Function

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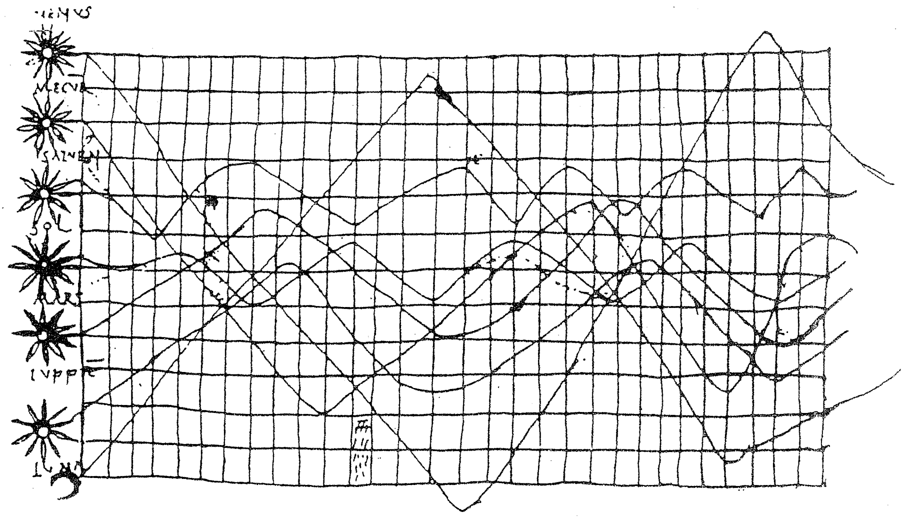

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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For more information about *Function* see the journal home page at `http://www.maths.monash.edu.au/~cristina/function.html`.

* $14 for *bona fide* secondary or tertiary students.

The Front Cover

The rather strange diagram reproduced on our front cover for this issue is thought to be the first example of a graph, in something like the modern sense of that term. It is reproduced from a tenth (or possibly eleventh) century manuscript, which now belongs to the Bayerische Staats-Bibliothek in Munich, where it is catalogued as *Codex Latinus 14436*. It was titled by the original scribe as *Macrobius Boetius in Isagog. Saec. X.*

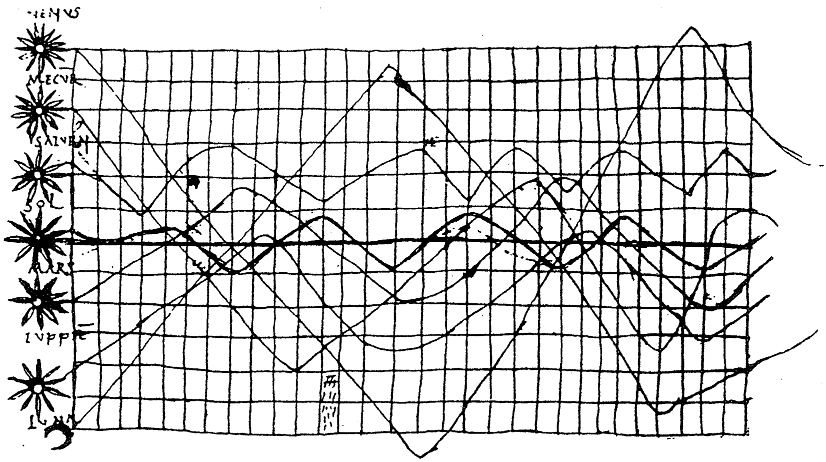
Macrobius is Ambrosius Theodosius Macrobius a Roman grammarian and scholar who lived in about AD 400). Very little is known of his life and work, but we do know that he wrote a commentary on Cicero's book *In Somnium Scipionis* ("The Dream of Scipio") and that in this work Macrobius reviewed the Physics and Astronomy of his day.

The manuscript was first brought to modern scholarly attention by Sigmund Günther who published an account of it (in German) in 1877. Attached to the work proper is an appendix titled *De cursu per zodiacum* ("The Path through the Zodiac"). This may have been added by a later scribe, and its thrust is a brief description of the paths of the planets through the zodiac. Our diagram is an illustration from this appendix.

To understand what is involved here, we need to backtrack a little. The stars in the night sky form a fixed pattern, which we see from different viewpoints, but whose overall appearance remains the same. The different parts of this pattern form the constellations, and these always stay the same and also keep the same relations with one another. Through this fixed pattern, there travel a number of "wandering stars" or *planets* (the name means "wanderer"). In the ancient world there were five known planets: Mercury, Venus, Mars, Jupiter and Saturn (those visible with the naked eye), and to these five were added two others: the Moon and the Sun. The Moon shares this property of moving against the fixed backdrop of the stars, and so does the Sun, although this is less obvious as its powerful light hides the stars during the day.

These seven heavenly bodies were all termed planets by the ancients, and they also share other properties. Their movements are not random but follow regular (if complicated) paths through the backdrop of fixed stars. Indeed they progress through a relatively small number of constellations (twelve in all) and these constellations are known as the houses of the zodiac.

Thus what Macrobius (or a later scribe) was writing about was some aspect of the paths taken by the seven planets through the night sky. These paths are the subject of the diagram reproduced here. Just visible at the left, above the stylised pictures of the planets are their names, which read (from the top down and in Latin): Venus, Mercury, Saturn, Sun, Mars, Jupiter, Moon.



Clearly, there is an attempt to illustrate for each planet some aspect of its movement against the background. Just what is being illustrated is, however, unclear.

Our version of the "graph" comes from a note published in English in 1936 and written by H Gray Funkhouser. The "graph" itself may also be viewed at

http://www.york.ac.uk/depts/maths/histstat/old_graph.gif

Funkhouser sees as a distinguishing feature the use of a grid to aid in the display of quantitative data. He states that this convention (nowadays very familiar to us all) is actually relatively recent as a standard practice. He draws attention to the 3rd edition of Jevons' *Principles of Science*, published in 1879. In this work, the author gives instructions in how to use squared paper, and "[it] is evident that he is describing an unfamiliar procedure".

However, when we compare the tenth century version with modern practice, we notice an immediate problem: there are no scales on the axes! This makes it all the more difficult to interpret the figure. Funkhouser accepts a suggestion from S W McCuskey of the Harvard College Observatory to the effect that the horizontal axis represents time and the vertical axis plots the inclinations of the various planetary orbits to the ecliptic.

Here again let us pause for a little background. The earth orbits the sun, or as they would have expressed things back then, the sun orbits the earth, and it does so in a plane. This plane appears as a fixed path through the houses of the zodiac, and this path is termed the *ecliptic*. The (other) planets (including the Moon) move in complicated paths near, but not quite *on*, the ecliptic. This is because, as we would now see things, there are slight angles between the plane of the earth's orbit and those of the other planets and of the Moon.

