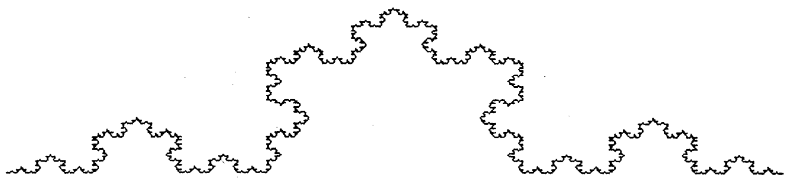
Function

A School Mathematics Journal

Volume 23 Part 3 June 1999



Department of Mathematics & Statistics – Monash University

Reg. by Aust. Post Publ. No. PP338685/0015

Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

* * * * *

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors, *Function*
Department of Mathematics & Statistics
Monash University
PO BOX 197
Caulfield East VIC 3145, Australia
Fax: +61 3 9903 2227
e-mail: function@maths.monash.edu.au

Function is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage): $20.00* ; single issues $5.00. Payments should be sent to: The Business Manager, Function, Department of Mathematics & Statistics, Monash University, Clayton VIC 3168, AUSTRALIA; cheques and money orders should be made payable to Monash University.

For more information about *Function* see the journal home page at `http://www.maths.monash.edu.au/~cristina/function.html`.

* $10 for *bona fide* secondary or tertiary students.

EDITORIAL

In this issue you will find quite an interesting collection of contributions.

Do you still remember when you first learned about adding two fractions? The usual mistake students make in primary school is to add the respective numerators and denominators. Our first feature article encourages you to "add" fractions taking this usually unacceptable approach in order to generate rather intriguing sequences known as *Farey fractions* which have some surprising properties.

Have you tried to solve the Three Churches Problem presented in last year's April issue? The problem gives the position of a boy with respect to three churches equally spaced from one another, and asks to determine the distance between the churches. Peter Grossman presents here an interesting generalisation of this problem. A nice article, specially for those readers who like geometry.

Vexatious Arithmetic is the topic of this issue's *History of Mathematics* column. The article starts with an old anonymous lament which expresses a mathematics student's frustration with multiplication, division, the Rule of Three, and Practice and looks at these techniques—some of which have already been forgotten by most readers—from a historical perspective.

The *von Koch's curve* on the front cover was generated with a computer program developed in the *Computers and Computing* section. This section also extends the related article published in the last issue of *Function* by describing further characteristics of the fractal which are useful for the design of the drawing algorithm.

The *Problem Corner* editor includes solutions to the problems in the February issue and new problems for the readers. He will also publish the best solutions received by September 1.

We wish you enjoy this issue of *Function*.

* * * * *

FAREY FRACTIONS

Rik King, University of Western Sydney

In your very earliest mathematical career in primary school you may perhaps have been reproved for doing the following sort of operations on some rational numbers.

$$\frac{1}{2}+\frac{2}{4}=\frac{3}{6}$$

No doubt you were not told then, but this is a perfectly legitimate way to define an operation on rational numbers, since these numbers (as distinct from the natural numbers) are our own creations, and any rules are applied solely at our own volition. The following definition of addition however,

$$\frac{a}{b}+\frac{c}{d}=\frac{a+c}{b+d}$$

does not produce results which agree with measurements in the real world. Thus, if you were to eat $\frac{1}{2}$ of a cake and your sister $\frac{1}{2}$, according to the above scheme of addition, you would have together have eaten only $\frac{2}{4}$ of the cake, whereas you know, in fact, that all the cake has been eaten. Thus the definition, while mathematically permissible, is not useful because it is not descriptive of reality. Oddly enough though, the subject matter of this article provides you with a little scope to indulge in the kind of addition used above, and, at the same time arrive at some quite interesting results.

John Farey was an English geologist of the early 1800's, whose name is forgotten in geology, but remembered in mathematics. In 1816 Farey published observations on the properties of ordered fractions in reduced form—and, as you would know already, a rational number $\frac{a}{b}$ is in reduced form if the greatest common divisor of a and b is 1. More specifically, Farey wrote down the ordered sequence of all non-negative fractions with denominators limited by some number of choice n.

