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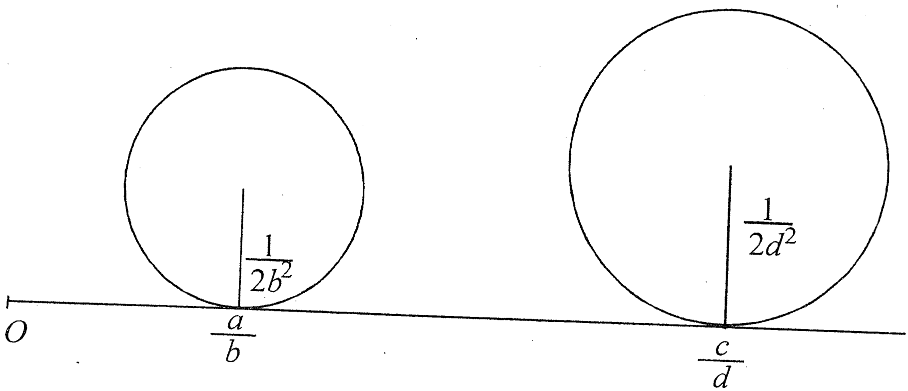

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* $13 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to our readers to our last issue of this millennium!

In this issue of *Function* there are articles for all mathematical tastes. The first article, by R King, deals with a not so well known ingenious geometrical representation of reduced fractions, and relates to a previous article on Farey fractions; the diagram on the front cover is from this article. For those interested in the mathematical aspect of games, we include an article by R D Coote, who analyses the hierarchies of the different types of hands in the game of poker and the likelihood of their occurrences. If you have a bent for algebra, then the article by Z Starc about inequalities involving the harmonic, geometric and arithmetic means is for you. The last feature article is a discussion by Peter Grossman about the controversial question of when does the millennium end; his conclusion is that we should keep the fireworks and champagne packed away for another year ...

The *History of Mathematics* column is about Euclidean and non-Euclidean geometries, and the remarkable results on geometries on curved surfaces given by the nineteenth-century mathematician Eugenio Beltrami.

The last article of this millennium of the *Computers and Computing* column had to be about the so called Millennium Bug. You will find there some background to the problem that has received so much media attention and is costing millions of dollars to fix. Reference is also made to a website with information about what you should do with your personal computer.

Finally, we also include solutions to previously included problems and a few new problems to keep you entertained until the next millennium.

Happy reading!

* * * * *

FORD CIRCLES

Rik King, University of Western Sydney

Readers would be familiar with the idea of reduced fractions – rational numbers where there is no possibility of cancelling to a lower form (e.g. 3/7 but not, of course, 3/9). Not so well known is an ingenious geometrical representation of reduced fractions, which was discovered in 1938 by an American, L R Ford. The diagrams constructed by him have become known subsequently as Ford circles.

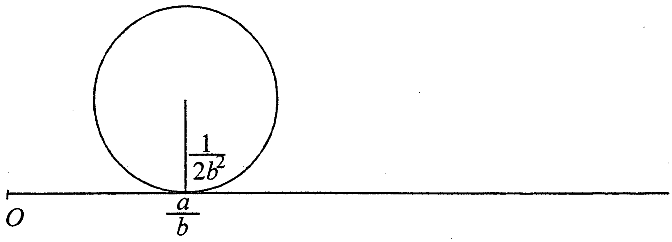

Figure 1

Let us begin to picture Ford circles in the following way. Suppose we have some positive fraction $\frac{a}{b}$ which is reduced, and that, without loss of generality $b > 0$. This fraction is laid out along the x-axis and a circle is drawn which is tangent to the x-axis at the point $\left(\frac{a}{b}, 0\right)$ and lies above the x-axis. The circle is chosen to be of radius $\frac{1}{2b^2}$, and consequently, it is centred on the point $\left(\frac{a}{b}, \frac{1}{2b^2}\right)$. Such a circle, shown in Figure 1, is known as the Ford circle corresponding to the fraction $\frac{a}{b}$. In a similar way, for another reduced fraction $\frac{c}{d}$, a circle of radius $\frac{1}{2d^2}$ and centre $\left(\frac{c}{d}, \frac{1}{2d^2}\right)$ is drawn with tangent the x-axis at the point

$\left(\frac{c}{d}, 0\right)$. Note, from Figure 2, that the circle corresponding to the second fraction $\frac{c}{d}$ has a larger radius, which means that, in the case of the illustration, the choice $b > d$ has been made: the larger the denominator of a fraction, the smaller the radius of its Ford circle.

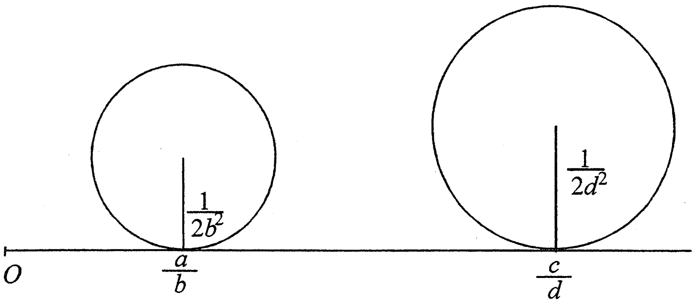

Figure 2

What is the distance between the centres of the Ford circles for $\frac{a}{b}$ and $\frac{c}{d}$?

Suppose we call this distance L. The following expression for L^2 follows from Pythagoras' theorem:

$$L^2 = \left(\frac{a}{b} - \frac{c}{d}\right)^2 + \left(\frac{1}{2b^2} - \frac{1}{2d^2}\right)^2$$

Now let us compare L^2 with the square of the sum of the radii of the circles, that is, we form the expression

$$L^2 - S^2 \text{ where } S = \frac{1}{2b^2} + \frac{1}{2d^2}$$

Thus

$$
\begin{aligned}
L^2 - S^2 &= \left(\frac{a}{b}-\frac{c}{d}\right)^2 + \left(\frac{1}{2b^2}-\frac{1}{2d^2}\right)^2 - \left(\frac{1}{2b^2}+\frac{1}{2d^2}\right)^2 \\
&= \left(\frac{a}{b}-\frac{c}{d}\right)^2 - \frac{1}{b^2d^2} \\
&= \frac{(ad-bc)^2-1}{b^2d^2}
\end{aligned}
$$

All the numbers involved are integers, which means then that $ad-bc$ will also be some integer. Furthermore $ad-bc \neq 0$ for if $ad-bc = 0$, then $ad = bc$, and so

$$\frac{ad}{bd} = \frac{bc}{bd}$$

hence

$$\frac{a}{b} = \frac{c}{d}$$

which contradicts our assumption.