McCuskey's interpretation is that the central horizontal line of the "graph" is intended as a representation of the ecliptic. That line has been thickened for prominence in the version shown here on p 34 – this is not a feature of the original, reproduced on the Front Cover itself.

But now there is a problem, in fact the first of many encountered in the detailed interpretation of the diagram. The path of the Sun should coincide with the ecliptic itself; in the diagram, it doesn't. The graph of the Sun's movement is also thickened in the version reproduced on p 34, in order to make it more visible through the "spaghetti" of all the other graphs. We see that it lies near to but not on the ecliptic.

Funkhouser suggests that this is a result of observational error. To observe the Sun against a background of stars, we need to make our observations at either dusk or at dawn, when the Sun is low in the sky and thus near the horizon. When observations are made under these conditions, there are two sources of error. First, the Sun is seen through a thicker layer of atmosphere than at other times. Second, there are a number of well-documented psychological tricks that our eyes and brains play on our perceptions in such circumstances.

The observations of Venus suffer the same problems. Because Venus is near the Sun, we see it only at dawn or at dusk. (Indeed, it is sometimes referred to as either "the morning star" or as "the evening star".) Funkhouser states that the text accompanying the "graph" hints at exactly such difficulties. It should also be noted that the same problems arise (even more acutely) with Mercury, but on this matter, Funkhouser is silent. (Mercury is actually very difficult to see – Copernicus died without ever having observed it because it isn't visible at all from the latitude where he lived. It could be that the "graph" for Mercury is theoretical rather than the product of actual observation.)

The lack of any explicit scale, either on the vertical or the horizontal axis, makes for other difficulties of interpretation. Even if McCuskey's suggestion is incorrect, we would expect to find some periodicity in the apparent orbits, and the various lines exhibit nothing like what we should see, *whatever* aspect of the planetary appearance is being represented.

Take as an example the Moon. This orbits the earth 13 times in the course of a year, and what is shown seems to be about one lunar cycle. On this interpretation, the horizontal scale represents a time of about four weeks. But now look at (say) Venus. Venus takes about 7 months to orbit the Sun, and what with the Earth taking 12 months, the combined motion takes about 84 months to repeat itself. But this would make the horizontal axis represent a time of about seven years. Similar analyses for the other planets complicate the picture yet further.

In the end, Funkhouser admits defeat in his attempt to produce a detailed and still consistent interpretation. He finds that the entire work was apparently "compiled as a text for use in monastery schools", and continues:

> "Considering the limited means of observation of the time and the absence of objective data, the graph can be considered scarcely more than a schematic diagram such as a teacher today might sketch on a blackboard for illustrative purposes."

Yet he pauses in this easy dismissal, and notes that:

> "... there is some interest in speculating upon what the maker had in mind for there is evidence that he exercised some care in drawing. There can be noticed an erasure and a correction of one of the curves [that for the Sun] in the center of the diagram. It is possible that a further study of the graph may reveal other advanced ideas in addition to that of use of a grid."

So either the earliest graph we know about is a rather crude example of its type or else we still lack the detailed knowledge of how it is meant to be interpreted.

For more detail on all this, see the references given at the website referred to above.

ooooooooooooooooooooooooooooo

Funfact

Take a pizza and pick an arbitrary point in it. Suppose you cut the pizza into 8 slices by cutting at 45 degree angles through that point, and shade the alternate pieces black and white.

Surprising theorem: the total area of the black slices and the total area of the white slices will always be the same!

In fact, this theorem is true for any multiple of 4 slices cut by using equal angles through a fixed arbitrary point in the pizza.

Alternatively, if instead of equal angles, you decide to use equal arcs on the circumference and slice from a fixed arbitrary point in the pizza, the theorem is still true!

Adapted from the math funfacts website of Harvey Mudd College.

THE PART-BURIED PIPE

Michael A B Deakin, Monash University

Some time ago, there was a discussion in another Mathematics journal of a problem that readers of *Function* could easily appreciate and learn from. Dr Michael Hirschhorn of the University of New South Wales wrote about it; apparently, one of his students had put it to him. It concerns the determination of the diameter of a pipe, circular in cross-section, and which only partly protrudes above the ground. See Figure 1, which shows the situation in cross-section. Ground-level is indicated by the line AB; the section ACB lies above the ground and the section ADB is buried. We can access, and so measure (with greater or less accuracy), three quantities:

x, the distance AB along the straight line APB,
y, the distance AB along the circular arc ACB, and
z, the distance CP giving the height of C above the ground.

The problem is to determine the radius, r let us call it, of the pipe.

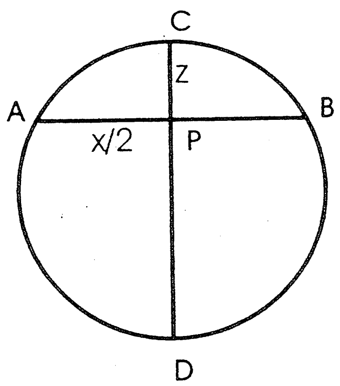


Figure 1

The discussion of this problem may be found in the *Gazette of the Australian Mathematical Society* (Dec 1997 and June 1998). The second of these contributions is from Professor E R Love of the University of Melbourne, and it is most elegant. Professor Love's solution of the problem uses a theorem in Euclidean Geometry (Theorem 35 of Book III of Euclid's *Elements*). If two chords *AB* and *CD* of a circle intersect at a point *P*, then the product of the lengths *AP* and *BP* will always be equal to the product of the lengths *CP* and *DP*.

Applying this to the special case of Figure 1, we have

$$z(2r - z) = \left(\tfrac{1}{2}x\right)^2$$

and so

$$r = \frac{1}{2}\left(z + \frac{x^2}{4z}\right)$$

This formula certainly gives a ready answer to the question, and it relies on our being able to measure both x and z with reasonable accuracy. Clearly the accuracy of the answer we arrive at depends on the accuracy with which the data can be known. I will get back to this point later.

However, Hirschhorn's original version of the problem made no use of z. There may be a good reason for this. In real life, we don't see the cross-section as drawn in Figure 1, but rather look down from above on a piece of pipe protruding from the ground. A flexible tape can measure y to considerable accuracy, and x can also be determined; in the case of a small pipe, a pair of callipers would give a good measure, and if the pipe is larger, then more sophisticated surveying equipment could be used. z may be rather harder to measure, although with a small pipe and reasonably level ground, a carpenter's gauge should give a reasonable estimate.

But now let us see what can be done to determine r from x and y alone.

