the "upward spikes" are really very small loops.

But Murphy is not quite beaten yet. Look very closely at the front cover and see if you can't discover the last trace of his baleful influence!

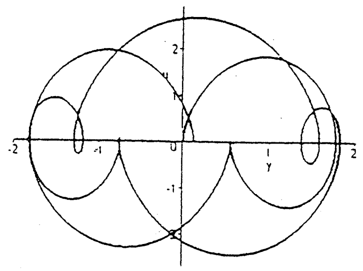

Figure 3. The first attempt at a graph of Murphy's Eyeballs

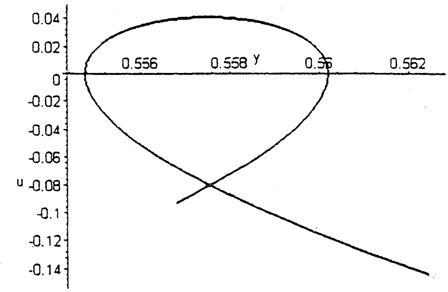

Figure 4. Close-up of the right-hand "upward spike"

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Pseudo-Induction

The following identity may be proved in several different ways, but perhaps none is more elegant than this, a method known as pseudo-induction.

Theorem: If a, b, c, d are any four different real numbers, then

$$\frac{d-a}{c-a}\cdot\frac{d-b}{c-b}+\frac{d-b}{a-b}\cdot\frac{d-c}{a-c}+\frac{d-c}{b-c}\cdot\frac{d-a}{b-a}\equiv 1.$$

Proof: The left-hand side is a quadratic in d. So is the right. If two quadratics agree at three separate values of the independent variable, then they are identical. (Prove this as an exercise, and also its generalisation to polynomials of higher degree.)

But they agree for $d = a$, $d = b$, $d = c$. Hence the result.

THE ANT ON THE BOX

Peter Grossman, Gerald St, Murrumbeena

I came across the following problem on the Web recently. It was proposed by Professor Yoshiyuki Kotani from the Tokyo University of Agriculture and Technology, and it has a surprising solution.

A box has dimensions 2 units by 1 unit by 1 unit. There is an ant at one vertex. Which point on the surface of the box is furthest from the ant, measured by the distance the ant would have to crawl on the surface of the box? See Figure 1.

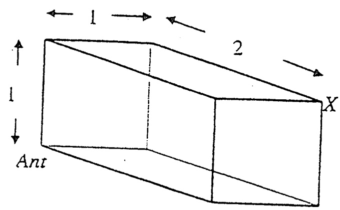

Figure 1

When I first read the problem, my immediate reaction was that the vertex diagonally opposite the ant, labelled *X* in the figure, is the furthest point. I imagine most people would react in the same way. On analysing the problem, however, I discovered that this is not the solution! Certainly, *X* is furthest in straight line distance from the ant, but that is not what the problem is asking. Let's take a closer look at what's going on.

As a first step towards solving the problem, we can calculate the distance the ant would have to crawl in order to reach *X*. There is an infinite number of paths from the ant to *X*; naturally, we take the *distance* to mean the length of the shortest path. If we unfold the box and flatten it out, the shortest path is a straight line on the flattened box. What makes the problem more difficult than it might otherwise be is that there is more than one way of carrying out the unfolding. One unfolding is shown in Figure 2, and another in Figure 3, together with straight line paths from the ant to *X*.

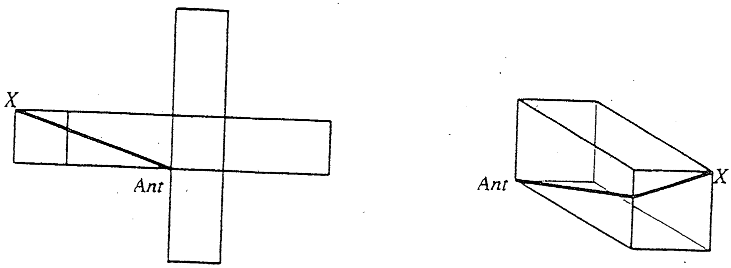

Figure 2

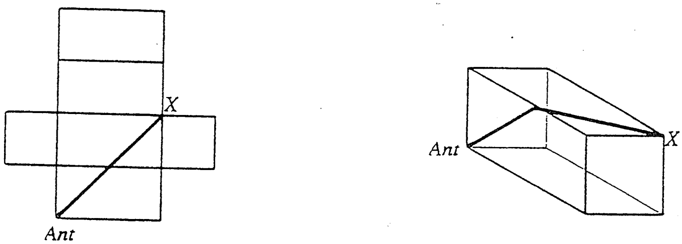

Figure 3

Both paths are locally minimal, by which we mean that, if the ant were to stray from the path by a small amount, it would end up travelling further. Another way of thinking about it is to imagine each path as a piece of elastic cord, with its ends fixed at the ant and at X, and stretched across the box. The cord would naturally stay in either of the positions shown, and would snap back to that position if we displaced it by a small amount. There are four other locally minimal paths from the ant to X, all of which are rotations or reflections of the two paths already given.

You can easily see that the path in Figure 2 has length $\sqrt{10}$ units, while the path in Figure 3 has length $2\sqrt{2}$ units. Since $\sqrt{10} > 2\sqrt{2}$, the distance from the ant to X on the surface of the box is $2\sqrt{2}$ units ≈ 2.828 units.

Now, imagine a point, Y, that starts at X, and moves along the diagonal of the square front face of the box as shown in Figure 4.

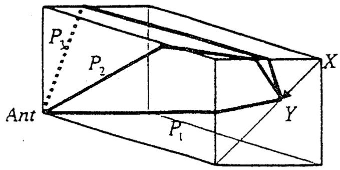

Figure 4

Think of the locally minimal paths from the ant to Y as elastic cords that move as Y moves. Three of these paths are shown in Figure 4, and three others can be obtained by symmetry from these three. As Y moves away from X, path P_1 (which, together with P_3, is initially the equal longest path) becomes shorter, and paths P_2 (initially the shortest) and P_3 become longer. While P_1 remains longer than P_2, the distance from the ant to Y is the length of P_2. Therefore, the distance from the ant to Y must be increasing as Y starts out from X and moves along the diagonal. So X is not the point furthest from the ant!

The point furthest from the ant occurs where P_1 and P_2 have the same length. At this point, all six locally minimal paths are of this length or longer. Moving on the surface of the box in any direction from this point causes at least one of the "short" paths (P_1 or P_2 or the mirror image path of P_1 or P_2) to become shorter, and hence the distance from the ant to decrease.