For the sake of simplicity, we confine our discussion to the interval $0 \leq \frac{a}{b} \leq 1$, although this restriction is not strictly necessary. Then, following Farey, a table of fractions may be constructed in the following way:

For the first row write

$$\frac{0}{1} \qquad\qquad\qquad\qquad \frac{1}{1}$$

for the second row write

$$\frac{0}{1} \qquad \frac{0+1}{1+1} \qquad \frac{1}{1}$$

that is,

$$\frac{0}{1} \qquad \frac{1}{2} \qquad \frac{1}{1}$$

for the third row write

$$\frac{0}{1} \qquad \frac{0+1}{1+2} \qquad \frac{1}{2} \qquad \frac{1+1}{2+1} \qquad \frac{1}{1}$$

The rows we have constructed so far correspond to $n = 1, 2, 3$ and are as follows

$$\frac{0}{1} \qquad\qquad\qquad\qquad \frac{1}{1}$$

$$\frac{0}{1} \qquad\qquad \frac{1}{2} \qquad\qquad \frac{1}{1}$$

$$\frac{0}{1} \qquad \frac{1}{3} \qquad \frac{1}{2} \qquad \frac{2}{3} \qquad \frac{1}{1}$$

You can see that the rule we are using is this: from the n^{th} row by copying the $(n-1)^{\text{th}}$ row in order, but insert the fraction $\frac{a+c}{b+d}$ between the consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ of the $(n-1)^{\text{th}}$ row, provided that $b+d \le n$.

You should check that the 4th row will be

$$\frac{0}{1} \quad \frac{1}{4} \quad \frac{1}{3} \quad \frac{1}{2} \quad \frac{2}{3} \quad \frac{3}{4} \quad \frac{1}{1}$$

Note that we have not included the terms $\frac{2}{5}$ and $\frac{3}{5}$ for the row $n = 4$, since that would have violated the condition $b+d \le n$. Before continuing, you should construct the 5th row and check you answer from the last page of this article.

The rows shown above are known as Farey sequences of order n; and any reduced fraction with a positive denominator smaller than or equal to n is known as a Farey fraction of order n.

Farey observed the following feature of the sequences: the difference between adjacent terms was always 1 divided by the product of their denominators e.g. for $n = 4$ we have

$$\frac{1}{3}-\frac{1}{4} = \frac{1}{3.4} \text{ and } \frac{2}{3}-\frac{1}{2} = \frac{1}{2.3}$$

By now you would have noted the occurrence of the following pattern: if $\frac{a}{b}$ and $\frac{c}{d}$ are adjacent fractions, we form a new fraction, $\frac{a+c}{b+d}$ and the value of this appears to lie between $\frac{a}{b}$ and $\frac{c}{d}$. The new fraction, which is formed by the process discussed at the start of this article, is called the *mediant*. Now it is necessary to establish, for all values of n, the relative value of the mediant i.e. is it always the case that

$$\frac{a}{b} \le \frac{a+c}{b+d} \le \frac{c}{d}\ ?$$

Since we are dealing with positive numbers, we have that

$$\frac{a}{b} \le \frac{c}{d} \text{ implies } ad \le bc$$

Therefore

$$ad + ab \le bc + ab$$

and then $a(b+d) \le b(a+c)$. So $\frac{a}{b} \le \frac{a+c}{b+d}$.

Likewise, examination of the right hand side of the initial inequality returns us to the starting point. Thus, we can affirm that the value of the mediant always lies between the values of the two original fractions.

What of the property (1/product of denominators) observed by Farey, and easily verifiable in the first four sequences; may this property be associated with a general n?

Let us assume that 2 successive terms $\frac{a}{b}$ and $\frac{c}{d}$ do have a difference of the form $\frac{1}{bd}$. That is, $\frac{c}{d} - \frac{a}{b} = \frac{1}{bd}$.