The inference which may be taken from the above is that the Ford circles representing any two reduced fractions can never cross, which is, in its own right, a quite interesting result. The circles, however, might be tangent to each other: the special case when tangency occurs requires that the distance between their centres is equal to the sum of their radii lengths, i.e. $L = S$, and so $L^2 - S^2$ should equal zero, which in turn implies that $|ad-bc| = 1$.

But as discussed in the June issue of *Function* on pp. 74–79 this is precisely the same condition as found for the adjacency of two fractions in a Farey sequence! This special case where tangency of the circles occurs is certainly one of some interest, so that although the reduced fractions $\frac{a}{b}$ and $\frac{c}{d}$ for our starting point were chosen arbitrarily, for further investigation we now narrow the field so that they are restricted to be the adjacent terms in some Farey sequence.

Suppose that a third circle is inserted between the twocircles of Figure 2, one that is tangent to both of them and the x-axis, as shown in Figure 3. Actually, this configuration is a simplified version of a famous problem due to Apollonius, the "Great Geometer", who was born in 262 BC in southern Asia Minor, and who was a pupil of Euclid. The mechanics of how to carry out the drawing of the new tangent circle using just a ruler and compass is not the main concern here, and we shall merely presume that the required circle has been drawn without inquiring into the method used. The pertinent question here is: what will be the x-coordinate of its centre?

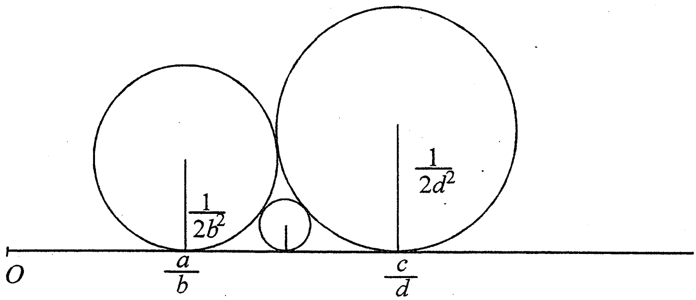

Figure 3

Let the x-coordinate be $\frac{c}{f}$. Because of the adjacency of the newly created circle to both of the circles tangent to the x-axis at $\left(\frac{a}{b},0\right)$ or $\left(\frac{c}{d},0\right)$ plus the Farey relationship, it is clear that $\frac{e}{f}$ is also some reduced fraction.

Recalling the previously derived tangency conditions, we have

$$ad - bc = -1$$

and now re-applying the same condition, because the new circle is also a tangent, we have

$$af - be = -1$$

Subtracting gives:

$$a(d-f) - b(c-e) = 0.$$

so

$$a(d-f) = b(c-e)$$

Now since we are dealing only with reduced fractions, since b does not divide a, it must be the case that

$$b \text{ divides } d-f$$

Therefore f may be written as d plus some multiple of b i.e.

$$f = d + nb,$$

where n is some integer, positive or negative. At this stage, in order to impose simplicity on the final result, we make the decision to set $n = 1$, so that

$$f = d + b,$$

Substituting for f gives

$$a(d-d-b) - b(c-e) = 0$$

so that $e = a + c$. That is, we now have

$$\frac{e}{f} = \frac{a+c}{b+d}$$

This is, however, the 'mediant' of the Farey fractions $\frac{a}{b}$ and $\frac{c}{d}$. The circle which is tangent to the circles correspondingto $\frac{a}{b}$ and $\frac{c}{d}$ has for its x-coordinate the mediant of $\frac{a}{b}$ and $\frac{c}{d}$. And, further, the mediant has been demonstrated previously to be also in reduced form.

What will be the radius of this new circle? Intuition would suggest that it might be $\frac{1}{2(b+d)^2}$, and this is, in fact, correct. There are a few ways to prove this result, the use of Pythagoras' theorem being one of them, and it is left to interested readers to find their own way through this proof.

For those wishing to explore further, it needs to be remarked that a number of simplifications were indulged in on the way through this article (e.g. the discarding of absolute value signs, and the choice of the value of n). the purpose of these shortcuts was to ensure a final result which was simple. With a more rigorous treatment, it can be found that, for any Ford circle, there are, in fact, an infinity of tangential Ford circles and whenever just a few of these are drawn, very intriguing geometrical patterns are formed.

References

Rademacher, H., 1983. *Higher Mathematics from an Elementary Point of view,* Birkhauser, New York.

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My idea of a joyful Christmas vacation was different [from my father's]. I arrived at the cottage on the coast with my precious [copy of] Piaggio's [*Differential Equations*] book and did not intend to be parted from him. I soon discovered that Piaggio's book was ideally suited to a solitary student. It was a serious book, and went rapidly enough ahead into advanced territory. But unlike most advanced texts, it was liberally sprinkled with "Examples for Solution". There were more than seven hundred of these problems. The difference between a text without problems and a text with problems is like the difference between learning to read a language and to speak it ... I started at six in the morning and stopped at ten in the evening, with short breaks for meals. ... Never have I enjoyed a vacation more.

–Freeman Dyson
in *Disturbing the Universe*, New York: Harper and Row, 1979.

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THE PITFALLS OF PLAYING POKER WITH A DEPLETED DECK – OR THE CASE OF THE ELUSIVE FLUSH

R D Coote, Wentworth Falls

Some people are aware of the hierarchy of the different types of hands in the game of poker: straight flush, four of a kind, full house, flush, straight, three of a kind, two pairs, one pair, and high card.

By looking at the number of ways each of these hands can occur, this order can be seen to be justifiable and fair.

There are ${}^{52}C_5 = 2598960$ different five-card hands that can be dealt and the numbers of each of the above types are as follows:

Straight flush	(Five consecutive cards in the same suit)	40
Four of a kind	(Four cards of the same kind and one other)	624
Full house	(A three of a kind and a pair)	3774
Flush	(Five cards from the same suit but not consecutive	5108
Straight	(Five consecutive cards not all from the same suit)	10 200
Three of a kind	(A three of a kind and two others)	54 912
Two pairs	(Two pairs and one other)	123 552
One pair	(One pair and three others)	1 098 240
High card	(None of the above)	1 302 540

Here are three examples of how these numbers were calculated; we leave to the reader to prove the rest:

- **Straight Flush**: five consecutive cards from the same suit (eg 3, 4, 5, 6, 7 all of clubs)

$$10 \times 4 = 40$$

(5 high through to A high) × (suits)

- **Flush**: five cards (not consecutive) from same suit (eg 2, 5, 6, 8, K all of hearts)

$$4 \times \left[{}^{13}C_5 - 10\right] = 5108$$

(diff. Suits) × (combs. of 5 from 13 – straight flushes)

- **Three of a kind**: three of a kind and any two other cards (eg Q, Q, Q, 4 S, 9C)

$$13 \times {}^{4}C_{3} \times 48 \times 44 \ / \ 2! = 54912$$

(type of 3 of kind) × (combs. of 3 from 4) × (1st other card) × (2nd other card)/(unorder)

In this article I will only consider what is initially dealt. I will not take into account discarding unwanted cards and drawing replacements.