To find the point we are looking for, let d be the distance of the point from each of the two edges of the square face that meet at X. Then we have the situation depicted in the unfoldings shown in Figure 5.

Setting the lengths of P_1 and P_2 to be equal, we obtain the following equation:

$$\sqrt{(1-d)^2+(3-d)^2}=\sqrt{(2+d)^2+(2-d)^2}$$

The solution is $d=\frac{1}{4}$, and the distance of this point from the ant is $\frac{\sqrt{130}}{4}\approx 2.850$, just a little further than X. We can now state the solution to the Kotani puzzle as follows:

Starting from the vertex, X, diagonally opposite the ant, draw a diagonal of the square face opposite the ant. Then the point furthest from the ant is on the diagonal, one quarter of the way along from X.

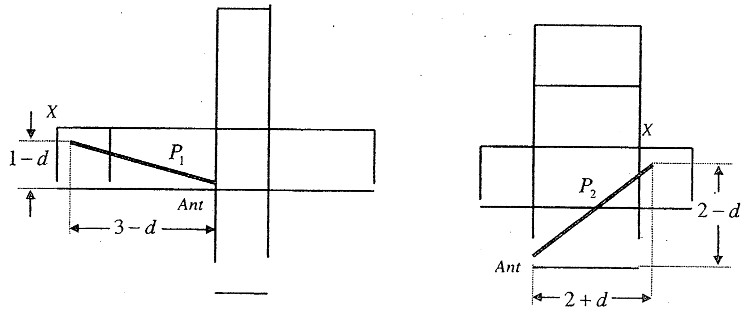

Figure 5

The problem can of course be generalised to boxes with dimensions other than $2\times1\times1$, and you may wish to explore what happens. A related, and more difficult, problem was proposed by Donald Knuth: find the pair of points on a $2\times1\times1$ box whose distance apart, measured on the surface of the box, is as large as possible. Knuth's problem can also be generalised to boxes with arbitrary dimensions, and this problem is apparently unsolved at present.

You can find more information about this and related problems at the following website:

www.btinternet.com/~se16/js/cuboid.htm

The site contains a neat Java applet for generating graphics that you can use in your explorations.

Acknowledgment:

I would like to thank the anonymous referee who drew the above website to my attention.

An Elementary Solution to Pell's Equation

Julius Guest, Alexander St, East Bentleigh

Pell's Equation is named after John Pell (1610-1685), an English mathematician. He was first a professor of Mathematics in Amsterdam for three years, and later in Breda for another six. His main interests were in algebra, and in particular Diophantine equations. Strangely enough, he never may never have studied 'his' equation in detail. He did edit and publish Brounker's "Translation of Rhonius's Algebra", where it *is* discussed, and he seems to have had other connections with it as well. It was our friend L Euler who attributed the equation to Pell. According to one theory, he confused Pell with another English mathematician of the period, Wallis;[1] however, modern scholarship disputes this and holds that Euler was correct.

However, neither Pell nor Wallis was by any means the first to study our topic. The study of such equations goes right back to Archimedes (i.e. some 2300 years). It was he who invented the famous Sicilian cattle problem.[2] It goes roughly like this. Once upon a time in Sicily there was a herd of black, white, yellow and dappled bulls as well as black, white, yellow and dappled cows. The unknown numbers of each of these types of creature were related by seven linear equations. This ultimately reduced of course to a Diophantine equation in two unknowns. This alone was pretty heavy stuff to solve in those times. But there was more .... . He further demanded that the numbers of the black and the white bulls should sum to a square number and that the numbers of the yellow and the dappled bulls should sum to a triangular number (i.e. of the type $t_n = \frac{n(n+1)}{2}$). These last two restrictions led to the *first* Pell equation:

$$x^2 - 472949y^2 = 1. \qquad (1)$$

It is just possible that Archimedes himself was able to find the least solution of Equation (1) in positive integers, but frankly I very much doubt it, for it took over 2000 years before anybody could find it. Equation (1) was finally solved in the latter part of the Twentieth Century

[1] See *Function, Vol* **22**, *Part* **4**.
[2] See *Function, Vol* **16**, *Part* **3**, p 68.

using a Cray I, then the fastest supercomputer in the world. It is also of interest to realise the size of Archimedes' herd. His total herd contained numbers whose expression ran into hundreds of thousands of digits! Thus the Archimedes problem showed great ingenuity on his part, but could scarcely have been based on facts relevant to a relatively small island in the Mediterranean Sea.

It is however an instance of the equation

$$x^2 - Ny^2 = 1 \tag{2}$$

to which I now turn.

According to Fermat, but still not proved, this equation has an infinite number of solution pairs (x, y), where the x and y are positive integers, and where N is another given positive integer, which is not a perfect square.

Here is a way to find solutions of Pell's Equation. For illustration I have chosen the case $N = 3$, and searched for all those solutions for which y lies in the interval [1, 8000000].

We rewrite Equation (2) for this case in the more convenient form

$$x^2 = 1 + 3y^2. \tag{3}$$

This has twelve solution pairs in the range chosen. They are found by the following Qbasic program:

```
SCREEN9: COLOR 14, 1:CLS
N = 3: k = 1
FOR y# =1 TO 8000000
z# = N*y# * y# + 1: x# =SQR(z#)
IF INT(x#) = x# THEN
LPRINT
LPRINT SPC(28); k; "."; "x ="; x#; SPC(3); "y = "; y#
k = k + 1
END IF
NEXT y#
END
```

We then obtain our twelve solution pairs:

1.	x = 2	y = 1
2.	x = 7	y = 4
3.	x = 26	y = 15
4.	x = 97	y = 56
5.	x = 362	y = 209
6.	x = 1351	y = 780
7.	x = 5042	y = 2911
8.	x = 18817	y = 10864
9.	x = 70226	y = 40545
10.	x = 262087	y = 151316
11.	x = 978122	y = 564719
12.	x = 3650401	y = 2107560

References

1. H Davenport, *The Higher Arithmetic*, p 106
2. P Hoffman, *Archimedes' Revenge*, pp 28-32

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OLYMPIAD NEWS

The 2001 Australian Mathematical Olympiad was held on Tuesday, February 13 and Wednesday, February 14. Each of the two sessions was 4 hours long, and posed 4 questions each worth seven points. No calculators were allowed. Here are the questions put to the olympians.

1. Let $L(n)$ be the least common multiple of 1, 2, ..., n. Determine all pairs (p, q) of prime numbers such that $q = p + 2$ and $L(q) > qL(p)$.