Since $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$ it follows that $ad - bc = -1$.

We are seeking to show that the mediant $\frac{a+c}{b+d}$ always shares the same property with both of its adjacent fractions.

From the left hand side of the inequality, viz: $\frac{a}{b} \le \frac{a+c}{b+d} \le \frac{c}{d}$

we have $$\frac{a+c}{b+d} - \frac{a}{b} = \frac{b(a+c) - a(b+d)}{b(b+d)} = \frac{bc - ad}{b(b+d)}$$

and since $$b(a+c) - a(b+d) = bc - ad = 1$$

we have $$\frac{a+c}{b+d}-\frac{a}{b} = \frac{1}{b(b+d)}.$$

There will be a similar relation stemming from $\frac{a+c}{b+d} \le \frac{c}{d}$.

Thus, the mediant does indeed preserve the form of the difference between adjacent fractions, as noted by Farey.

A few more questions and uses of Farey fractions:

1. Is it the case that all proper fractions can be found by the process of forming mediants? The answer to this questions is in the affirmative. It is easy enough to prove but will not be followed through here.

2. Although interesting in their own right Farey fractions have also proved a useful stepping stone to some higher theoretical results in number theory. Also, with applied ends in view, they have been extensively tabulated and are very useful in forming rational approximations, as the following example shows.

   Suppose there is a practical problem of constructing a set of gears in the ratio of $\frac{1}{\pi}$, but using fewer than 100 teeth on the smaller of the two gears. You can easily check that the well known approximation for $\frac{1}{\pi}$, which is $\frac{7}{22}$, has an error of 10^{-4}. The lesser known $\frac{113}{355}$ is much better with an error of 10^{-8}, but it has more than 100 on the top line and is therefore of no use in this situation. In the $n = 1025$ Farey sequence however, there appears the ratio $\frac{113}{355}$ and, near it, the ratios $\frac{99}{311}, \frac{92}{289}, \frac{85}{267}$, as well as a number of others, all of which are better approximations to $\frac{1}{\pi}$ than $\frac{7}{22}$, and all of which satisfy the requirements of the problem in hand.

3. For a given value of n, how many Farey fractions are there? The exact answer to this has not been found. It turns out that for the sequence of order n, the number of fractions is about

$$3\left(\frac{n}{\pi}\right)^2$$

and that the approximation gets better as n becomes larger; the exact answer, however, is not known at the present time.

References

1. Conway, J H, Guy R K, 1996, *The Book of Numbers*, Springer-Verlag, New York

2. Rademacher, H, 1983, *Higher Mathematics from an Elementary Point of View*, Birkhauser, New York.

3. Schroeder, M R, 1982, *Number Theory in Science and Communication*, (2nd Edn), Springer-Verlag, New York.

Answer to question on page 76 is: $\left[\frac{0}{1}\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{2}{5}\ \frac{1}{2}\ \frac{3}{5}\ \frac{2}{3}\ \frac{3}{4}\ \frac{4}{5}\ \frac{1}{1}\right]$

* * * * *

A Minimum Problem

The Age "Odd Spot" recently consisted of this morsel.

> "A British quiz show contestant won $A319,000 despite getting a question wrong. He said 24 was the minimum number of strokes a tennis player needed to win a set. The producers later said the answer was 12 if the opponent double faulted every serve."

In what sense were the producers right?

* * * * *

> "I said, 'Lincoln, you can never make a lawyer if you do not understand what demonstrate means', and I left Springfield, went home to my father's house, and stayed there until I could give any proposition in the six books of Euclid at sight. I then found what 'demonstrate' means, and went back to my law studies."

–Abraham Lincoln

EQUILATERAL TRIANGLES AND THE THREE CHURCHES PROBLEM

Peter Grossman

The following problem (Problem 22.2.3) appeared in Problem Corner in the April 1998 issue of *Function*:

"Three churches A, B and C are equally spaced from one another, ie., they lie at the vertices of an equilateral triangle. George is standing at a point which is 8 km from A, 5 km from B and 3 km from C.