Write $x / y = k$ and also introduce the angle θ as a shorthand for the angle *AOC*, where O is the centre of the circular cross-section of the pipe.

If we measure θ in *radians* (which is a good habit to develop!), then $y/2 = r\theta$ and we also have $x/2 = r\sin\theta$.

So now we can write

$$\frac{\sin\theta}{\theta} = k. \qquad (*)$$

k can be found from our measurements of x and y, and so we have an equation in the unknown angle θ. If we can solve this equation for θ, then r can be determined, because $r = \dfrac{y}{2\theta}$ and y is assumed known.

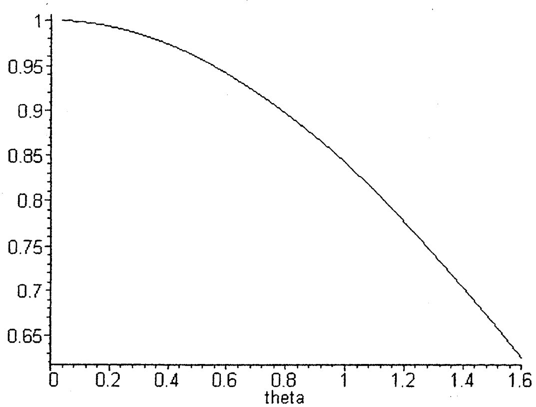


Figure 2

The Problem is that Equation (*) is not a standard type of equation. However, it is quite possible to solve it numerically if we are told the value of k. Look at Figure 2, which maps the left-hand side of Equation (*) over the range $0 \le \theta \le \pi/2$. (This is as far as we need to go, since when $\theta > \pi/2$, the radius is accessible to direct measurement; however, some of the formulae continue to apply all the way to $\theta = \pi$, at which point the pipe ceases to be buried at all!) This means that k varies between 1, when $\theta = 0$, and $\frac{2}{\pi} (\approx 0.637)$, when $\theta = \frac{\pi}{2}$.

Either the value of θ could be read off the graph or else it would not be difficult to tabulate the function and so approximate the correct value. Of course with a computer package such as Maple, the answer can readily be found to great accuracy.

Another approach is to use the little-known but handy formula

$$\frac{\sin\theta}{\theta} \approx 1 - 0.166605\theta^2 + 0.007610\theta^4$$

which produces errors less than 0.0002 over the range $0 \leq \theta \leq \pi/2$. However the solution of the resulting quadratic is probably harder than the tabular approach!

The question of determining the accuracy of the answer, given that our measurements themselves may not be particularly dependable is typical of the type of problem that arises when we apply Mathematics to the real world.

In this case, if θ is small, the graph shown in Figure 2 is nearly flat, and so we would expect the solution of Equation (*) to be relatively imprecise. This corresponds to our seeing very little of the pipe, so that our estimate of the radius might also and on this ground alone be expected to be rather imprecise. But even for quite large values of θ this can still be a problem.

Take a numerical example. Suppose we measured y to be 1.1 in some unit of length, and x to be 1.0 in this same unit. Then k is 0.909, and we find θ to be 0.746 (radians) and so r equals 0.737 units of length. However if our measurement of x were in error by 1% and the true value of x was really 0.99 units of length, then we would get θ = 0.746 (radians) and find a value of 0.699 units of length as our estimate of r. The error is over 5%. We would get a more accurate answer if more of the pipe were exposed, but otherwise matters could be even worse. The reader may like to consider what would happen if k were even closer to the value 1.

For a real-life example, I buried a metal can buried as an experiment in my back garden. The radius of the can was accurately determined as 8.9 cm. When the can was suitably in place, with just a small amount of its surface visible above the ground, I took a flexible tape measure and measured y to find with good accuracy a value of 16 cm. I then tried to measure x but with rather less success. I didn't have a pair of callipers which might have done the job very accurately, so I took a stiff ruler and tried to line it up as best I could. My various attempts gave values that seemed to vary between 14 and 15 cm, and so in the end I decided that 14.5 cm was the best result I could achieve. I made no attempt to measure z. This would have been quite beyond my available resources.

My value of x gave k = 0.906 and when I substituted this into Equation (*), and solved using Maple, I got $\theta = 0.761$ (in radians, of course), so that the result was r = 10.5, a pretty inaccurate result. Had I used x = *14*, I would have got on much better. In this case, k = 0.875, $\theta = 0.883$, and so we estimate r = 9.06. Calculating backwards, I deduced that the true value of x was 13.9. I would have had no hope of measuring it to this accuracy.

We can also use the known value of r to find the value of z. I leave this pleasure to the reader. The answer is z = 3.36, and I certainly would have had no hope of measuring it to anything like this accuracy. In view of my problems with x, I would probably have struggled to get even the approximation z = 3, and had I used this with my best value (14) for x, the computed value of r (as given by Professor Love's formula) would have been r = 9.7, which is worse than the 9.06 mentioned above. The problem with the use of z instead of y is that our estimate of r then involves two uncertain measurements instead of one.

Perhaps one final remark is in order. I checked the approximation

$$\frac{\sin\theta}{\theta} \approx 1 - 0.166605\theta^2 + 0.007610\theta^4$$

in each of my various solutions. In each case θ was very slightly overestimated.

NEWS ITEMS

1. The Clay Challenge Problems

Back in 1900, an International Congress of Mathematicians was held in Paris. One of the speakers was David Hilbert (1862-1943), who is often regarded as the greatest mathematician of the last century. His address posed 23 problems which he thought would influence the thrust of mathematical research in the new (ie, the twentieth) century. Hilbert's enormous influence, and the insight informing his choice of problem ensured that his problems did indeed exert great influence on the Mathematics of the twentieth century.

Over the past hundred years there has been much progress.

Out of the original twenty-three problems, eight were of a purely investigative nature. For example, the sixth was to produce an axiomatisation of Physics. To date, twelve of the remaining fifteen have been completely resolved. Quite remarkably, only one problem, the eighth of Hilbert's original list, the so-called Riemann Hypothesis remains as mysterious and challenging as ever, being now widely regarded as the most important open problem in Pure Mathematics.

To mark the turn of another century, the Clay Mathematics Institute, affiliated with Harvard University, has issued a more modest list of seven problems, not so much to influence the direction of research, as to draw attention to long outstanding unsolved mathematical questions. "Rather these problems focus attention on a small set of long-standing mathematical questions, each central to mathematics, that also have resisted many years of serious attempts by experts to solve them." To add spice to the challenge, each problem is associated with a $US1million prize for its solution. Readers can find out more detail from the website

http://www.ams.org/claymath/

which gives details of the different problems. Six of the seven are accompanied by detailed technical discussions, and the list of the authors responsible reads like a Who's Who of contemporary Mathematics. The seven problems are: 1. The P versus NP Problem, 2. The Hodge Conjecture, 3. The Poincaré Conjecture, 4. The Riemann Hypothesis, 5. Yang-Mills

Theory, 6. The Navier-Stokes Equations, 7. The Birch and Swinnerton-Dyer Conjecture.