2. Let ABC be an isosceles triangle, with $AC = BC$. Let P, Q, R be points on AB, BC and AC, respectively, such that PQ is parallel to AC and PR is parallel to BC. Further, let O be the circumcentre of ABC. Prove that the quadrilateral $CROQ$ is cyclic.

3. A town has c celebrities and m magazines. One week, each celebrity was mentioned in an odd number of magazines and each magazine mentioned an odd number of celebrities.

   (a) Prove that m and c are either both even or both odd.
   (b) In how many different ways can that mentioning of celebrities happen? Express this number in terms of c and m.

4. Prove that the polynomial $4x^8 - 2x^7 + x^6 - 3x^4 + x^2 - x + 1$ has no real root.

5. Determine all functions f, defined for all real numbers and taking real numbers as values, for which

$$f\left((x-y)^2\right) = x^2 - 2yf(x) + \left(f(y)\right)^2.$$

6. Let ABC be a triangle with $AC > BC$. On the circumcircle of the triangle ABC, let D be the midpoint of arc AB that contains C. Let E be the point on AC such that DE and AC are perpendicular.

   Prove that $AE = EC + CB$.

7. Prove that there are no four positive integers w, x, y, z such that

$$x^2 = 10w - 1$$
$$y^2 = 13w - 1$$
$$z^2 = 85w - 1.$$

8. The country of Senso Unico has an airline whose flight routes are arranged as follows:

   (i) whenever there is a direct route from city A to city B, then there is no direct route from city B to city A;
   (ii) there is a route out of every city in Senso Unico;
   (iii) whenever there is a direct route from city A to city C, then there is a city B such that there is a direct route from A to B and a direct route from B to C.

The population of Senso Unico is very proud to have the smallest possible number of cities that allows such an arrangement.

Determine this number.

In March, the XIII Asian Pacific Mathematics Olympiad was held. There was one 4-hour session, with 5 questions asked, each carrying seven points. Here are the questions.

1. For a positive integer n let $S(n)$ be the sum of the digits in the decimal representation of n. Any positive integer obtained by removing several (at least one) digits from the right-hand end of the decimal representation of n is called a *stump* of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$.

2. Find the largest positive integer N so that the number of integers in the set $\{1, 2, ..., N\}$ which are divisible by 3 is equal to the number of integers which are divisible by 5 or 7 (or both).

3. Let two equal regular n-gons S and T be located in the plane such that their intersection is a $2n$-gon ($n \geq 3$). The sides of the polygon S are coloured in red and the sides of T in blue.

   Prove that the sum of the lengths of the blue sides of the polygon $S \cap T$ is equal to the sum of the lengths of its red sides.

4. A point in the plane with a cartesian coordinate system is called a *mixed point* if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed point.

5. Find the greatest integer n such that there are $n+4$ points A, B, C, D, X_1, ..., X_n in the plane with $AB \neq CD$ that satisfy the following condition: for each i = 1, 2, ..., n triangles ABX_i and CDX_i are equal.

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HISTORY OF MATHEMATICS

A Result in Number Theory

Michael A B Deakin

In my last column, I talked about some unsolved problems in Number Theory, the study of the higher arithmetical properties of the numbers, especially the integers. Of particular interest in this branch of Mathematics are the *prime numbers*, which are not divisible by any number other than themselves and one. The sequence of primes begins:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots\ldots,$$

and, as we saw last time, it goes on forever.

One route to this result is a theorem first proved by Leonhard Euler (1707-1783), one of the very greatest of mathematicians. Euler demonstrated that the series formed from the reciprocals of the primes, i.e.

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\ldots\ldots$$

diverges. That is to say, if we take sufficiently many terms, we may make the sum as large as we like. [It diverges very slowly: the sum of the first thousand terms is still well short of three, but given any positive number, we may raise the total to exceed it by taking sufficiently many terms.]

I mentioned this result in my previous column, but delayed a full discussion of it until this one. Here I will show Euler's original proof, and compare it with a modern and more careful one. The modern one is shorter, and in a sense is more elementary, but most readers will still need guidance in following it.

Euler began by forming, for each prime p, a related number $1-\frac{1}{p}$ and so formed the sequence $1-\frac{1}{2}$, $1-\frac{1}{3}$, $1-\frac{1}{5}$, etc. He now supposed that the reciprocals of all these new numbers were multiplied together, and so formed the infinite product $\left(\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\ldots\ldots\right)^{-1}$. He next expanded each of the terms in this product as an infinite series to get

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\ldots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots\ldots\right)\left(1+\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\ldots\right)\left(1+\frac{1}{7}+\frac{1}{7^2}+\frac{1}{7^3}+\ldots\right)$$

Now imagine all these brackets being multiplied together. Each term in the product will look like $\frac{1}{N}$, where N is a positive integer decomposed into its prime factors. But each positive integer N can be so expressed in precisely one way. [This is the fundamental theorem of Arithmetic, discussed in my previous column.] So the product so produced will have the terms of the series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots\ldots,$$

but possibly in a different order, which shouldn't matter. Thus Euler claimed that

$$\left(\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)\left(1-\frac{1}{7}\right)\ldots\ldots\right)^{-1}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots \qquad (1)$$

Now we saw last time that the series on the right diverges, and so therefore does the product on the left. Euler's first deduction was that there must be infinitely many primes, for otherwise the product would be finite, but the series infinite. This is an alternative proof of a result proved in my previous column, but there a simpler (and earlier) proof was given.

Unfortunately there are problems with Euler's result (1). A naïve objection may easily be formulated: if both sides diverge, then aren't we merely saying $\infty=\infty$, which is hardly news? Clearly Euler meant to say more than this, and modern thought has clarified the sense in which his equation is to be understood.

Euler was able to prove, using the same argument, a generalisation of Equation (1). It goes like this:

$$\left(1-\frac{1}{2}\right)^{-s}\left(1-\frac{1}{3}\right)^{-s}\left(1-\frac{1}{5}\right)^{-s}\left(1-\frac{1}{7}\right)^{-s}\ldots = 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\frac{1}{5^s}+\frac{1}{6^s}+\ldots\ldots, \qquad (2)$$

where s is a number, which can even be complex. But for our purposes, we will continue to think of s as real and positive, which is how Euler saw matters. When $s>1$, both sides of the generalised equation (2) converge, and so we may adopt the modern view of seeing Equation (1) as a limiting case as s gets ever closer to 1.

[The right-hand side of Equation (2) in such a case defines a function, now known as a the ζ-function. This is defined for complex values of ζ, and the so-called Riemann Hypothesis concerns its zeroes. Its proof is one of the outstanding problems in Mathematics. Equation (2) is the basis for its importance in Number Theory.]