(a) Show that George must be outside the triangle.

(b) How far apart are the churches?"

Part (a) is easily solved using the triangle inequality. Part (b) requires more work, and eventually yields an answer of 7 km. (You can find a complete solution to the problem in the August 1998 issue.)

Can we obtain a more general result? It turns out that we can, and the answer is rather intriguing. The results presented here are based on information provided by our regular contributor K R S Sastry.

In the general problem, we have an equilateral triangle, ABC, with side length s. Let P be a point in the plane of the triangle, and let $PA = p$, $PB = q$, and $PC = r$. We seek a formula connecting the distances p, q, r and s. In order to find such a formula, let $\angle ABP = \alpha$, so that $\angle PBC = 60° - \alpha$. See Figure 1.

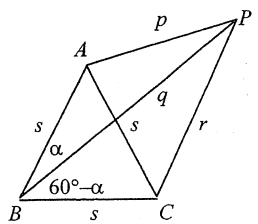


Figure 1

We apply the cosine rule to the triangle *ABP*:

$$p^2 = q^2 + s^2 - 2qs\cos\alpha \tag{1}$$

Likewise, applying the cosine rule to the triangle *PBC* yields:

$$r^2 = q^2 + s^2 - 2qs\cos(60° - \alpha) \tag{2}$$

By applying the identity $\cos(a-b) = \cos a\cos b + \sin a\sin b$ to equation (2), and using the exact values of $\cos 60°$ and $\sin 60°$, we obtain:

$$r^2 = q^2 + s^2 - qs\left(\cos\alpha + \sqrt{3}\sin\alpha\right) \tag{3}$$

Our task now is to eliminate α from equations (1) and (3). From (1), we get:

$$\cos\alpha = \frac{q^2 + s^2 - p^2}{2qs}$$

Using the fact that $\sin^2\alpha + \cos^2\alpha = 1$, we obtain an expression for $\sin\alpha$:

$$\sin\alpha = \sqrt{1 - \left(\frac{q^2 + s^2 - p^2}{2qs}\right)^2}$$

(We take the positive square root only, because we can assume that $0 \le \alpha \le 180°$.)

These expressions for $\cos\alpha$ and $\sin\alpha$ can now be substituted into equation (3). The details are messy, but after simplifying as much as possible, we eventually obtain:

$$p^4 + q^4 + r^4 + s^4 = p^2q^2 + p^2r^2 + p^2s^2 + q^2r^2 + q^2s^2 + r^2s^2 \tag{4}$$

In fact, (4) can be expressed in the following remarkable form:

$$3(p^4 + q^4 + r^4 + s^4) = (p^2 + q^2 + r^2 + s^2)^2 \tag{5}$$

You can easily show that equation (5) is equivalent to equation (4) by expanding both sides of (5) and collecting like terms.

It should come as no surprise that equation (5) is symmetric in p, q and r, the distances from P to the three vertices of the triangle: permuting p, q and r in any order does not change the equation. What is surprising is that *the equation is symmetric in all four variables.* If we examine this in the light of the three churches problem, this means not just that $s = 7$ is a solution of (5) when p, q and r equal 8, 5 and 3 (in any order); it also means that we can construct three other "three churches" problems with the same four numbers. For example, George could be at distances 3 km, 5 km and 7 km from the churches, in which case the churches would be 8 km apart. In geometric terms, we have the following result:

Any one of p, q, r and s satisfying equation (5) may be taken as the side length of an equilateral triangle ABC. Then there is a point P in the plane of ABC such that the distances from P to A, B and C are the remaining three values.