Quite often the full deck of fifty two cards will be reduced to, say thirty two, by only using cards of face value seven and higher, This is referred to as sevens low.

In the case of a deck reduced to sevens low there are only ${}^{32}C_5$=201 376 different five card hands. The chance of being dealt a straight flush has increased from 40 in 2 598 960 to 16 in 201 376. This is an approximate five-fold increase. When we look at the number of ways all the various hands can occur, two anomalies appear.

Straight flush	16
Four of a kind	224
Full house	1344
Flush	208
Straight	4080
Three of a kind	10752
Two pairs	24192
One pair	107520
High card	53040

Firstly, one is approximately twice as likely to be dealt a pair than to be dealt nothing. Secondly we have the case of the flush. It is the second most unlikely hand to be dealt but it is still fourth in the standard hierarchy.

These anomalies are not restricted to sevens low decks. In fact the hierarchy "according to Hoyle" is only justifiable when using the intact deck. Even when using a deck with only the twos taken out, one is still more likely to be dealt a full house than a flush. Also in all but the standard deck and the threes low deck, one is more likely to be dealt a pair than nothing. With these problems should we change the rules when playing with a reduced deck?

The results are set out in the table below.

Full Deck

Type of hand	Ways	% of total	Rank
Straight flush	40	0.0015	1
Four of a kind	624	0.0240	2
Full house	3744	0.1441	3
Flush	5108	0.1965	4
Straight	10200	0.3925	5
Three of a kind	54912	2.1128	6
Two pairs	123552	4.7539	7
One pair	1098240	42.2569	8
High card	1302540	50.1177	9
Total	**2598960**		

7s Low

Ways	% of total	Rank
16	0.0079	1
224	0.1112	3
1344	0.6674	4
208	0.1033	2
4080	2.0261	5
10752	5.3393	6
24192	12.0133	7
107520	53.3927	9
53040	26.3388	8
201376		

6s Low

Type of hand	Ways	% of total	Rank
Straight flush	20	0.0053	1
Four of a kind	288	0.0764	2
Full house	1728	0.4584	4
Flush	484	0.1284	3
Straight	5100	1.3528	5
Three of a kind	16128	4.2781	6
Two pairs	36288	9.6257	7
One pair	193536	51.3369	9
High card	123420	32.7381	8
Total	**376992**		

5s Low

Ways	% of total	Rank
24	0.0036	1
360	0.0547	2
2160	0.3283	4
984	0.1495	3
6120	0.9301	5
23040	3.5015	6
51840	7.8783	7
322560	49.0207	9
250920	38.1333	8
658008		

4s Low

Type of hand	Ways	% of total	Rank
Straight flush	28	0.0026	1
Four of a kind	440	0.0405	2
Full house	2640	0.2431	4
Flush	2275	0.2095	3
Straight	7140	0.6575	5
Three of a kind	31680	2.9171	6
Two pairs	71280	6.5635	7
One pair	506880	46.6737	9
High card	463645	42.6926	8
Total	**1086008**		

3s Low

Ways	% of total	Rank
32	0.0019	1
528	0.0308	2
3168	0.1850	4
3136	0.1831	3
8160	0.4766	5
42240	2.4669	6
95040	5.5504	7
760320	44.4033	8
799680	46.7020	9
1712304		

I would strongly encourage readers to investigate other popular games. Ask yourself the question: What happens if you change the conditions?

This is real mathematics. You may even be doing some original research!

* * * * *

WHEN DOES THE MILLENNIUM END?

Peter Grossman

As the year 2000 approaches, the debate about when the current millennium really ends and the new one begins is becoming more heated. Many people argue that we should wait until 1st January 2001 before popping the champagne corks. Popular sentiment, however, favours 1st January 2000 as the start of the new millennium, and those who insist otherwise are accused of being tiresome pedants or party-poopers. The (admittedly ambitious) aim of this article is to lay the matter to rest once and for all, at least as far as readers of *Function* are concerned. This is not by any means the first article to appear on the subject, and it undoubtedly won't be the last. However, perhaps its appearance in *Function* will lend it some authority. If you have the misfortune to get into an argument with someone about when the millennium really ends, at least you can defend your position by saying you read it in *Function*!

Let's set the millennium question aside for a moment, and take a look at centuries. The word "century" is used in two slightly different ways. On the one hand, it can refer to any 100-year period, so in that sense we could say the current century started at any time we choose in the past 100 years. However, if we follow the usual practice of referring to the current century as the *twentieth* century, we are using "century" in its other sense: "one of the successive periods of 100 years reckoned forwards or backwards from a recognised chronological epoch", as *The Macquarie Dictionary* puts it.

For the calendar now in common use, the chronological epoch in question was the birth of Jesus, as calculated by the sixth century monk Dionysius Exiguus, using the information available to him at the time. (Modern scholarship actually puts Jesus' birth about five years earlier.) In this system, the year commencing with the birth of Jesus was AD 1, and the preceding year was 1 BC[1]. (There was no year 0.) It follows that the first century was the 100-year period beginning with the birth of Jesus, i.e., from the beginning of the year 1 to the end of the year 100. Then the second century was the next 100-year period (101–200), and so on. Continuing in this way, we find that our own century, the twentieth, started on 1st January 1901, and will end on 31st December 2000. The twenty-first century will begin on 1st January 2001. In particular, the year 2000 clearly belongs to the twentieth century.

[1] AD = *Anno Domini* (Latin for "in the year of our Lord"), BC = Before Christ. The alternative terms CE (Common Era) and BCE (Before the Common Era) are also used.

Notice that this conclusion is drawn from the fact that we refer to the current century as "the twentieth century". If, instead, we called it "the 1900s", it would run from 1st January 1900 to 31st December 1999 (which, after all, is also a 100-year period, so we are entitled to call it a century). A difficulty with this approach is that the period immediately preceding the 100s (the 00s?) would comprise only 99 years. (Either that, or 1 BC would have to be included to bring the total up to 100 years, which would be even more awkward.)