Euler went on to derive some consequences of Equation (1). Again, by modern standards, his proofs leave out some necessary steps. Here I will give only the gist of Euler's original argument, and will not attempt to fill in these gaps.

Go back to Equation (1), and write its right-hand side as *M*. Then, expanding the left-hand side explicitly, Euler had

$$M = \frac{2.3.5.7\ldots\ldots}{1.2.4.6\ldots\ldots} = \frac{1}{\frac{1}{2}\times\frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\ldots.}$$

His next move was to take (natural) logarithms of both sides to find

$$\begin{aligned}\ln M &= -\ln\left(\frac{1}{2}\right)-\ln\left(\frac{2}{3}\right)-\ln\left(\frac{4}{5}\right)-\ln\left(\frac{6}{7}\right)-\ldots\ldots \\ &= -\ln\left(1-\frac{1}{2}\right)-\ln\left(1-\frac{1}{3}\right)-\ln\left(1-\frac{1}{5}\right)-\ln\left(1-\frac{1}{7}\right)-\ldots\ldots\ldots\end{aligned}$$

But now Euler knew that

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\ldots ,$$

and so he wrote out his formula for $\ln M$ as

$$\begin{aligned}\ln M = & \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{4}\left(\frac{1}{2}\right)^4 + \frac{1}{5}\left(\frac{1}{2}\right)^5 + \ldots\ldots \\ & + \frac{1}{3} + \frac{1}{2}\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{4}\left(\frac{1}{3}\right)^4 + \frac{1}{5}\left(\frac{1}{3}\right)^5 + \ldots\ldots \\ & + \frac{1}{5} + \frac{1}{2}\left(\frac{1}{5}\right)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{4}\left(\frac{1}{5}\right)^4 + \frac{1}{5}\left(\frac{1}{5}\right)^5 + \ldots\ldots \\ & + \frac{1}{7} + \frac{1}{2}\left(\frac{1}{7}\right)^2 + \frac{1}{3}\left(\frac{1}{7}\right)^3 + \frac{1}{4}\left(\frac{1}{7}\right)^4 + \frac{1}{5}\left(\frac{1}{7}\right)^5 + \ldots\ldots \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

His next trick was to sum this double series in columns, and so he found

$$\begin{aligned}\ln M = & \quad \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots\ldots \\ & + \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{7}\right)^2 + \ldots\ldots\right] \\ & + \frac{1}{3}\left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{5}\right)^3 + \left(\frac{1}{7}\right)^3 + \ldots\ldots\right] \\ & + \frac{1}{4}\left[\left(\frac{1}{2}\right)^4 + \left(\frac{1}{3}\right)^4 + \left(\frac{1}{5}\right)^4 + \left(\frac{1}{7}\right)^4 + \ldots\ldots\right] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

which he abbreviated to

$$\ln M = A + \frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \ldots\ldots .$$

Now look carefully at this formula. M is known to be infinite, as is very easy to show, so $\ln M$ is also infinite. A is what we want to find out about. But Euler already knew that B, C, D, etc however had finite values (although he did not stop to explain this point fully). He then went on to state that the sum $\frac{1}{2}B + \frac{1}{3}C + \frac{1}{4}D + \ldots\ldots$ was also finite. This is true, but it needs proof, which again he did not stop to supply. However, the conclusion was that A could not be finite, and so had to be infinite.

Euler is thus credited with the discovery that the series formed by summing the reciprocals of the primes is divergent. I would now like to take you through a modern proof of this result. Originally I intended to use the proof in Hardy and Wright's *An Introduction to the Theory of Numbers*, but I was shown a better proof, which I shall use instead. It is due to Ivan Niven, whom I mentioned at the end of the previous article. [Niven made great mileage from looking anew at old problems: he gave the first elementary proof that π is irrational.]

In this case, Niven's proof is elementary (in the sense that it uses only concepts from basic Number Theory) and it is also short (11 lines long). This is not to say that it is easy. It isn't; it takes work to follow it.

It begins with the observation (a corollary of the Fundamental Theorem of Arithmetic) that *every number may be expressed as either a square, a squarefree integer (one containing no square (>1) as a factor) or else as the product of a square and a squarefree integer; furthermore this expression is unique – it can be done in only one way.* This is also the starting point for Hardy and Wright's proof. [So, if we start out with some simple examples, 1, 4, and 9 are squares. Of the other numbers up to ten, 2, 3, 5, 6, 7, 10 are squarefree, while 8 is a product: $8 = 2 \times 2^2$.]

Now consider the sum of all the reciprocals of the squares:

$$\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\ldots\ldots ,$$

and the sum of all the reciprocals of squarefree integers:

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\ldots\ldots .$$

Call the first sum s and the second sum S, say.

Then form the product $s(1+S)$. Then, just as Euler proved Equation (1), then we may establish that

$$s(1+S)=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\ldots\ldots , \tag{3}$$

the sum of the reciprocals of all the positive integers.

Well, that is how Euler would have proceeded, but Niven was more careful. Instead of using infinite sums (which in some cases can lead to error), he took only finite ones. So, for some number N, define $s(N)$ as the result of stopping the series s at the largest square before N and $S(N)$ the similarly defined finite version of the series S. Then we have, instead of the simple Equation (3), the more complicated *inequality*

$$s(N)(1+S(N)) \geq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+....+\frac{1}{N-1} \qquad (4)$$

This may be proved in various ways, and this is a detail Niven left to the reader. I will do likewise, but as an illustration consider the case $N=10$. Then what Niven claimed is that

$$\left(\frac{1}{1}+\frac{1}{4}+\frac{1}{9}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right) \geq \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}$$

and this is easily checked. When you do this, you will see the principle on which the general claim is based.

Now Niven used an argument not unlike another of Euler's, but still using only finite sums. Because it is known that

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+....+\frac{1}{N-1} \to \infty \quad \text{as } N \to \infty,$$

and because it is also known that $s(N)$ tends in these circumstances to a finite limit (found by Euler to be $\frac{\pi^2}{6}$), it follows that $S(N) \to \infty$ as $N \to \infty$. Thus Niven established the preliminary result that the series S diverges.

In order to complete his proof, that the series A diverges, he supposed that it does not, but rather tends to a finite limit β. Then β will be larger than the sum of the first few terms of the series A, no matter how large that "few" might be. Specifically suppose we determine all the primes less than N and add up their reciprocals. Let $p(N)$ be the largest prime less than N. Then $\beta > \frac{1}{2}+\frac{1}{3} \;....\; +\frac{1}{p(N)}$, so that

$$\exp(\beta) > \exp\left(\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{p(N)}\right).$$

Now use the basic property of the exponential function to write the right-hand expression as $\exp\left(\frac{1}{2}\right)\exp\left(\frac{1}{3}\right)\ldots\ldots\exp\left(\frac{1}{p(N)}\right)$.