In order to obtain an explicit solution to the general three churches problem, we need to solve either equation (4) or equation (5) for s. For this purpose, equation (4) is easier to work with. Notice that it is a quadratic equation in s^2, because $s^4 = (s^2)^2$. We rewrite (4) as follows:

$$(s^2)^2 - (p^2+q^2+r^2)s^2 + (p^4+q^4+r^4-p^2q^2-p^2r^2-q^2r^2) = 0$$

We can now solve for s^2 using the quadratic formula:

$$s^2 = \tfrac{1}{2}\left(p^2+q^2+r^2 \pm \sqrt{(p^2+q^2+r^2)^2 - 4(p^4+q^4+r^4-p^2q^2-p^2r^2-q^2r^2)}\right) \qquad (6)$$

While this expression for s^2 is rather complicated, it does have an interesting feature. The discriminant of the quadratic (the expression under the square root) can be factorised to $3(p+q+r)(-p+q+r)(p-q+r)(p+q-r)$. By Heron's formula[1], this expression equals 48 times the square of the area, Δ, of a triangle with side lengths p, q and r. We may therefore write:

$$s^2 = \tfrac{1}{2}\left(p^2+q^2+r^2 \pm 4\sqrt{3}\Delta\right)$$

[1] Heron's formula states that the area of a triangle with side lengths a, b and c, and semiperimeter s, is $\sqrt{s(s-a)(a-b)(s-c)}$.

Equation (6) is a particular case of a theorem proved by H Eves in 1982. Eves' theorem provides a formula for the side length of a regular n-gon, in terms of the distances from a point in the plane of the n-gon to three consecutive vertices of the n-gon.

Equation (6) actually gives two values for s^2, depending on whether we take the positive or the negative square root of the discriminant. How do we know which one to take? Eves gave an answer to that question, too. We take the positive root if the point P lies inside the circumcircle of triangle ABC, and the negative root if P lies outside the circumcircle. (The discriminant is zero if and only if P lies on the circumcircle. This is the case in the three churches problem in the form stated at the beginning of this article.)

Can we be sure that the solution given by (6) is always real? The answer is yes; this follows from a theorem proved in 1852 by A F Möbius (the same mathematician who gave his name to the well-known Möbius strip). Möbius proved that the distances from the vertices of an equilateral triangle to a point in the plane of the triangle (our p, q and r) must themselves be the side lengths of a triangle or a degenerate triangle (the latter case occurring, for example, in the original three churches problem). This is sufficient to ensure that all the factors in the factorised form of the discriminant above are non-negative.

Rather than assuming at the outset that the equilateral triangle exists, we can vary the problem a little by assuming only that we have been given three distances, p, q and r. Provided that these three distances are the side lengths of a triangle (i.e., provided that the criterion proved by Möbius is satisfied), we can use equation (6) to evaluate s. In this situation, however, we will generally end up with two values of s, with no basis on which to choose between them. Eves showed that each of these values is the side length of an equilateral triangle that satisfies the problem.

Let's see how this works in a specific case. Let $p = 3$, $q = 5$, and $r = 7$. You will recognise these numbers as the example, from earlier in this article, of another "three churches" problem, constructed using the same four numbers as in the original problem. As expected, $s = 8$ is one of the solutions provided by equation (6). The other solution is $s = \sqrt{19}$. The two cases are depicted in Figure 2.

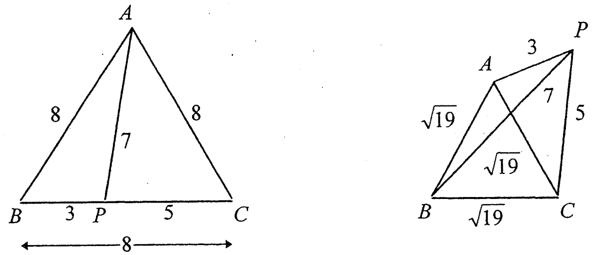


Figure 2

There are many ways in which you could explore this problem further. Drawing careful diagrams of particular cases, especially of related cases involving the same set of four numbers, may lead you to discover other patterns and make some conjectures. We have not even touched here on the problem of finding natural number solutions for p, q, r and s. One such set of solutions is $p = m^2 - n^2$, $q = 2mn + n^2$, $r = m^2 + 2mn$, $s = m^2 + mn + n^2$. The discriminant is zero for all solutions of this form, because $p + q - r = 0$. (Putting $m = 2$ and $n = 1$ gives the values in the original three churches problem.) Can you find any natural number solutions that are not of this form?