These problems are all too technical to describe in detail in *Function*.

The first was the subject of a partial exposition in *Vol* **4**, *Part* **3**. Very Roughly, it concerns the computational complexity of mathematical problems. For example, we may not know the factors of 992 084 173 133 857, or whether it is prime or not. But *if we are told* that

$$992\,084\,173\,133\,857 = 9\,918\,851 \times 10\,002\,007,$$

then it is quite a simple matter to check whether this is correct or not.

The Hodge Conjecture concerns the ways in which volumes may be cut up. It is a distant relative of Hilbert's third problem, which was the subject of an article in *Function*, *Vol* **2**, *Part* **1**. [This problem, incidentally, was the first of the Hilbert problems to be solved: only a few months after it was posed.]

The Poincaré Conjecture concerns the 4-dimensional analogue of a relatively simple result in 3-dimensional topology. It arose from an incorrect statement in a 1900 paper by the great mathematician Henri Poincaré, and has resisted many subsequent attempts to solve it.

The Riemann Hypothesis is the one left over from Hilbert's list of 1900. It concerns the zeroes of a special function called the ζ-function and its solution would tell us a lot about the distribution of prime numbers. It has been checked and found to hold for the first $1.5 billion$ cases, but little progress has been achieved towards a rigorous solution. It is often described as the most important outstanding problem in all of Mathematics.

Yang-Mills Theory concerns our descriptions of fundamental particles. However, there are paradoxes with it. In particular "the 'mass-gap hypothesis', which most physicists take for granted and use in their explanation for the invisibility of 'quarks', has never received a mathematically satisfactory justification". This problem is the only one not accompanied by a detailed mathematical description on the website given above.

The Navier-Stokes Equations are very complicated and difficult partial differential equations that describe the flow of fluids. Their importance lies in their practical applications, and despite almost 200 years of intensive study, they remain only poorly understood.

The Birch and Swinnerton-Dyer Conjecture (named after two number theorists) is a technical result in number theory and is formulated in terms of the ζ-function.

So there are the points of challenge. A successful attack on any one will bring the solver not only a very tidy sum of money, but also enduring mathematical fame!

2. Other Millennial Challenges and Programmes

The Clay Challenge is only one among a number of such responses to the new century. Fernando Q Gouvea, an editor of the Mathematical Association of America's *Focus* and *MAA Online*, advises that during the year 2000, the American Mathematical Society published a book, *Frontiers and Perspectives* and that a conference on *Mathematical Challenges* was held last August in Los Alamos. There is an online report on this at

http://www.maa.org/news/math_challenges.html

and it is expected that this will also result in a book.

The publishing firm Springer-Verlag is understood to be planning a "Year 2000" book in their extensive series on Mathematics.

Gouvea writes, "All of these have in common the fact that they rely not on one person, but on a whole group of mathematicians. There probably is no one person who can survey the whole wide range of mathematics today."

However, this has not prevented one mathematician from trying to do exactly that. Stephen Smale is one of the most eminent of living mathematicians, and he has written an article "Mathematical Problems for the Next Century". Smale got in early; his article was published by the journal *Mathematical Intelligencer* in 1998 (*Vol* **20**, *No* **2**). It is available online; go to

http://www.city.edu.hk/ma/staff/smale/bibliography.html

and select Item 104.

It remains to be seen if future generations will judge Smale to be as pre-eminent in today's mathematical world as Hilbert was in his!

3. More on the Archimedes Palimpsest

In February 1999, *Function* carried a report on the discovery and sale at auction of two previously unknown mathematical manuscripts by the ancient Greek mathematician Archimedes. The works came to light only recently and were auctioned and knocked down to an anonymous bidder, but are now in the custody of a gallery in Baltimore, which has made them available to scholars (carefully selected scholars).

The works in question are those we know today as *On Floating Bodies* and *The Method of Mechanical Theorems*. These works had been originally written on parchment made from scraped and dried animal skins. Because such material was rare and expensive, it was often recycled. So the original writing, believed to have been copied in the 10^{th} Century from Archimedes' (c 300BC) original, was scraped off and the parchment written over. In this case, the rewrite was a monastery prayer-book, itself dated to the 12^{th} Century.

Such a work is referred to as a *palimpsest* and it is a matter of some technical difficulty to recover the original (scraped off) text. Two different teams of scientists are using highly sophisticated modern technology to do just this. The first team, from Johns Hopkins University, is using "hyperspectral imaging". This technique involves bombarding the text with ultraviolet light, which causes the parchment to fluoresce in spots where the original 10^{th} Century ink (now itself long gone) has altered the chemistry of the underlying parchment. In addition, this team is experimenting with confocal microscopy, a technique developed for biomedical research. It employs a scanning laser. The document is moved up and down in the microscope as the laser scans each page in an attempt to determine the underlying 3-dimensional structure of the remaining text.

The second team is from the Rochester Institute of Technology. They too are involved with high-tech approaches to the hidden text. As a report in the [American] Associated Press (13/10/'00) has it: "The two teams are using .... techniques .... developed for medicine and space research".

The initial effort is concentrated on the text of *On Floating Bodies*. The reason is that this work has hitherto been known *only* through translations. What all the high-tech methods reveal is the very first record for us of what Archimedes actually wrote.

Professor Reveil Netz, of Stanford University (California) notes that that the palimpsest is changing our notion of Archimedes' math from that based on a heterogeneous 1907 translation to something playful and creative, "and it can provide a glimpse into the head of someone who stands at the foundation of modern science".

The dual approach is a competition between the two teams (Hopkins and Rochester) to see who will be awarded the sole right to complete the task of analysing the entire manuscript. The initial aim is to produce a new edition of *On Floating Bodies* and this is promised for next September. By the end of the year, the fuller investigation will be under way, led by the successful team.

4. Yet More Approximations to Pi

Two further approximations to the number π have been sent to us, following our article in the last issue. The first is

$$\pi \approx \sqrt{\frac{40}{3} - \sqrt{12}} = 3.141533\ldots ,$$

and is known as "Kochansky's Approximation. (See p. 64.)

The other is very simple formula is

$$\pi \approx \sqrt{2} + \sqrt{3} = 3.146\ldots .$$

The relative error in Kochansky's formula is less than 6 parts in 100,000. The simple formula is much less accurate, with a relative error below 1.5 in 1000. But it is much simpler than Kochansky's formula and also simpler than many of the approximations given in our last issue. It is however less accurate than the familiar 22/7, which has a relative error of just over 4 in 10,000.