Next Niven used another property of the exponential function: for positive values of x, $\exp(x) > 1+x$. Thus this last expression is in turn greater than $\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\ldots\left(1+\frac{1}{p(N)}\right)$.

And now we get a second echo of Euler's thought. For the expansion of this product will contain only squarefree numbers (after the initial 1). [Euler would probably have used an infinite product and claimed that the result of the expansion was S. Here, following Niven, we use the more secure argument based on finite products.] But now, as before, we can replace an equality by an inequality, which will do the job for us. The product is in its turn greater than $S(N)$. Again, Niven leaves this detail to the reader, and so do I. But here is a special illustrative case to guide you.

Return to the case $N=10$. $p(10)=7$, so the product is

$$\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{5}\right)\left(1+\frac{1}{7}\right)$$

which is to be greater than $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}$. Once again it will reveal the general principle if you work out this case and maybe a few others.

Now let us sum up. Niven assumed that the series A converges, so that $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\ldots+\frac{1}{p(N)} < \beta$ for all N, no matter how large. But then, by means of a chain of inequalities, he showed that $\exp(\beta)$ must exceed $S(N)$.

Now we can do as Euler did and take logarithms to find $\beta > \ln S(N)$, no matter how large we make N. But $S(N) \to \infty$ as $N \to \infty$ and so also does $\ln S(N)$. Thus we have a contradiction, and β cannot be finite. This proves the assertion that the series A diverges after all.

Further Reading

The account of both proofs derives from William Dunham's *Euler: The Master of us all*, a very readable presentation of Euler and his work. See pp. 70-74. Dunham also includes the details necessary to supplement the gaps in Euler's original proof. The original of that proof (in Latin) is to be found scattered throughout *Volume* **14** of the First Series of Euler's collected works. Niven's proof first appeared as a short note in the *American Mathematical Monthly*, *Vol* 78 (1971), pp 272-273. We have only had the benefit of it for 30 years or so!

FIGURATE NUMBERS

Here we summarise a number of points raised by Problem 25.1.1 and related material. Some of this was sent in by Garnet J Greenbury and Lachlan Harris, and more was supplied by *Function* editor Malcolm Clark. We also refer readers to an earlier *Function* article by John Stillwell in *Vol* **10**, *Part* **2**. This material also relates to the "20-years ago" segment published in our last issue.

Let us begin with the proof that $1+2+3+...+n=\frac{n}{2}(n+1)$. There are many possible ways to proceed, but the one we give here is probably the simplest. Write out the sum in two ways:

$$S=1+2+3+.....+(n-2)+(n-1)+n$$

$$S=n+(n-1)+(n-2)+...+3+2+1,$$

and now add the two in vertical columns. There are n such columns and the sum in each is n + 1. Thus $2S=n(n+1)$, and the result follows.

Using this formula, we may generate the triangular numbers in the sequence:

1 3 6 10 15 21 28 36 45 55 66 .....

The triangular numbers get their name from the triangular shapes into which they may be arranged:

```
*        *        *        *
         * *      * *      * *
                  * * *    * * *
                           * * * *     and so on.
```

Garnet J Greenbury and Lachlan Harris both pointed out that the sequence of square numbers:

1 4 9 16 25 36 49 64 81 100 121 ....

with the associated patterns

```
*        * *      * * *    * * * *
         * *      * * *    * * * *
                  * * *    * * * *
                           * * * *     and so on
```

form a related sequence, and that there is an entire set of such sequences, the next one being the pentagonal numbers:

1 5 12 22 35 51 70 92 117 145 ..... .

After that come the hexagonal numbers:

1 6 15 28 45 66 91 120 153 .... and so on.

The members of these different sequences are often termed *figurate numbers*, or sometimes *polygonal numbers*. Our two correspondents pointed out that each of these sequences has a formula which generates it, and the sequence of formulas itself has a nice pattern.

If t_n represents the n th triangular number, then $t_n = \frac{n}{2}(n+1)$, and if s_n represents the n th square number, then $s_n = \frac{n}{2}(2n+0)$, and if p_n represents the n th pentagonal number, then $p_n = \frac{n}{2}(3n-1)$, and so it goes. For the hexagonal numbers h_n we have $h_n = \frac{n}{2}(4n-2)$. Etc.

More generally, if we speak of the nth number in the mth sequence, where $m=3$ for the triangles, 4 for the squares, etc, then that nth number, $N(n,m)$ is given by the formula

$$N(n,m)=\frac{n}{2}[(m-2)n+(4-m)]$$

Lachlan Harris sent us a proof of this result, but we won't go into that here. It is not particularly difficult if you know the technique of mathematical induction (see *Function*, *Vol* **22**, *Part* **3**), and we leave it to readers. [It is also an interesting exercise to see what happens if we substitute $m=1$ or 2 into this formula. We leave this to readers too.]

We may also relate the different numbers in the different sequences to one another. Thus the relation $t_n + t_{n-1} = s_n$ was noted in our last issue, and it formed the basis of Jeannette Hilton's proof of the formula for the sum of the first n cubes. Lachlan Harris also noted this formula as did Malcolm Clark, whose geometric proof was the same as Jeannette Hilton's. Malcolm Clark went on to note the sequence of similar results, which can be proved by similar geometric arguments:

$$s_n = t_n + t_{n-1}$$
$$p_n = s_n + t_{n-1}$$
$$h_n = p_n + t_{n-1}$$
etc.

In our more general notation, we find

$$N(n,m)=N(n,m-1)+N(n-1,3),$$

for $m \geq 4$. This has the simple interpretation that the adjoining of a triangle to a figure increases the number of sides by one. Application of this formula gives another approach to a proof of the formula for $N(n,m)$.

In the earlier *Function* article, John Stillwell noted an ancient Greek interest in figurate numbers. They knew the result concerning the relation of square numbers to triangular numbers, and other such relationships. However, he noted that "apart from Diophantus' work, which contains impressive results on sums of squares, Greek results on polygonal numbers were of this elementary type".

He went on to comment: "On the whole, the Greeks seem to have been mistaken in attaching much importance to polygonal numbers. There are no major theorems about them, except perhaps for the following two". The two theorems he mentions are:

(1) A theorem that every positive integer is the sum of four integer squares. This was first stated by Bachet in 1621 in an edition of Diophantus' works, and was later proved by Lagrange in 1770. A minor generalisation was stated by Fermat in 1670, and asserts that every integer is expressible as the sum of n n-gonal numbers. This was proved by Cauchy in 1815, but it turns out to be a disappointment as all but four of the numbers can be either 0 or 1.