* * * * *

At a certain reception in his [Einstein's] honor at Princeton, when asked to comment on some dubious experiments that conflicted with both relativistic and prerelativistic concepts, he responded with a famous remark—a scientific credo—that was overheard by the American geometer, Professor Oswald Veblen, who must have jotted it down. Years later, in 1930, when Princeton University constructed a special building for mathematics, Veblen requested and received Einstein's permission to have the remark inscribed in marble above the fireplace of the faculty lounge. It was engraved there in the original German: "Raffiniert ist der Herrgot, aber boshaft ist er nicht," which may be translated "God is subtle, but he is not malicious." In his reply to Veblen, Einstein explained that he meant that Nature conceals her secrets by her sublimity, and not by trickery.

–Banesh Hoffman and Helen Ducas

in *Albert Einstein, Creator and Rebel*, New York: Viking Press, 1972

HISTORY OF MATHEMATICS

Michael A B Deakin

Vexatious Arithmetic

"Multiplication is vexation,
Division is as bad;
The Rule of three doth puzzle me,
And Practice drives me mad."

This anonymous 1570 lament encapsulates the reaction of a frustrated mathematics student and in so doing tells us a lot about the mathematics studied back then.

Four topics are mentioned – all unfavourably! However let me take them in turn.

Not all that long ago, certainly in my lifetime, calculating machines were not only expensive, but also very cumbersome and not readily available. In order to carry out even routine computations, such as multiplications or divisions, one had either to do them in one's head (see my column of August 1995), or else use pencil and paper.

To assist with this, every primary school pupil was required to memorise the "multiplication table": the list of all the products of one-digit numbers by one-digit numbers from $2\times 2=4$ to $9\times 9=81$. (In practice, the list was extended to $12\times 12=144$, as there were 12 pennies in a shilling in the old currency, but this detail is not needed for the discussion that follows.) Learning the "times tables" was a major preoccupation of the Year 4 Arithmetic syllabus in my own childhood. They actually comprise only 36 different facts, as $n\times m=m\times n$. (I recall that it greatly aided my own learning of the tables to be told this. My mother told it to me with just this aim in view, but in so doing she also sparked my lifelong interest in mathematics. Thanks Mum!)

Armed with these facts, one could multiply, first of all, multi-digit numbers by single digit numbers ("short multiplication") and later multi-digit numbers by one another ("long multiplication").

Here is an example of "short multiplication".

$$\begin{array}{r} 2345 \\ \times 7 \\ \hline 16415 \end{array}$$

The calculation proceeded from right to left as follows. $7 \times 5 = 35$. The 5 was written down (at the right of the bottom line) and the 3 was "carried". $7 \times 4 = 28$. 28+3=31. The 1 was written down and again the 3 "carried". $7 \times 3 = 21$. 21+3=24. The 4 was written down and the 2 "carried". The final multiplication was $7 \times 2 = 14$. $14 + 2 = 16$, and **this time** the entire number was written down. The rationale is:

$$7 \times 2345 = 7 \times (5 + 40 + 300 + 2000) = 35 + 280 + 2100 + 14000 = 16415.$$

"Long multiplication" proceeded by means of a number of "short multiplications" with the successive results added up.

Here is an example of "long multiplication".

$$\begin{array}{r} 2345 \\ \times 67 \\ \hline 16415 \\ \underline{14070\_} \\ 157115 \end{array}$$

The third line is exactly the calculation shown above. The fourth is generated by multiplying the first line by 6, but with an offset (to make the multiplier in fact 60). The results are then added.

Primary school students of old had a lot of practice in doing "long multiplications"! Possibly many did indeed find this "vexation"!