Both of the approximations given above are constructible with ruler and compass.

LETTER TO THE EDITOR

Complex Trigonometry

I have heard again from my wayward and eccentric Welsh correspondent Dai Fwls ap Rhyll, which is surprising as it means that I have heard from him directly and for the third successive year. A record, by all accounts!

For those who are new to *Function* and the letters I have written on Dr Fwls and his work, let me briefly summarise. Dr Fwls has achieved a reputation for calling into question established mathematical wisdom. He has found fault with prevailing orthodoxy in many fields of Mathematics; algebra, geometry, calculus and perhaps especially statistics have all fallen prey to his penetrating criticisms.

His habit is to write to chosen correspondents around the world. His letters pose some puzzle or other that shows the flaws in conventional Mathematics; he offers no resolution of any of the problems his analyses indicate. I am proud to say that I have been one of these trusted correspondents ever since 1980, when I first reported on his activities.

This year he has once again cast a critical eye on trigonometry, more especially the use of trigonometry in tandem with complex numbers. It has long been widely accepted that the trigonometric functions, sine, cos, tan etc, can be defined for complex numbers as well as for the more familiar real numbers. When the arguments of these functions are complex, then so may be their values.

Dr Fwls' latest investigation concerns the attempt to solve the complex trigonometric equation

$$\tan\theta = i,$$

where θ is the unknown and i is the square root of -1.

He writes, where A is any angle,

$$\tan(A+\theta) = \frac{\tan A + \tan\theta}{1 - \tan A \tan\theta}$$

and this step is clearly acceptable because it is known that the addition formulae that apply to real arguments also hold when those arguments are complex.

But now he uses the information that $\tan\theta = i$ to reach the equation

$$\tan(A+\theta) = \frac{\tan A + i}{1 - i\tan A}$$

By multiplying this equation top and bottom by $1 + i\tan A$ and simplifying, he concludes that

$$\tan(A+\theta) = i,$$

so that we have

$$\tan(A+\theta) = \tan\theta,$$

whatever the value of A. So now we may deduce that

$$A + \theta = n\pi + \theta,$$

where n is an integer. [This step also is known to hold in complex trigonometry, as well as in real.]

But now we recall that A was *any* angle whatsoever. It thus follows that

Every angle is an integral multiple of π.

All angles are (in essence) straight angles. This startling conclusion certainly comes as a shock! True, this time, Dr Fwls has used somewhat advanced Mathematics to reach his conclusion, but the result is as paradoxical as ever he has achieved.

Kim Dean, Erewhon-upon-Yarra

HISTORY OF MATHEMATICS

How to make your Name Immortal

Michael A B Deakin

In this column, I want to discuss some as yet unsolved problems in Mathematics. These problems are all easy to *state*; it is quite another thing to solve them of course. That is why they are still "open". But fame if not fortune awaits anyone who solves any one of them! All of them concern Number Theory, which (in our context here) is to do with the properties of the counting numbers, or, what is essentially the same thing, the positive integers. (Most of the time, we shall talk about the divisibility properties of these numbers.) So here when I use the term "number", this is what I mean.

Numbers, in this sense, come in three types:

(a) there is the special number 1, which is accorded a category all of its own,
(b) then there are *prime numbers*, which are divisible by no numbers other than themselves and that special number 1,
(c) finally, there are the composite numbers, which are all the rest (and so have other divisors).

The first few primes are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, … . Although a lot is known about the prime numbers, there is still a lot we don't know. In some cases, we simply *don't* know; in others, we think we *do* know, but strict proof is lacking (so it is always still possible that our "knowledge" is mistaken). One long outstanding problem is *Goldbach's Hypothesis*. This is easy to state, but has eluded many attempts to prove it. It says:

> *Every even number (apart from 2) may be written as the sum of two primes.*

Try it. $4 = 2 + 2$, $6 = 3 + 3$, $8 = 5 + 3$, $10 = 7 + 3 = 5 + 5$, $12 = 5 + 7$, etc. The assertion has been tested for all even numbers up to 4×10^{11}, and no counterexample has ever been found: that is to say, no-one has ever found an even number that could not be so expressed. It has been proved that every even number can be expressed as a sum of no more that 300 000 primes, and there are other technical results, but this one and all the others leave much to be filled in!

If we take the reciprocals of the counting numbers and add them up, we obtain a *divergent series*. That is to say that if we form the sum

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{N}=S(N)\text{, say,}$$

we may make this sum as large as we like, merely by taking enough terms. [It diverges painfully slowly; the first hundred terms get us only to 5.18… . We need 12367 terms to take the sum beyond 10.] For more on this series, see *Function*, *Vol* **1**, Part **4**.

In fact, we may estimate the value of *S(N)* fairly easily. As N grows larger and larger, then *S(N)* is better and better approximated by $\ln N$, the natural logarithm of N. To be precise, we have

$$\lim_{N\to\infty}\left(S(N)-\ln N\right)=\gamma,$$

where γ is a constant, known as Euler's constant (with a value of 0.577…). And here is the next unsolved problem.

Is Euler's constant irrational?

Almost certainly it is, but nobody has managed to prove this.

But now let us look for a while at some things that *have* been proved.

The first and most basic of these is the so-called "Fundamental Theorem of Arithmetic", which states that

Every composite number may be expressed as a product of prime numbers, and (apart from the trivial matter of rearranging the order in which they occur) this product may be constructed in only one way.

So, for example, $360 = 2\times2\times2\times3\times3\times5$, or $2^3\times3^2\times5^1$, and no other factors will give the product 360. Moreover, any composite number may similarly be expressed as such a product.

So fundamental is this assertion that many people see no need to prove it at all. When I was in primary school, I and my fellow-students learned to factorise numbers [in those days before the electronic calculator this was a useful computational skill], and it never occurred to any of us to question whether the procedure always worked nor whether the answer was unique. It was only when I went to University that I was acquainted with the need for a proof. It is a little difficult to decide in view of all this who it was that first perceived the need for a strict proof. Of the various proofs available today, two find their origins in Euclid's *Elements*, and the parts that have been added since are actually rather simple.

The next result was definitely proved a long time ago, for a complete and explicit proof appears in Euclid's *Elements*.

The number of primes is infinite.

There are several ways to prove this assertion. Euclid's original proof is, to my mind, still the best. It supposes that there are only finitely many primes. In other words, that list I wrote out up above comes to an end somewhere. In today's notation, we would write p_1 for the first prime, p_2 for the second, etc. [So $p_1 = 2$, $p_2 = 3$, and so on.] If there were only a finite number of primes, then we could, after a sufficiently diligent search, reach the last one, p_N, let us call it. Now form the number $p_1p_2p_3\ldots p_N + 1$.