(2) Euler's pentagonal number theorem. This states that a certain infinite sum involving the pentagonal numbers is equal to a certain infinite product. It involves extending the concept of a pentagonal number $p_n = \frac{n}{2}(3n-1)$ to cases where n is negative. The details are here omitted, but interested readers may wish to consult the text *Number Theory* by George Andrews, pages 169-177.

For yet more on polygonal numbers, see *Function*, *Vol* **18**, *Part* **5**.

ooooooooooooooooooooooooooooooo

Even in the examples [of calculating probabilities] which are susceptible of numerical treatment the chance is reckoned *relative to some particular body of knowledge* stated or implied.

For example consider the statement:

'January 1st, 2501 will be a Sunday .... .'

To [someone] who only knows that January 1st will be one of the seven days of the week, the chance that the statement is true is $\frac{1}{7}$. But to one who ... happens to know that a century never begins on a Sunday ... there is no chance that the statement is correct.

Clement V Durell

COMPUTERS AND COMPUTING

A Model for Blood Alcohol Concentration

Cristina Varsavsky

In the last few years, people in Victoria have been bombarded with messages highlighting the nasty consequences drink-driving can have, and have been subjected to stricter on-the-road controls. These are strategies devised by the Victoria Road Traffic Authority, based on research showing that an unacceptably high proportion of car accidents around the world are due to drink-driving.

The legal alcohol limit for full licence holders in Victoria is 0.05. This means that the blood alcohol concentration (BAC) of the driver should not be higher than 0.05 grams per 100ml of blood. This limit of 0.05 is based on research around the world which correlates BAC levels with accident rates and impairment of psychomotor, visual, perception and concentration skills.

Even when the limit is related to the concentration of alcohol in blood, the on-the-road controls involve a breath test on the drivers. This is done for practical reasons, and is based on studies which indicate that the concentration of alcohol in breath is, in general, a good indicator of the concentration of alcohol in blood. When there is doubt, or with serious offences, a blood test is conducted to verify the breath readings.

The BAC is calculated as the ratio of mass of alcohol to the volume of body fluids. This volume depends on the weight of the driver. It also depends on the gender as the ratio of fluid contents to weight is different for men than women; this is known to be around

0.55 litres/kilograms for women and 0.68 litres/kilograms for men.

So

$$\mathrm{BAC} = \frac{\text{mass of alcohol}}{\text{weight} \times \mathrm{F}}$$

where F is the gender factor.

For example, if 7 grams of alcohol made its way to the blood stream of a male driver of 85 kg, his BAC would be

$$\text{BAC} = \frac{7\text{ g}}{85\text{ kg} \times 0.68\text{ l/kg}}$$

But since the units of BAC are grams per 100ml, we write this as

$$\text{BAC} = \frac{7\text{ g} \times 0.1\text{ l/100ml}}{85\text{ kg} \times 0.68\text{ l/kg}}$$

which gives a BAC of 0.012 g/100ml.

The purpose of this article is to use a spreadsheet to model BAC of drivers after they take alcoholic drinks. A typical situation occurs when you organise parties, where you would like to entertain your guests and make sure everybody has a good time, but you would certainly not like to feel responsible for any accidents due to drink driving after the party.

Before setting up the spreadsheet, we need to know a bit more about what happens after alcohol enters the stomach. Although some of the alcohol is absorbed directly from mouth, the stomach and the oesophagus, most of it passes through the pyloric valve into the upper of the small intestine where it is absorbed by the blood vessels. These capillaries feed the large vein which passes through the liver. The absorption via the small intestine to the blood stream is proportional to the amount of alcohol present in the stomach, while the degradation in liver is linear. Experiments show that these two processes can be modelled as follows:

Absorption from the stomach to the blood stream: $A = A_0\, e^{-m t}$
Degradation in liver: $A = A_0 - k\, t$ (1)

where A_0 is the initial amount of alcohol, and t is measured in minutes. The absorption rate m and the degradation rate k vary from individual to individual, but usually they are around 0.04636 and 0.15 respectively.

With this information we are then ready to model the BAC of a driver. All we have to do is combine the exponential and the linear processes.

Let us start with a simple example. Suppose Bill has just had a beer. How will his BAC vary over time?

First we start with calculating the amount of alcohol ingested by Bill. A standard can of light beer contains 375 ml with an alcohol concentration of about 3%. The density of alcohol is approximately 0.785 g/ml. Therefore

$$\begin{aligned}\text{mass of alcohol} &= \text{volume} \times \text{density} \\ &= (375\ \text{ml} \times 0.03) \times 0.785\ \text{g/ml} \\ &= 8.7\ \text{g}\end{aligned}$$

These 8.7 grams of alcohol are absorbed into the blood stream in an exponential fashion. The amount of alcohol in Bill's stomach t minutes after he drank the can of beer is $A = 8.7\ e^{-0.04636t}$, so the amount of alcohol in the blood stream t minutes after the drink is

$$\begin{aligned}B &= 8.7 - 8.7\ e^{-0.04636t} \\ &= 8.7\left(1 - e^{-0.04636t}\right)\end{aligned}$$

At the same time, as soon as the alcohol is in the blood stream, this degrades in the liver at a constant rate of 0.15 grams per minute. Here is where the spreadsheet becomes very handy, because we can simulate the combination of these two processes, and hence calculate Bill's BAC minute by minute.

	A	B	C	D	E
1	t min	From stomach	Degraded in liver	Alcohol left	BAC
2	0	0			
3	1	0.394125582	0.15	0.244125582	0.000422
4	2	0.770396569	0.3	0.470396569	0.000814
5	3	1.129621807	0.45	0.679621807	0.001176
6	4	1.472573498	0.6	0.872573498	0.00151
	...	...	...	...	...
48	46	7.668758903	6.9	0.768758903	0.00133
49	47	7.715475971	7.05	0.665475971	0.001151
50	48	7.760076673	7.2	0.560076673	0.000969
51	49	7.802656883	7.35	0.452656883	0.000783
52	50	7.843308134	7.5	0.343308134	0.000594
53	51	7.88211781	7.65	0.23211781	0.000402
54	52	7.919169338	7.8	0.119169338	0.000206
55	53	7.954542365	7.95	0.004542365	7.86E-06

Figure 1

Figure 1 shows the layout of the spreadsheet. The contents of the cells are as follows:

```
B3:   =$J$1*(1-EXP(-0.04636*A3))
C3:   =0.15*A3
D3:  =+B3-C3
E3:  =+D3*0.1/($H$1*0.68)
H1:  85 (Bill's weight)
J1:  8.7 (alcohol ingested)
L1:  0.68 (gender factor)
```

These cells are copied down until all the alcohol is degraded by the liver.