The situation in regard to decades raises a difficulty. The problem is that the popular nomenclature for decades is inconsistent with that used for centuries. By analogy with centuries, we should call the current decade "the 200th decade", but we rarely do; rather, we refer to it as the 1990s, or simply "the nineties". When the 1990s finish at the end of 1999, the 200th decade will still have another year to run. (If we continue to identify decades in this way, we will need a name for the decade from 2000 to 2009. Any suggestions?)

Returning to our original question, the key to answering it is: how do we refer to the current millennium? The fact is, we don't often refer to it at all. For people who choose to treat the period from 1000 to 1999 as the current millennium ("the one thousands", by analogy with the nomenclature we use to identify decades), the next millennium will start on 1st January 2000. No doubt this view enjoys popular support because of the strong psychological impact made by the rare change in the first digit of the year. If, however, we want to refer to the current millennium as the *second* millennium (and the next one as the third), the answer is unequivocal: the third millennium does not begin until 1st January 2001.

So keep the fireworks and champagne glasses packed away for another year. And, let's face it, concerns about the millennium bug are probably going to put a bit of a dampener on celebrations at the end of 1999 in any case. (Incidentally, even setting aside the question of when the new millennium starts, the term "millennium bug" is inappropriate for another reason. If computers had been around in 1899, we would have had the same problem then, so "century bug" would be a better term.) For now, those of us who are going to wait for the real turn of the millennium might have to put up with being called pedantic killjoys, but our time will come. When 31st December 2000 rolls around, we'll be partying like there's no tomorrow!

* * * * *

TWO INEQUALITIES FOR THE MEAN

Zdravko F Starc, Yugoslavia

Let a, b be positive real numbers. Let us denote the harmonic mean of a and b by $H = \dfrac{1}{\dfrac{1/a+1/b}{2}} = \dfrac{2ab}{a+b}$, the geometric mean by $G = \sqrt{ab}$, the arithmetic mean by $A = \dfrac{a+b}{2}$ and the quadratic mean by $K = \sqrt{\dfrac{a^2+b^2}{2}}$.

The following inequalities

$$\frac{2ab}{a+b} \le \sqrt{ab} \le \frac{a+b}{2} \le \sqrt{\frac{a^2+b^2}{2}}$$

that is,

$$H \le G \le A \le K$$

are valid, and equalities occur for $a = b$.

In this note we shall prove the following inequalities

$$\frac{a+b}{2} \cdot \sqrt{\frac{a^2+b^2}{2}} + \frac{2ab}{a+b} \cdot \sqrt{ab} \ge 2ab, \tag{1}$$

i.e.

$$AK + AG \ge 2G^2$$

and

$$\frac{a+b}{2} \cdot \sqrt{ab} + \frac{2ab}{a+b} \cdot \sqrt{\frac{a^2+b^2}{2}} \le \frac{(a+b)^2}{2} \tag{2}$$

i.e.

$$AG + HK \le 2A^2$$

We notice, that in the inequalities (1) and (2), equality occurs if and only if $a = b$.

We need the following result:

If $x_1 \geq x_2$ and $y_1 \geq y_2$ then

$$x_1 \cdot y_1 + x_2 \cdot y_2 \geq x_1 \cdot y_2 + x_2 \cdot y_1 \tag{3}$$

where equality occurs for $x_1 = x_2$ or $y_1 = y_2$. The inequality (3) immediately follows from $(x_1 - x_2)(y_1 - y_2) \geq 0$.

Letting $x_1 = \dfrac{a+b}{2}$, $x_2 = \dfrac{2ab}{a+b}$, $y_1 = \sqrt{\dfrac{a^2+b^2}{2}}$, $y_2 = \sqrt{ab}$ in (3) we obtain

$$\frac{a+b}{2} \cdot \sqrt{\frac{a^2+b^2}{2}} + \frac{2ab}{a+b} \cdot \sqrt{ab} \geq \frac{a+b}{2} \cdot \sqrt{ab} + \frac{2ab}{a+b} \cdot \sqrt{\frac{a^2+b^2}{2}} \tag{4}$$

In the first case, by virtue of $\dfrac{a+b}{2} \geq \sqrt{ab}$ and $\sqrt{\dfrac{a^2+b^2}{2}} \geq \dfrac{a+b}{2}$ the inequality (4) becomes (1). In the second case, by virtue of

$$\sqrt{\frac{a^2+b^2}{2}} \geq \frac{a+b}{2} \text{ and } \sqrt{ab} \geq \frac{2ab}{a+b}$$

the inequality (4) becomes (2).

* * * * *

Leibnitz believed he saw the image of creation in his binary arithmetic in which he employed only two characters, unity and zero. Since God may be represented by unity, and nothing by zero, he imagined that the Supreme Being might have drawn all things from nothing, just as in the binary arithmetic all numbers are expressed by unity with zero. This idea was so pleasing to Leibnitz that he communicated it to the Jesuit Grimaldi, President of the Mathematical Board of China, with the hope that this emblem of the creation might convert to Christianity the reigning emperor who was particularly attached to the sciences.

—Laplace
in *Essai Philosophique sur les Probabilités Oeuvres*

HISTORY OF MATHEMATICS

Going Non-Euclidean and being Euclidean all along

Michael A B Deakin, Monash University

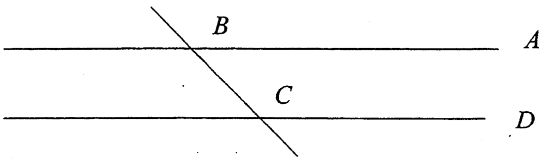
Euclid's *Elements* (his major work on Geometry) begins with 23 **Definitions** that give the meanings of the technical terms he uses, terms like "point", "straight line", "parallel", etc. It goes on to assert five **Postulates** (or **Axioms)** and a further five **Common Notions**. The Common Notions are Axioms also in the sense that they are assumptions that are not justified by reference to other material. They are often described as "self-evident". The first says that "Things which are equal to the same thing are also equal to one another", and I suppose we may think of this as "self-evident". The other four are in similar vein. What distinguishes the Common Notions from the Postulates is that the Postulates are specifically geometric in character while the Common Notions are general principles.

For the most part, the Postulates are also "self-evident". The first states that "[it is always possible] to draw a straight line from any point to any [other] point". The next two are similarly non-controversial. The fourth states "that all right angles are equal to one another", and this he need not have stated because it may be proved by Euclidean methods from the other Postulates, the Definitions and the Common Notions. So this too is non-controversial. It could even have been left out!

But a problem arises with his fifth postulate, which has been the source of great controversy and which has also led to much wonderful mathematics over the centuries since Euclid wrote. That Postulate reads:

> "That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced [ie. extended] indefinitely, meet on that side on which the angles are less than two right angles."