This number cannot be divisible by any of the $p_1, p_2, \ldots p_N$ for it is exactly 1 greater than a multiple of each of these numbers. It is also not itself equal to 1. Therefore it is either a new prime not on the original list or else a composite number with prime factors not on this list (because of the Fundamental Theorem).

[Both these possibilities can actually occur. For $N \le 5$, the number so generated is prime, so that, for example, $2\times3\times5\times7\times11+1=2311$, which is prime, but $2\times3\times5\times7\times11\times13+1=30031=59\times509$ and

$$2\times3\times5\times7\times11\times13\times17+1=5105111=19\times97\times277,$$

where the factors displayed on the right of these expressions are all prime.]

The fact that the supposedly complete list turned out not to be complete after all is a contradiction, and so the assumption on which it was based (that the number of primes was finite) must be false.

Not only is the number of primes infinite, but the series formed by summing their reciprocals diverges. That is to say, the sum

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\ldots\ldots+\frac{1}{p_N} \quad \text{(where } p_N \text{ is the } N \text{ th prime)}$$

may also be made as large as we please simply by making N large enough. [This series is even more painfully slow to diverge than the earlier one. The sum of the first thousand terms is only 2.547… .]

I won't prove this result here. The proof will be the subject of my next column. But the result does show that the primes are sufficiently numerous to take the series (albeit very slowly) off to infinity. The result relates to another, one of two closely related results both known as the *Prime Number Theorem*. This states that (for large values of N) the N th prime is approximately $N \ln N$. This theorem is very hard to prove and I won't attempt it here (but see *Function, Vol* **11**, *Part* **1**, p 28). It gives us a (very crude) estimate of the sum of the first thousand terms of this latter series as 1.932… . [Quite a long way from the exact figure, but then in this context 1000 is a small number!]

Many primes occur in pairs, so that the difference between successive primes is exactly 2. We have (3, 5), (5, 7), (11,13), (17, 19), (29, 31), ….. . And the list goes on. Does it stop? Most people believe not. So here's another unsolved problem.

Is the number of prime pairs finite or infinite?

Most number theorists believe the list goes on forever. For example, Shanks in *Solved and Unsolved Problems in Number Theory* says of this conjecture that "the evidence is overwhelming"; Hardy and Wright in *An Introduction to the Theory of Numbers* note very strong numerical evidence from tables of primes, "evidence, [which] when examined in detail, appears to justify the conjecture".

On the other hand, if we add the reciprocals of the pairs of primes, then this time the series *converges*. This was first proved in 1919 by the Norwegian mathematician Viggo Brun. The sum

$$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\ldots\ldots$$

does not increase indefinitely as the number of terms increases. It approaches a finite limit. If there are only finitely many prime pairs, then this follows at once, because we would run out of terms to add; but the theorem is known to hold even if the number of such pairs should turn out to be infinite. Even then, as the number of terms increases, the sum tends towards a constant, known as B (after Brun, and called *Brun's Constant*).

Finding the value of Brun's Constant is a difficult and challenging problem. If we simply add finitely many values of the series then (unless it eventually terminates) we necessarily underestimate the value. There is no complete theory to supply an exact value (as there is for example with

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots,$$ which is known to sum to $\frac{\pi^2}{6}$).

So we need some way to estimate the error we incur when we stop. Unfortunately, this leads us to use further unproved (but generally believed) results, one of them being "the first Hardy-Littlewood conjecture", which estimates the value of the error (in the very long run) in terms that do not involve computing all those prime pairs.

I won't go into the exact statement of this new conjecture, but it does not do away with the need to compute a lot of primes. In 1976, the Australian mathematician (and Monash graduate) Richard Brent used the conjecture and also calculated all the twin primes up to 100 billion to find

$$B \approx 1.90216054.$$

More recently (1996), T Nicely computed all the twin primes up to ten thousand billion and also estimated Brun's Constant as

$$B \approx 1.9021\,6057\,78 \pm 2.1\times10^{-9}$$

(thus correcting Brent's final digit). [It was in the course of this work incidentally that Nicely found the bug in Intel's Pentium chip, so such work as this is not without its uses!]

Just to show how necessary it is to go to such massively large numbers, I computed for myself the first 50 prime pairs, which involved finding all the primes under 1500. The resulting estimate of Brun's Constant is 1.54… , which is woefully inadequate! If I add a correction of $1/\ln 1487$ (1487 being the *second largest* prime on my list), then I get up to 1.68, which is still short of the mark. That the addition of this "fudge term" improves matters is yet another conjecture awaiting proof. In fact much of this area of Mathematics consists of a web of interconnected conjectures, whose precise statements rapidly lose that immediacy that first confronted us when we first set out on this journey.

It is this aspect of the matter that makes it so difficult to make your name immortal; if life was easier, then these unsolved problems would not still be unsolved!

Further Reading

I have based this article on a number of sources. One is the article on Number Theory in the *Encyclopedia Britannica* by Tom Apostol and Ivan Niven, which is very clear and accessible. [Niven is the same Niven who wrote a nice article for *Function* some years back — see *Vol* 8, *Part* 1.] Rather more difficult are the articles once posted on a cluster of related websites, currently unavailable as explained in our last issue. See

http://mathworld.wolfram.com/TwinPrimes.html

and

http://mathworld.wolfram.com/UnsolvedProblems.html

if and when they once more become available.

A good general book on Number Theory is *An Introduction to the Theory of Numbers* by G H Hardy and E M Wright. [This is the same Hardy whose life and work were discussed in my column of June 1995.] The fifth (1979) edition is the most recent. For more specific information on unsolved problems, see R K Guy's *Unsolved Problems in Number Theory.*

COMPUTERS AND COMPUTING

A Journal of Online Mathematics

Cristina Varsavsky

There is new online journal for lovers of mathematics, and its focus is, precisely, online mathematics. The first issue of the *Journal of Online Mathematics and its Applications* (*JOMA*) appeared in January 2001 at

`http://www.joma.org`

JOMA is a publication of the Mathematical Association of America (MAA) devoted to online teaching and learning that complements other MAA existing journals. It is a refereed journal, and has the same high standards for which all MAA publications are known.

JOMA's Chief Editor is David A. Smith, based at Duke University, was one of the pioneers of the *Calculus Reform* movement in the United States, and is known by his strong commitment to providing relevant and exciting learning experiences to mathematics students. He sees *JOMA* as the grandchild of the prestigious *American Mathematical Monthly* with the mission of making the modern tools, curricula, and teaching and learning environments accessible to teachers and students everywhere.