Division was, for most students, not merely "as bad" but actually worse. It too came in two flavours: "short" and "long". "Short" division was the division of (usually) a multi-digit number by a single digit number.

Here is an example of "short division":

$$\begin{array}{l} 7\overline{)4321} \\ \quad 617:2 \end{array}$$

The argument proceeds by first noticing that 7 will not divide an integral number of times into 4, but that we do have $43 = 6\times 7+1$. The 6 is written down and the 1 "carried". This "carried" 1 is combined with the 2 to make 12, and now $12 = 1\times 7+5$. The 1 is written down and the 5 "carried". The next step is $51 = 7\times 7+2$. The 7 is written down and a remainder of 2 is "left over". The calculation shows us that $4321 = 617\times 7+2$. The rationale is:

$$4321 = 4200+70+49+2 = 600\times 7+10\times 7+7\times 7+2 = 617\times 7+2.$$

"Long division" was rather more tedious. Its general principle was much the same as that of "short division", that is to say the successive subtraction of integral multiples of the "divisor" (7 in the above example) from the "dividend" (4321 in that example). What made it "hard" however was that with a multi-digit divisor we are taken outside the range of the memorised multiplication tables.

Here is an example.

$$\begin{array}{r} 310:21 \\ 27\overline{)8391} \\ \underline{81} \\ 29 \\ \underline{27} \\ 21 \end{array}$$

The first step is to notice that $83 = 3\times 27+2 = 81+2$. The 3 was written in the top line, the 81 under the 83, and the 2 discovered by subtraction. Next the 9 was "brought down" to form 29 and the process repeats. This time $29 = 1\times 27+2 = 27+2$, the 1 is entered into the top line, the 27 at the bottom of the calculation and the 2 discovered by subtraction. The 1 was"brought down, but the final number 21 remains less than 27 and thus forms the remainder. So we end up with $8391 = 310\times 27+21$. In this example the product $81 = 3\times 27$ might perhaps have been known to some of the better students, but where such products were not known they had to be derived by subsidiary calculations in a "work column" or scratch pad.

Now this much is not entirely unfamiliar. We all at least *know what is meant* by "multiplication" and "division". The processes I have outlined won't be entirely alien to readers of *Function*, even if (like me) they rarely use them today.

But "the Rule of three" and "Practice". What are these?

Well I learned of both when I went to school, even if today it's hard to find any textbook which discusses either. They are closely related, and both refer to the study of Ratio and Proportion.

The basic equation of the Rule of three is the equality

$$\frac{a}{b}=\frac{c}{d} \tag{1}$$

although this fact was often somewhat obscured by the use of an older notation that read

$$a:b::c:d.$$

This old notation was still used in my childhood, but by then it was recognised as obsolete. It is easier in what follows to use our more usual one. However the older notation explains why a and d were referred to as "the extremes" and b and c were termed "the means".

According to Webster's Dictionary, the Rule of three is the simple theorem that if Equation (1) holds, then

$$b\times c=a\times d,$$

"the product of the means equals the product of the extremes". However, Webster's is the only dictionary in which I have managed to find precisely this definition. Webster's goes on to say that the term refers to the method of finding the fourth term in a ratio when three are given; that is to say to the solution of Equation (1) for any one of a, b, c or d when the other three are given. The Macquarie Dictionary gives this latter as the meaning of the term and goes on to say that it is also referred to as "the golden rule". There *are* old Arithmetic books that use *this* term, but Webster confines it to its moral sense[1] and to popular derivatives of that. The Oxford Dictionary tells us that one of the meanings of "golden rule" is the rule of Three, and it attributes this terminology to Robert Recorde (1510?–1558), whose Arithmetic text was one of the first such in English.

[1]"Do unto others as you would have them do unto you." The maxim derives from two passages in the Gospels, Matthew 7:12 and Luke 6:31. It became abridged to "Do as you would be done by" and under this form provides the name of a character in Charles Kingsley's children's classic *The Water Babies*, which carries a strong line in Victorian morality (Kingsley was a parson).