Rather a mouthful. A diagram explains it better. It is given on the next page, together with the relevant description.



If angles *ABC* and *BCD*, both lying to the right of the line *BC*, add to *less than* two right angles, then the lines *BA* and *CD* will meet if extended far enough to the right. On the other hand, if these angles addedto *more than* two right angles, then it could be shown from the Postulate that the lines would meet if extended far enough to the left.

In the case drawn, the angles add to exactly two right angles and so the lines are parallel. It will be clear to readers that this Postulate is much more meaty stuff than the other four. It strikes many readers as the sort of thing that really ought to be *proved*. After all it's hardly self-evident; in fact it's considerably *less* self-evident than many things Euclid *did* bother to prove (for example that any two sides of a triangle are together longer than the third).

Nor is this merely a modern perception. From the days of Euclid onwards there have been attempts to prove the fifth Postulate either from the others or else from some other simpler Postulate that Euclid failed to state. The later Greek mathematician Proclus (411–485AD) wrote about the fifth Postulate: "This ought to be struck out of the Postulates altogether; for it is a theorem involving many difficulties".

It was Proclus who found a better expression of the Postulate:

> "If any straight line cuts one of two parallels, it will cut the other also".

This is equivalent to a much later version, now named in honour of the Scottish mathematician and geologist John Playfair (1748–1819). Playfair's Axiom, as it is now called, reads:

> "Through any point not on a given line one and only one line can be drawn parallel to the given line".

There are even those who profess to find this statement in the work of Ptolemy (85–165AD), whose attempts to prove the Postulate are known to us from descriptions by Proclus. This is the form of the Postulate I will use for the rest of this article.

It is now known that Playfair's Axiom may be negated in two ways. Let the given line be called l and let the point be P. Euclid's assertion is equivalent to the statement that exactly *one* line through P will be parallel to l. All other lines through P will meet l. The negation could take the form of saying either that *no lines through P are parallel to l*, or else of sayingthat *more than one line through P is parallel to l*.

By the late 18th Century, and certainly by the early 19th, the suspicion began to grow that it was not actually possible to prove Playfair's Axiom from the other Postulates and that one or other or both of the negations might actually be possible. That is, we could deny Playfair's Axiom and yet produce self-consistent geometries.

If we suppose that more than one parallel may be drawnthrough l, then if there are two such there are infinitely many because we may draw any other line through *P between* these two lines and this new line can't meet l either. This possibility leads to what is now called Lobachevskian Geometry, after Nikolai Lobachevski (1792–1856), one of three founders of this research. (The others were Carl Gauss, 1777–1855, and János Bolyai, 1802–1860.) An account of this geometry was published in *Function*, *Volume* 21, *Part* 4, p. 110, and for earlier articles on the same topic, see *Volume* 12, *Part* 4, p. 107 and *Volume* 3, *Part* 1, p. 15.

The other possibility (no parallel lines) is now the basis of what is called Riemannian Geometry, after Bernhard Riemann (1826–1856). I will start with this case. It may at first seem odd that we could deny the existence of parallel lines, and if we stick to the familiar world of Euclidean Geometry, it seems quite silly. But reflect that we live on a curved surface, that of our planet, which to a good approximation may be regarded as a sphere.

Now, of course, there can be *no* straight lines on a spherical surface, but there are curves that may be drawn that are the *straightest possible*. Such "straight*est* curves" are called "geodesics" in the case of a general curved surface, and "great circles" in the particular case of the sphere. Great circles are those circles drawn on the surface of the sphere, and whose centres coincide with the centre of the sphere. On the earth, for example, the equator is a great circle, as is any circle drawn on the surface and passing through both poles. [For more on great circles, see *Function*, *Volume* 4, *Part* 1, p. 16; *Volume* 6, *Part* 4, p. 19 and *Volume* 6, *Part* 5, p. 8.]

When we study the geometry of objects drawn on the surface of a sphere, it is the great circles that take the place of the straight lines of ordinary flat (Euclidean) geometry. This new, spherical, geometry is very like our familiar Euclidean Geometry, but differs in one important respect. For there are *no parallel lines* as any two great circles must intersect (twice in fact). All of Euclid's other Postulates, however, apply to this new geometry, once we change the words "straight line" to read "great circle".

The great circles on a sphere, substituting for Euclidean lines, thus provide an example of Riemannian Geometry. Because we have an actual model for this Riemannian Geometry, we are now assured that it is self-consistent. Before we knew about the application to the sphere, we had no way of knowing that someone might not come along and find a contradiction in this new set of Postulates (and so end up giving a partial proof of Euclid's fifth Postulate).

But this important insight can be generalised: it is only a small part of a much larger story. In 1868, a mathematician called Eugenio Beltrami (1835–1899) proved some remarkable results concerning non-Euclidean geometries.

Beltrami was interested in finding how geometries could be set up on curved surfaces. He reasoned as follows. Suppose we set up a surface and use on it some sort of co-ordinate system embedded in that surface. This co-ordinate system will have to use geodesics in place of our familiar straight lines, because in general there won't be straight lines available.

So, on the sphere for example we could take the north and south poles and two diametrically opposite points on the equator (call them for convenience the east and the west poles). Then join the north and south poles by great circles (meridians) and do likewise for the east and the west poles. This gives us a handy set of co-ordinates for the sphere.

Beltrami then investigated measures of the curvature of the surface, and found that, in the case of the sphere, along each of these "co-ordinate lines", the curvature will be constant (equal in fact to the reciprocal of the sphere's radius). As there are two such co-ordinate lines at each point of the surface, and both share this property, this means that the product of the curvatures will be constant and positive – positive because both in this case the co-ordinate "lines" bulge outward. This product is called the gaussian curvature of the surface, after Gauss, whom we met briefly above.