According to the information provided in the journal's home page, *JOMA* will publish the following type of resources for students and teachers:

- innovative, class-tested, web-based learning materials,
- articles on design and use of online materials,
- original research articles on student learning via online materials and other technology-rich environments,
- surveys of existing online materials,
- high-quality "mathlets" (self-contained, dynamic, single-purpose learning tools), and
- other articles on related subjects, with particular emphasis on applications.

However, David Smith states in his welcoming article that this is only a starting list. This list will evolve to reflect changes in the mission of the journal to adapt to emerging technologies and the needs of the "readers". The content will be that of the college level mathematical sciences; given that this overlaps to some extent with the content corresponding to the upper secondary and lower undergraduate levels, the journal is quite suitable for senior high school and first and second-year university students and their teachers.

JOMA is the journal of the Mathematical Sciences Digital Library (MathDL) which is an exciting initiative funded by the National Science Foundation under the program titled Science, Mathematics, Engineering and Technology Education Digital Library. This American digital library of which MathDL is part, will be linked system of sites devoted to online resources.

All very well, but—you might wonder—what will you find on this web site? How will it be different from any of the other websites?

After reading David Smith's welcoming article, I went on to the next one where the *Mathlets* editor, Tom Roby, presents an overview of the *JOMA Mathlets Project.* After being harvested from known collections and everywhere around the world using search engines, the mathlets undergo a scholarly review following strict criteria. This first issue includes calculus mathlets which passed this scrutiny. I got quite absorbed with visualising polynomial approximations to functions, various graph plotters and explorers, a predator-prey simulation and other interactive applets, and made a few bookmarks to use these resources with my students. You do not only get to play with the these interactive teaching and learning tools, in many cases you can also download the code which is freely available to everyone.

Next I browsed through Xiao Gang's article on the Interactive Mathematics Server at the University of Nice (France), where I clicked on a link that caught my eye and took me to a world of games and puzzles, proofs without words, eye openers, things impossible, and other mathematical curiosities and exciting resources. I spent some time playing with and marvelling at the wealth of applets presented by Alex Bogomonly in the interactive column *Cut the Knot!* with magic squares, Nim games, freaky links, the Sierpinski gasket, bicolor towers of Hanoi, and many more. I could have stayed there for hours, but I had to come back to the real world ... otherwise this issue of *Function* would never have reached you!

I also had to have a look at the paper by Franz Embacher and Petra Oberhuemer about the Maths Online Collection at the Institute for Theoretical Physics in Vienna. I ended up solving linear equations, visualising sequences, calculating integrals, and again, made a few new bookmarks for my collection.

JOMA also has a Developers corner that provides services and a forum for discussion for those involved in building such educational tools.

I could fill pages describing the exciting things I found by reading this first issue of *JOMA*, but I leave it up to you to lurk down your own paths and perhaps even make a contribution to this fantastic resource available to all lovers of mathematics. The intention of this article was only to whet your appetite.

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TWENTY YEARS AGO

We reprint from time to time articles from the *Function* archives, and this time we are pleased to reprint a gem we discovered and included in the issue for April 1981.

THE HILTON SUMMATION

The *rth* triangular number t_r is defined as the sum of the first r positive integers:

$$t_r = 1 + 2 + 3 + ... + r.$$

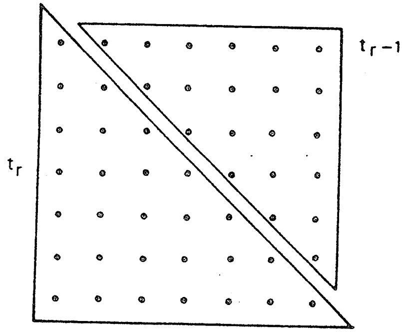
Clearly the triangular numbers satisfy the equation

$$t_r - t_{r-1} = r.$$

They also satisfy another equation

$$t_r + t_{r-1} = r^2.$$

The proof of this statement is geometric. [See the diagram below, which also makes clear the reason for calling t_r "triangular".]



Multiply these two equations together to find

$$t_r^2 - t_{r-1}^2 = r^3.$$

If we now add up the first n cubes, $1^3, 2^3, 3^3, \ldots, n^3$, we find

$$1^3 + 2^3 + 3^3 + \ldots. + n^3 = \left(t_1^2 - t_0^2\right) + \left(t_2^2 - t_1^2\right) + \left(t_3^2 - t_2^2\right) + \ldots\ldots + \left(t_n^2 - t_{n-1}^2\right)$$
$$= t_n^2, \text{ as } t_0 = 0.$$

It follows that

$$1^3 + 2^3 + 3^3 + \ldots. + n^3 = (1 + 2 + 3 + \ldots. + n)^2.$$

This proof (of a standard result) was first produced by Jeannette Hilton, then a 13-year old schoolgirl. It was published in *The Mathematical Gazette* in 1974.

PROBLEMS AND SOLUTIONS

SOLUTION TO PROBLEM 24.5.1

The problem asked for a proof that $(mn)! \geq (m!)^n (n!)^m$, where m and n are positive integers.

SOLUTION

The problem came from our sister magazine *Parabola* (NSW). Their solution is as follows.

Splitting the products into blocks of m factors we find that

$$\begin{aligned}
(mn)! = & (1\times 2\times \ldots \times m) \\
& \times((m+1)\times(m+2)\times\ldots\times 2m) \\
& \times((2m+1)\times(2m+2)\times\ldots 3m) \\
& \times\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
& \times(((n-1)m+1)\times((n-1)m+2)\times\ldots\times nm)
\end{aligned}$$

$$\begin{aligned}
& \geq (1\times 2\times\ldots m)\times(2\times 4\times\ldots\times 2m) \\
& \qquad \times(3\times 6\times\ldots\times 3m)\times\ldots.\times(n\times 2n\times\ldots\times mn) \\
& = m!\times\left(2^m\times m!\right)\times\left(3^m\times m!\right)\times\ldots\times\left(n^m\times m!\right) \\
& = m!\times m!\times m!\times\ldots.\times m!\times(1\times 2\times 3\times\ldots\times n)^m \\
& = (m!)^n (n!)^m
\end{aligned}$$

Solutions were also received from Keith Anker, Julius Guest and Carlos Victor.

SOLUTION TO PROBLEM 24.5.2

The problem (from *Crux Mathematicorum* and *Mathematical Mayhem*) read: The sequence $\{a_n\}_{n=1}^{\infty}$ is defined by $a_1 = 2$ and a_{n+1} = the sum of the 10^{th} powers of the digits of a_n, for all $n \geq 1$. Decide whether any number can appear twice in the sequence $\{a_n\}_{n=1}^{\infty}$.