The graph in Figure 2 shows that Bill's BAC reaches its peak of 0.004 g/100ml about 20 minutes after the drink, and the alcohol is completely eliminated from the body before the hour.

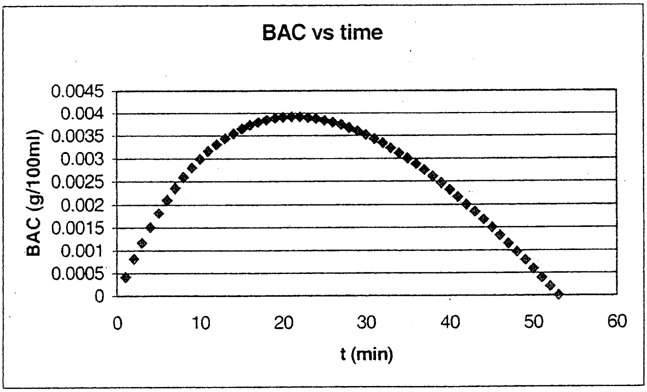

Figure 2. BAC for 85 kg male who has ingested 8.7 g of alcohol.

Now you can use this simulation for different values of the parameters representing the weight, alcohol ingested, and gender factor. For example, if Sarah, who weighs 60kg takes three beers at once, her BAC curve would look like the one in Figure 3. She would be above the legal limit between 30 and 70 minutes after the drink and her body will eliminate all alcohol in about 3 hours.

Now, what if the drinks taken by your guests are spaced out? For example, if Gabriela, who weighs 75kg, has a beer every 20 minutes, how many will she have to have to reach the legal limit? What would her BAC curve look like?

While there is alcohol in the blood stream, the degradation in the liver is always 0.15 g/min, so this part of the process will not change. But the absorption from the stomach to the blood stream is exponential and the amount of alcohol absorbed every minute will depend on the amount of alcohol present in the stomach during that minute.

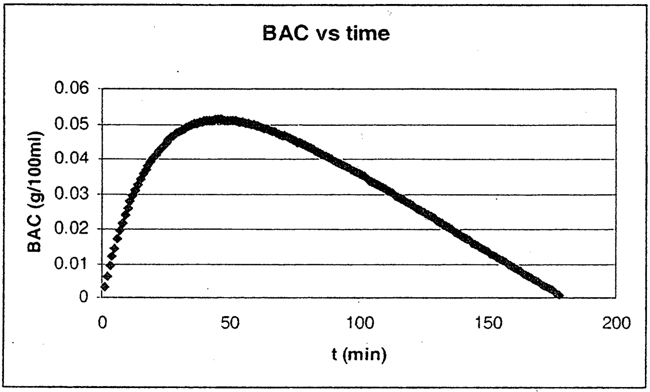

Figure 3. BAC for 60 kg female who has ingested 27 g of alcohol.

For this simulation, it is helpful to have a separate column to keep track of the amount of alcohol in the stomach; so we use column B for the alcohol in stomach, column C for the alcohol that passes from the stomach to the blood stream, column D for the amount of alcohol degraded in the liver, and column E for the amount of alcohol present in the blood stream. These are the formulas we put in the cells:

```
B3:  =+$J$1*EXP(-0.04636*A3)
C3:  =+$B$2-B3
D3:  =0.15*A3
E3:  =+C3-D3
```

We copy these formulae down into the first 20 rows (up to row 21), and in row 22, which corresponds to the second beer, we put

```
B22: =+($B$21+$K$1)*EXP(-0.04636*(A24-20))
C22: =+$C$21+$B$22-B24
```

These and the other formulae in row 22 are again, copied down to the 20 rows below, and so on. The result for Sarah after 5 drinks is shown in Figure 4. Sarah reaches the 0.05 limit soon after the 5th beer and remains well above it for about 80 minutes. Pretty bad for driving! But you can check with the first model that if she took the five beers at once it would have been much worse. Her BAC would have reached the limit in only 20 minutes and it would have escalated beyond the 0.08 level. Spacing out the drinks helps to keep the BAC at more manageable levels!

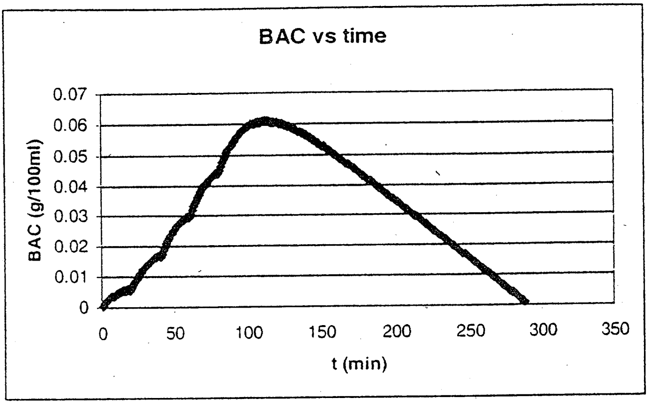

Figure 4. BAC for 75 kg female who has ingested 5 beers at 20 minutes intervals .

Reference
Spiccia, L "A Case Study on Metabolism" in Varsavsky, C (ed) *SCI1020 The Design of Science Resource Book*, Monash University, 2001.

* * * * *

PROBLEM SECTION

SOLUTION TO PROBLEM 25.1.1

The problem asked for a proof of the identity

$$T_n + T_{n-1}T_{n+1} = (T_n)^2,$$

where T_n is the n th triangular number.

Solutions were received from Keith Anker, J A Deakin, Garnet J Greenbury, Julius Guest and Lachlan Harris. All proceeded along similar lines, and the gist of the argument is given here.

Begin with the formula for the n th triangular number. This is

$$T_n = 1 + 2 + 3 + \ldots + n = \frac{n}{2}(n+1)).$$

The first part of this equality is obvious, the second may be proved in various ways. It is a standard result, and we give a proof on p 86. Three of the solvers did in fact supply proofs. Once we have the formula then it is merely a matter of substituting into the left-hand side of the identity and simplifying.

There are many results for triangular numbers, and Trotter's website, from which we took the problem, has many more. Most have been long-known but this particular one was new to Trotter. So far no-one has answered his other question: was Trotter the first to notice this identity?

For other, related material, see pp 86-89.