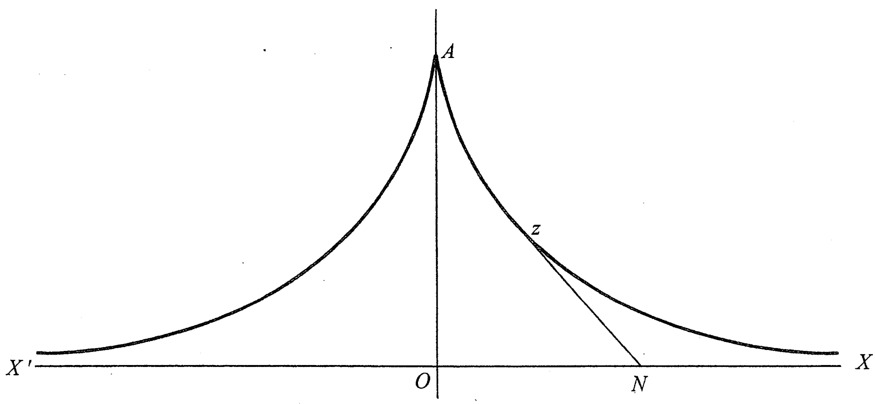
Beltrami next considered other cases, and proved two remarkable theorems:

1. If the gaussian curvature of a surface is constant, then any geodesic can be represented in a well-chosen co-ordinate system by means of an equation $y = mx + c$, where x and y are the co-ordinates, but this cannot be done if the gaussian curvature is not constant. (In the special case when the gaussian curvature is zero, we have the familiar x and y of co-ordinate geometry.)

2. If the constant gaussian curvature is positive, then the resulting geometry is Riemannian; if the constant gaussian curvature is negative, then the resulting geometry is Lobachevskian. If the gaussian curvature is zero, the geometry is Euclidean.

The interesting point now is to try to find a surface on which the curvature is such as to provide a model for Lobachevskian Geometry. This too Beltrami succeeded in finding. Here is a description of what he found.

On our Front Cover of *Volume* 22, *Part* 5, we showed a curve known as the **tractrix**. (For convenience it is reproduced below.) As was said then, the easiest way to think of the tractrix is to imagine someone dragging a weight along the ground by means of a string. Suppose the person set out from thepoint O and walked towards X. Suppose the weight was initially at A. When the person reaches the point N, the weight will have reached the point Z. The distance NZ remains constant and equal to OA, as it is assumed that the string does not stretch. The path traced out by the weight (i.e. the point Z) is the tractrix. It is usual to complete it by also including the mirror image formed if the weight was dragged to the left (toward X') instead of to the right.



Now suppose the tractrix is rotated about the line $X'OX$; a surface is generated. If we take any two points on this surface, it is of course not possible to connect them by a straight line. However there will be a "straightest" path joining them; this path will also be the shortest and these curves arethe *geodesics*. The surface so constructed is today called the "pseudo-sphere".

Each point on the pseudo-sphere lies on both a copy of the original tractrix and on a circle produced by the rotation. These are the co-ordinate lines we use. The tractrix is concave from the point of view of the line and thus its curvature is negative; the circle however has a positive curvature, as it "bulges outward". Thus these curvatures have a negative product and it turns out that this product is constant. The pseudo-sphere thus provides a model for Lobachevskian Geometry.

Beltrami's results showed that the non-Euclidean Geometries have the same claims to mathematical truth as has the Euclidean. Because we can find (on curved surfaces that exist in three-dimensional Euclidean space) cases of Riemannian and Lobachevskian Geometries, we learn that these cannot entail any contradiction (assuming that Euclidean Geometry itself is consistent, of course).

Since Beltrami's day there have been other theories reducing non-Euclidean geometries to versions of the standard Euclidean one. See for example *Function, Volume* 3, *Parts* 2 & 4. However, this is a nice and very convincing demonstration of the validity of Lobachevskian geometry. We now know that the alternatives to Euclid's fifth Postulate have exactly the same status as that postulate itself. And along the way Beltrami showed us a lot more about the structure of geometry itself. Perhaps the surprising aspect of the story is the role of that unusual curve, the tractrix.

* * * * *

Here I am at the limit which God and nature has assigned to my individuality. I am compelled to depend upon word, language and image in the most precise sense, and am wholly unable to operate in any manner whatever with symbols and numbers, which are easily intelligible to the most highly gifted minds.

—Goethe
In *Letter to Naumann (1826)*
Vogel: Goethe's Selbstzeungnisse, 1903.

COMPUTERS AND COMPUTING

That Bug: `ru y2k ok?`*

The Millennium Bug (also known as the "Year 2000 Problem" and the "Y2K Bug") has received wide media attention and is costing millions of dollars to fix.

It is the inability of "date sensitive" computer systems to operate before, during or after Saturday, 1st January 2000. Most problems will occur as a result of the year-date values being internally handled within the computer systems as only two digits: e.g. 01/01/2000 may be stored in the computer system as 01/01/00.

However, the problem is not limited to devices that you would normally have thought of as computers. Computer chips and computer technology are part of many things we take for granted on a daily basis, including lifts, electricity supply and supermarket checkouts.

In the 1950s and 1960s, memory for mainframe and "legacy computers" was expensive and very small in comparison to today. So any programs written in this era were optimised to require as little work-space as possible. By convention, dates were written with two digits for the year instead of the full four digits, with the 19 assumed. For example, a date stored as 01/01/75 was assumed to be in 1975. The developers of these early computer systems believed their systems would be obsolete and no longer in use by the turn of the century. But the practice continued well into the 1980s and 1990s.

A huge remedial effort has been under way across the world to fix the Year 2000 Problem and to check current computer systems. A large number of the computer programs developed in the 1990s are Year-2000 ready, but almost everything has to be checked.

Like many other large organisations, Monash University has a Year 2000 project under way to check that our major computer systems and technology are Year 2000 ready. You may care to check our website at:

http://www.adm.monash.edu.au/y2k/aware

* This article is an extract from http://www.adm.monash.edu.au/y2k/aware.

The millennium bug may cause a number of problems for your computers and/or applications. These include:

Rollover: Computers or applications may not be able to advance from one date to the next. The most common example is December 31, 1999 to January 1, 2000.

Century: Computers or applications may not be able to deduce the first two digits (of the four-digit year) given only the last two digits. For example 02/14/01 would be read as February 14, 1901, rather than February 14, 2001.

Leap Year: Applications may not be able correctly to calculate leap years. The year 2000 is a leap year. [A leap year occurs if the year is divisible by 4, but not if it is divisible by 100, unless it is also divisible by 400. The year 2000 is one of those years divisible by 400 and so is a leap year.] The complications involved in the formula mean that there is some doubt about how computers will calculate in the case of the year 2000.

Computation: Computer programs may not correctly determine the day of the week or other calendar matters when the interval involved spans across January 1, 2000.

Transfer: Computers may not be able to exchange information when one system is Y2K compliant and the other isn't, or if the two parties involved have used different methods to fix their Y2K problems.

Despite these problems, it is important to keep in mind that:

The World will not come to an end on 1st January 2000!

Monash University, like everybody else, will be faced with potential problems associated with the way devices such as computers have been programmed to handle dates.

Many of the problems will simply evaporate on closer inspection and others may be trivial and able to be safely ignored. However, it is important to recognise the scope of the problem and to be prepared.