SOLUTION (From Keith Anker)

Note first that

$$11 \times 9^{10} = 11 \times 9 \times 9^9 < 100 \times 10^9 = 10^{11}$$

Thus if we start examining the sequence where $a_n < 10^{11}$, then $a_{n+1} < 10^{11}$ also. $a_1 = 2$ is such a term.

Hence, after at most $10^{11} - 1$ terms, there is a duplicate, and terms repeat from there on.

[A different solution was sent to *Crux Mathematicorum* by Mansur Boase, a student at St Paul's School, London. Carlos Victor points out that no term can appear just twice.]

SOLUTION TO PROBLEM 24.5.3

For what natural numbers n is $7^n - 1$ a multiple of $6^n - 1$? (From the Czech Mathematical Olympiad, 1993.)

SOLUTION (From Carlos Victor)

No such n exists. Let $a = 7^n - 1$ and $b = 6^n - 1$. Then the possible last digits of a are 6, 8, 2 and 0. The last digit of b is 5. Thus for b to divide a, we need $n = 4k$. Then if $7^n - 1 = \lambda(6^n - 1) = \lambda(6^{4k} - 1) = \lambda((6^2)^{2k} - 1) = \lambda \times 35\psi$, where λ and ψ are integers. But this is impossible as $7^n - 1$ cannot be divisible by 7.

A different solution was received from Keith Anker.

SOLUTION TO PROBLEM 24.5.4

This problem (from *Mathematical Spectrum*) asked: There are n sheep in a field, numbered 1 to n, and some integer m (> 1) is given such that $m^2 \le n$. It is required to separate the sheep into two groups such that:

(1) no sheep has m times the number of a sheep in the same group, and
(2) no sheep has a number equal to the sum of the numbers of two sheep in its group.

For which values of m, n is this possible?

SOLUTION

A solution was received from Keith Anker and another from Carlos Victor. The solution printed in *Mathematical Spectrum*, where the problem first appeared, was due to Andrew Lobb, and proceeded along similar lines. We here print a composite of these solutions.

We first show that $n < 9$. To do this suppose that $n \ge 9$, and denote by N the group containing Sheep Number 9 and by N' the other group. There are then four possibilities:

> (1) $1 \in N$ and $2 \in N$. Then $3 \in N'$, by Condition (2), and 2 + 1 = 3. Similarly $7 \in N'$ and $8 \in N'$ (7 + 2 = 9 and 8 + 1 = 9). But then we note that, because 3 + 4 = 7 and 3 + 5 = 8, we need $4 \in N$ and $5 \in N$ for otherwise we have a contradiction. But now we deduce that $9 \in N'$ and this is a contradiction.
>
> (2) $1 \in N$ and $2 \in N'$. Then we may argue as above to show that $8 \in N'$, so that $6 \in N$, $5 \in N'$ and $7 \in N'$. From this it follows that $2 \in N$, and this is a contradiction.
>
> (3) $1 \in N'$ and $2 \in N$. Then $7 \in N'$ and so $6 \in N$ and $8 \in N$. Then $2 \in N'$, and once more we have a contradiction.
>
> (4) $1 \in N'$ and $2 \in N'$. Then $3 \in N$, $6 \in N'$, $4 \in N$, $5 \in N$, and so $9 \in N'$, again a contradiction.

But now $9 > n \geq m^2 > 1$, so $m = 2$. We now show that $n < 5$. For suppose otherwise: $n \geq 5$. Denote by O the group containing Sheep Number 1 and by O' the other group. Then $2 \in O'$ by Condition (1), and so $4 \in O$. Then, by Condition (2), $3 \in O'$ and $5 \in O'$. So $2 \in O$, and this is a contradiction.

Hence $n < 5$. But now $4 \leq m^2 \leq n < 5$ and so $n = 4$.

Thus $m = 2$ and $n = 4$. With these values, assign Sheep Numbers 1, 4 to one group and Sheep Numbers 2, 3 to the other to satisfy all the conditions imposed.

SOLUTION TO PROBLEM 24.5.5

This problem was also taken from *Mathematical Spectrum.* It supposed a triangle whose angles were measured in degrees as α, β and γ. We were also told that $\alpha^2 + \beta^2 = \gamma^2$. The problem asked for the possible values of α, β and γ.

SOLUTION

A solution was received from Keith Anker and another from Carlos Victor. *Mathematical Spectrum* received four answers and printed one by Can A Minh of the University of California at Berkeley. The solutions proceeded along similar lines and we here print a composite.

We have $\alpha^2 + \beta^2 = \gamma^2$ and $\alpha + \beta + \gamma = 180$. It follows that

$$\alpha^2 + \beta^2 = (180 - \alpha - \beta)^2 = 180^2 + \alpha^2 + \beta^2 - 360\alpha - 360\beta + 2\alpha\beta,$$

which simplifies to

$$\alpha = 180 - \frac{90 \times 180}{180 - \beta}$$

Now $0 < \alpha < 180$ and so $0 < \beta < 90$. Also $180 - \beta$ must divide 90×180, that is to say $2^3 \times 3^4 \times 5^2$, and $180 - \beta$ must be greater than 90. So the possibilities for $180 - \beta$ are 2×3^4, $2^2 \times 3^3$, $3^3 \times 5$, $2 \times 3 \times 5^2$, $2^3 \times 3 \times 5$ and $2^2 \times 5^2$. This gives possible values of β as respectively 18, 72, 45, 30, 60 and 80. Thus the possible triplets (α, β, γ) are (18, 80, 82), (30, 72, 78) and (45, 60, 75) and three others with α and β interchanged.

Here are some new problems.

PROBLEM 25.2.1 (Submitted by David Halprin)

If a hole of length 6 cm is drilled right through the centre of a solid sphere, what is the volume of the remaining material?

PROBLEM 25.2.2 (Submitted independently by J A Deakin and Julius Guest)

Sum the infinite series

$$1-\frac{1}{5}+\frac{1}{9}-\frac{1}{13}+\ldots\ldots\ldots .$$

PROBLEM 25.2.3 (Submitted by Claudio Arroncher)

ABC is a triangle, right-angled at A. H is the foot of the perpendicular drawn from A to BC; J is the mid-point of BC and M is the point where the angle-bisector of A meets BC. Given H, J and M, construct the triangle.

PROBLEM 25.2.4 (From *Mathematical Digest*)

Prove that if both the sides and the angles of a triangle are in arithmetic progression, then the triangle is equilateral.

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Kochansky's approximation to π is named after Father Adam Kochansky (1635-1700), who was librarian to King John III of Poland. For more details on the approximation, including its ruler-and-compass construction, see the website (in Dutch)

http://www.pandd.demon.nl/werkbladen/benpi3work.htm

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