SOLUTION TO PROBLEM 21.1.2 (Submitted by Julius Guest)

Given a triangle ABC with sides a, b, c opposite the vertices A, B, C respectively, suppose that angle A is known as are the difference $b-c$ and the length h_A of the perpendicular drawn from A to BC. Find a, b, c.

The proposer sent us a solution, which we present here.

We have

$$ah_A = bc \sin A,$$

since both sides are equal to twice the area of the triangle, and we may rewrite this as

$$bc = \frac{ah_A}{\sin A}$$

Let the known value of $b-c$ be D, so that

$$b^2+c^2=D^2+bc$$

but by the cosine rule

$$a^2=b^2+c^2-2bc\cos A=D^2+2bc-2bc\cos A=$$

$$D^2+2bc(1-\cos A)=D^2+\frac{ah_A(1-\cos A)}{\sin A}$$

and this may be simplified to

$$a^2-2ka-D^2=0\text{, where } k=\frac{h_A}{\tan(A/2)}.$$

k and D are both known so we may solve the quadratic to find a. We have

$$a=k+\sqrt{k^2+D^2}\,,$$

a unique solution, since the other root is negative, and may be dismissed. Thus a is now known.

Now use the sine rule to find

$$b-c=a\frac{(\sin B-\sin C)}{\sin A}=a\frac{\sin[(B-C)/2]}{\cos(A/2)}$$

This equation enables us to find $B-C$, and we may also find $B+C$ from the equation

$$\cos(A/2)=\sin[(B+C)/2].$$

Thus B, C may be found and then we may determine b and c by the sine rule.

SOLUTION TO PROBLEM 21.1.3 (Submitted by Julius Guest)

The statement of this problem was the same as for the previous one, except that the known quantities were A, a and b^2-c^2.

Again the proposer sent us a solution, which we present here.

Let R be the circumradius of the triangle ABC. Then $c = 2R\sin C$, and $2R = a/\sin A$. So R may be calculated.

Now apply the cosine rule, to find

$$2ac\cos(A+C) = k\text{, where } k = b^2 - c^2 - a^2.$$

Notice that since $b^2 - c^2$ is given, as also is a, k is a known quantity. But now, we have

$$2\sin C\cos(A+C) = k/(2bR).$$

We may now expand and simplify this equation to find:

$$F\sin(2C) + G\cos(2C) = H\text{, where}$$
$$F = \cos A,\ G = \sin A,\ H = \frac{k}{2bR} + \sin B.$$

It is now helpful to put $t = \tan C$, so that we may use the standard identities:

$$\sin(2C) = \frac{2t}{1+t^2} \text{ and } \cos(2C) = \frac{1-t^2}{1+t^2},$$

so that we now have a quadratic in t

$$Pt^2 - 2Qt + R = 0\text{, where } P = G + H,\ Q = F,\ R = H - G.$$

So we may solve to find two solutions: $\tan C = \dfrac{Q \pm \sqrt{Q^2 - PR}}{P}$.

The solution may then be completed. It should be noted that in some cases, there are indeed two separate solutions, in others only one. In yet others, no solution exists. Readers may care to explore these matters further for themselves.

SOLUTION TO PROBLEM 21.1.4 (Submitted by J A Deakin)

The problem was to evaluate the integral

$$\int_0^{\pi} \frac{xdx}{1+\cos^2 x} .$$

Solutions were received from Keith Anker, J C Barton, Julius Guest and the proposer. Here is Barton's solution.

Let $J = \int_0^{\pi} \frac{xdx}{1+\cos^2 x}$.

Put $x = \pi - t$, $\frac{dx}{dt} = -1$, $\cos^2 x = \cos^2 t$. Then

$J = \int_0^{\pi} \frac{(\pi - t)dt}{1+\cos^2 t} = \pi \int_0^{\pi} \frac{dt}{1+\cos^2 t} - J$. Hence $J = \frac{\pi}{2}\int_0^{\pi} \frac{dt}{1+\cos^2 t}$.

Write $\int_0^{\pi} \frac{dt}{1+\cos^2 t} = \int_0^{\pi/2} \frac{dt}{1+\cos^2 t} + \int_{\pi/2}^{\pi} \frac{dt}{1+\cos^2 t} = J_1 + J_2$, say.

In J_2 , set $t = \pi - s$, $\frac{dt}{ds} = -1$, $\cos^2 t = \cos^2 s$ and find that $J_2 = J_1$.

Hence

$J = \pi J_1 = \pi \int_0^{\pi/2} \frac{dt}{1+\cos^2 t} = \pi \int_0^{\pi/2} \frac{\sec^2 t dt}{2+\tan^2 t} = \pi \int_0^{\infty} \frac{du}{2+u^2}$, where $u = \tan t$.

Then $J = \frac{\pi}{\sqrt{2}} \left[\tan^{-1}\left(\frac{u}{\sqrt{2}} \right) \right]_0^{\infty} = \frac{\pi^2}{2\sqrt{2}}$.

The other solutions proceeded along similar lines.

We conclude with the next batch of problems.

PROBLEM 25.3.1 (from *Mathematical Digest*)

An adventurer who went in search of treasure on a certain small island had as sole clue the following instructions:

> From the middle of the hut (H) make a line to the ash tree (A) and another to the beech tree (B). From A in the direction remote from B make a line AC at right angles to and equal in length to HA. Similarly from B make a line BD. The treasure is buried at the midpoint of the line CD.

The adventurer arrived at the island and was able to identify the trees, but all trace of the hut had vanished owing to the ravages of termites. How was the treasure found?

PROBLEM 25.3.2 (submitted by Keith Anker)

A set of nine tokens is arranged in a square as shown.

* * *

* * *

* * *

The following game is now played. Player A, moving first, takes some or all of the tokens in any one row or column; Player B then in turn takes some or all of the tokens left in any one row or column. And so the game continues until all the tokens have been taken. The winner is the player to take the last token. Which of the players can force a win, and what is the winning strategy?

PROBLEM 25.3.3 (submitted by Garnet J Greenbury)

Let ABC be a triangle with its inscribed circle centred at O and one of its three escribed circles centred at P. The mid-point of the line-segment OP is D. Prove that D lies on the circumcircle of the triangle ABC.

PROBLEM 25.3.4 (adapted from *Mathematical Spectrum*)

If m, n are positive integers, determine the final two digits of the number $(20n - 15)^m$.

Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage): $27.50* ; single issues $7.50. Payments should be sent to: The Business Manager, Function, Department of Mathematics & Statistics, PO Box 28M, Monash University VIC 3800, AUSTRALIA; cheques and money orders should be made payable to Monash University.

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