The Millennium Bug is not just computer related but has the potential to affect all systems that use embedded microprocessors (in-built computer chips). This includes many manufacturing, household and university items of equipment such as research, medical and laboratory systems.

These systems are not confined to desktop PCs and mainframe computers.

Rather, since the 1960s these systems have gradually crept into the routine of our daily lives to a point where they are practically ubiquitous. Hidden away in everything from FAX machines to machine control systems they are to be found, and we have come to depend upon them. No one can determine with any accuracy how many are in operation around the world today.

For as long as the Millennium Bug has been recognised, nearly all the warnings of its threat have been targeted toward data processing in the area of Information Technology, predominantly in the banking and financial sectors. This has led to a false sense of security within the manufacturing and processing industries. Many manufacturers thought that because their machinery was controlled by a black box that didn't even look like a computer, then it wouldn't be affected. Nowadays a lot of effort is going into identifying and solving potential Y2K problems in such embedded systems.

The types of system likely to be affected include land, sea, air transport and traffic systems, industrial monitoring systems, operating systems, finance and investment management systems, payroll systems, accounting systems, medical systems and various industrial, business and domestic appliances.

There are several things you should do about your own home computer. For more information check out the website given above. This gives information on how to protect and if necessary to modify your hardware, your operating system, your software, your spreadsheets and databases and your data files. It provides links to free downloads and also information on commercially available software. However, it should be noted that this material is for information only; Monash makes no recommendations in this area.

Like all major institutions, Monash has had to invest a lot of time, money and effort into minimising the impact of the Y2K bug. We are quietly confident that we have done a good job, but the real test lies ahead of us! Time will tell!

* * * * *

PROBLEM CORNER

SOLUTIONS

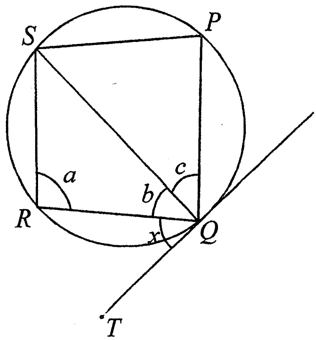
PROBLEM 23.3.1 (from Australian Mathematics Olympiad 1999)

Points P, Q, R and S lie, in that order, on a circle such that PQ is parallel to SR and $QR = SR$. Point T lies in the same plane as the circle such that QT is a tangent of the circle and the angle RQT is acute. Prove that

(a) $PS = QR$.

(b) angle PQT is trisected by QR and QS.

SOLUTION (by Ian Preston)



(a) Let angles $SRQ = a$, $PQS = c$, $SQR = b$, $RQT = x$

Due to $PQRS$ being a cyclic quadrilateral, opposite angles add to 180 degrees
$\therefore\ a + \text{angle SPQ} = 180.$

When traversing parallel lines the same side interior angles add to 180 degrees,
$\therefore\ a + b + c = 180$
$\therefore$ angle $SPQ = b + c$

Since the lines PS and QR are both at the same angle between 2 parallel lines, $PS = QR$.

(b) Angle RSQ is the corresponding angle to angle PQS
$\therefore$ angle $RSQ = c$
$RS = RQ$ (given)
$\therefore$ triangle SRQ is isosceles
$\therefore$ angle $RSQ =$ angle $RQS = b$
However angle $RSQ = c$
$\therefore b = c$.

The angle between a tangent and a chord equals the angle subtended at the circumference on the opposite side by the chord
If we take RQ to be the chord,
Angle $RQT = x =$ angle $RSQ = c$
$\therefore x = b = c$
$\therefore$ angle PQT is trisected by QR and QS.

Other solutions were received from Keith Anker, Carlos Victor and Frank Castro.

PROBLEM 23.3.2 (from Mathematical Spectrum)

Show that no prime number can be written as the sum of two squares in two different ways.

SOLUTION (Keith Anker)

We show that any number which can be expressed as the sum of two squares in two different ways is necessarily composite.

Suppose that $n = p^2 + q^2 = r^2 + s^2$.

If n is even, then it is already composite, for $1 + 1$ is the only way of writing 2 as a sum of two squares.

Therefore, n is the sum of an odd and an even square. From here, we follow a standard way of factorising an odd number expressed as a sum of two squares in two different ways.

Take p and r as odd, and q and s even (without loss of generality).

Let $p = ac + bd,\ q = ad - bc$; and
$r = ac - bd,\ s = ad + bc$.

We note that these definitions are reasonable, because with them we can check that

$$p^2 + q^2 = r^2 + s^2.$$

Thus, we have four equations for the four values, a, b, c, d, reducible to

$$ac = \frac{p+r}{2},\ bd = \frac{p-r}{2},\ ad = \frac{q+s}{2},\ bd = \frac{q-s}{2}$$

Take a as the *HCF* of ac and ad. This establishes the values of all four variables, each of which can be written as an *HCF*.

$$\begin{aligned}\text{Now, } (a^2+b^2)(c^2+d^2) &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + 2abcd - 2abcd \\ &= (ac+bd)^2 + (ad-bc)^2 \\ &= n.\end{aligned}$$

Hence n is composite.

So no prime number can be written as a sum of two squares in two different ways.

Other solutions were received from Carlos Victor and Frank Castro.

PROBLEM 23.3.3 (from Crux Mathematicorum with Math.Mayhem)

Find all real solutions of the equation

$$\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}$$

SOLUTION (Frank Castro, Brazil)

As $-1 \le x \le 1$ we put $x = \cos\theta$, $0° \le \theta \le 180°$.
So we get $\sqrt{1-\cos\theta} = 2\cos^2\theta - 1 + 2\cos\theta\sqrt{1-\cos^2\theta}$, which gives us $\sqrt{1-\cos\theta} = \cos(2\theta) + \sin(2\theta)$.

Squaring both sides of the last equality we have $1 - \cos\theta = 1 + \sin(4\theta)$ which can be written as $\sin(4\theta) = \sin(\theta - 90°)$. The only solution in the range is $\theta = 54°$. Hence $x = \cos 54°$ is the solution.

$$\left[\text{It is possible to show that } \cos 54^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4} \right]$$

Other solutions were received from Keith Anker, Carlos Victor and Julius Guest.

PROBLEM 23.3.4 (from Crux Mathematicorum with Math. Mayhem)

$ABCD$ is a square with incircle Γ. Let l be the tangent to Γ and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all perpendicular to l. Prove that

$$AA' \cdot CC' = BB' \cdot DD'.$$

SOLUTION

Without loss of generality we can take $A(-1,1)$, $B(1,1)$, $C(1,-1)$, $D(-1,-1)$, and the equation of the incircle is $x^2 + y^2 = 1$. Let $P(\cos t, \sin t)$ be any point on the incircle, and the equation of the line l tangent to the incircle at P, is

$$(\cos t)x + (\sin t)y = 1.$$

Now given a line with equation $ax + by - c = 0$, the perpendicular distance of a point (x_0, y_0) from the line is given by

$$\frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}.$$

Then we calculate

$$\begin{aligned} AA' &= d(A,l) = |-\cos t + \sin t - 1| \\ BB' &= d(B,l) = |\cos t + \sin t - 1| \\ CC' &= d(C,l) = |\cos t - \sin t - 1| \\ DD' &= d(D,l) = |-\cos t - \sin t - 1| \end{aligned}$$

and we easily obtain

$$AA' \cdot CC' = |\sin 2t| = |-\sin 2t| = BB' \cdot DD'.$$

Solutions were received from C. Victor, Frank Castro and Julius Guest.

PROBLEM 23.3.5 (K.R.S. Sastry, Bangalore, India)

Determine conditions on the integers b and c so that the three quadratic polynomials $x^2 + bx + c$, $x^2 + bx + c + 1$, $x^2 + (b+1)x + c$ factor over the integers.

SOLUTION (constructed from the solutions submitted by K. Anker and Frank Castro)

Let $P(x) = x^2 + bx + c$, $Q(x) = x^2 + bx + (c+1)$ and $R(x) = x^2 + (b+1)x + c$. Since both $P(x)$ and $R(x)$ factor over the integers we see that c must be even since two odd factors of c will always add to an even number, but b and $b + 1$ cannot both be even. Since $Q(x)$ factors over the integers and $c + 1$ is odd we deduce that b must be even. Let $c+1 = m \cdot n$ where m and n are odd, so using $Q(x)$ we have $b = m + n$.

Using $P(x)$ we see there must be an integer k such that $c = (m+k)(n-k)$ so that the sum of the factors is $m + n$.

We now have

$$1 = c+1-c = m \cdot n - (m+k)(n-k) = k(k+m-n)$$

and hence $k = \pm 1$ and $m = n$ so that $c = m^2 - 1$, $b = 2m$ where m is an odd integer.

Turning now to $R(x)$, for integer factors we require that

$$\Delta = (b+1)^2 - 4c = s^2$$

for some integer s. Since s and $b + 1$ have the same parity we have $s = 2t + 1$ for some integer t. Substitution of $b = 2m$, $c = m^2 - 1$ in the expression for Δ yields $m = \frac{1}{4}(s^2 - 5)$ and hence
$m = t^2 + t - 1$, $b = 2t^2 + 2t - 2$ and $c = t^4 + 2t^3 - t^2 - 2t$.

The polynomials $P(x)$, $Q(x)$ and $R(x)$ are then given by

$$P(x) = (x + t^2 + t - 2)(x + t^2 + t)$$
$$Q(x) = (x + t^2 + t - 1)^2 \text{ and}$$
$$R(x) = (x + t^2 - 1)(x + t^2 + 2t)$$

The proposer also provided a solution, and Julius Guest provided an algorithm for generating solution pairs (b, c).

PROBLEMS

PROBLEM 23.5.1 (from Mathematical Spectrum)

Let the complex numbers a, b, and c correspond to points A, B, and C in the Argand plane. Find an equation for the bisector of angle BAC.

PROBLEM 23.5.2 (from Crux Mathematicorum with Mathematical Mayhem)

The fraction $\frac{1}{6}$ can be represented as a difference in the following ways:

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}; \frac{1}{6} = \frac{1}{3} - \frac{1}{6}; \frac{1}{6} = \frac{1}{4} - \frac{1}{12}; \frac{1}{6} = \frac{1}{5} - \frac{1}{30}.$$

In how many ways can the fraction $\frac{1}{2175}$ be expressed in the form

$$\frac{1}{2175} = \frac{1}{x} - \frac{1}{y},$$

where x and y are positive integers?

PROBLEM 23.5.3 (from Crux Mathematicorum with Mathematical Mayhem)

Let $a = \sqrt[1992]{1992}$. Which number is greater,

$$a^{a^{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}} \text{ (1992 times)} \quad \text{or} \quad 1992?$$

PROBLEM 23.5.4 (from Mathematics and Informatics Quarterly)

A spider crawls randomly around the edges of a cube. At any vertex it moves to an adjacent vertex with probability $\frac{1}{3}$. The spider starts at one vertex. What is the expected value for the number of moves it will take the spider to reach the opposite vertex?

PROBLEM 23.5.5 (Julius Guest, East Bentleigh)

Determine

$$\int \frac{dx}{(1+\frac{1}{4}\cos x)^2}$$

* * * * *

Irrationality

The easiest number for which a proof of irrationality is possible is $\log_2 3$. For if

$$\log_2 3 = \frac{m}{n}$$

then,

$$3 = 2^{\frac{n}{m}}$$

and so

$$3^n = 2^m$$

But the left-hand side is odd and the right-hand side is even, which is impossible.

* * * * *

OLYMPIAD NEWS

Hans Lausch, Special Correspondent on Competitions and Olympiads

The 1999 Senior Contest Of the Australian Mathematical Olympiad Committee (AMOC)

The AMOC Senior Contest is the first hurdle for mathematically talented Australian students who wish to qualify for membership of the team that represents Australia in the following year's International Mathematical Olympiad. This year about seventy students took part in that four-hour competition on 11 august.

These are the questions:

1. Circle k_1 has its centre on another circle, k_2. The circles intersect at A and C. From any point B on k_2, draw BC, intersecting k_1 again at D. Prove that $AB = BD$.

2. Let x, y, z be integers with greatest comon divisor 1 such that $x^2 + y^2 = z^2$. Prove that exactly one of x, y, z is divisible by 5.

3. Let a_0, a_1, a_2 be real numbers such that $-1 \le a_0 + a_1 x + a_2 x^2 \le 1$ holds for all real numbers x which satisfy $-1 \le x \le 1$. Prove that
$$-2 \le a_2 \le 2.$$

4. Let A, B, C, D, E be points in the x-y-plane, whose coordinates are integers. Prove that among the line segments joining these points there is at least one with a midpoint whose coordinates are integers.

5. Let $ABCD$ be a cyclic quadrilateral whose diagonals intersect in a right angle at E. Let U, V, W, Z be on $AB, BC, CD, DA,$ respectively, such that $EU \perp AB$, $EV \perp BC$, $EW \perp CD$, $EZ \perp DA$. Prove that the quadrilateral $UVWZ$ is cyclic.

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