A somewhat later text, *Baker's arithmetick* (1670), has this to say:

> "The rule of Three, is the chiefeſt, the moſt profitable, and the moſt excellent Rule of all the Rules of Arithmetick. For all the other Rules have need of it, and it uſeth all the other, for the which cauſe it is said, that the Philoſophers did name it the Golden Rule."

By the time I studied Ratio and Proportion in school, the term "Rule of three" was still around, but only just. It was rarely mentioned. Even earlier, it was seen as already out of date. The I(nternational) C(orrespondence) S(chools) Reference Library's *Arithmetic* (1922) says, "In the arithmetics of our grandfathers it [Proportion] was called 'The Rule of Three'".

However, if the term "Rule of three" was almost dead in 1952 in Tasmania, where I studied the subject, certain other terms were still very much current. These were wonderful-sounding Italian words, and our Arithmetic text gave them all.

They referred to the equivalence of equation (1) to other, related equations.

$$\frac{b}{a}=\frac{d}{c}, \quad \frac{a}{c}=\frac{b}{d}, \quad \frac{a+b}{b}=\frac{c+d}{d}, \quad \frac{a-b}{b}=\frac{c-d}{d}, \quad \frac{a+b}{a-b}=\frac{c+d}{c-d}.$$

These results were known respectively as *invertendo*, *alternando*, *componendo*, *dividendo*, and the magnificent *componendo e dividendo*. The results were known to Euclid and appear in Book V of his *Elements*. Heath, the editor of the standard English version of this work, notes that *componendo* corresponds to what Euclid called *synthesis*; however Heath uses the term *separando* in place of *dividendo* and he makes it the equivalent of Euclid's Greek term *diairesis*.

These rules and the Rule of three itself may be proved by straightforward algebra, but the next rule has a little more oomph to it. It considers an equation more elaborate than equation (1). Suppose we have a whole string of equal ratios (numbers in proportion)

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\ldots$$

and suppose that $m, n, p, \ldots$ are any numbers. Then

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ \dots \ = \frac{ma + nc + pe + \dots}{mb + nd + pf + \dots}$$

This may be proved by setting each of the original ratios equal to t (say), so that $a = bt$, $c = dt$, $e = ft$, ... Then the extreme right term is also equal to t.

[I needed to teach this relation, which seems to have dropped out of the school syllabus, to a class of Final Year Honours students some years ago. It is needed in the solution of certain advanced problems in the calculus.]

If the Rule of three was the "Pure Mathematics" behind all this, Practice was the Applied Mathematics. It concerned the commercial applications of the Rule of three. Baker introduces Practice as "Certain brief Rules, called, *Rules of Practice*: With divers neceſſary *Questions* profitable not only for *Merchants*, but alſo for other *Occupiers* [i.e. people in other occupations]", and Chambers' *Cyclopædia*, in its 1727–1741 edition, has this to say:

> "*Practice*, in arithmetic, *Practica Italica*, or *Italian usages*; certain compendious ways of working out the rule of proportion [i.e. the Rule of three] ... . They were thus called from their expediting of practice and business; and because first introduced by the merchants and negotiants of Italy [this explains the Italian in the names of the theorems above]."

Much of the thrust of Practice, and the need for it, arose from the complicated systems of money and of weights and measures then in use. In the case of money, twelve *pennies* (or *pence,* abbreviated as *d*) made one *shilling* (abbreviated as *s*), and twenty *shillings* made one *pound* (£, an archaic form of the letter L).[2] When decimal currency was introduced, the value of the dollar was set at ten shillings.

All the measures of length, weight, volume, and the like were likewise complicated. When it came to length, for example, twelve *inches* made one *foot*, three *feet* made one *yard*, and the loveliest one was that five and a half *yards* made one *rod, pole* or *perch*. Four of these things made one *chain*, ten *chains* made one *furlong* and eight *furlongs* made one *mile*. [The Australian Government, incidentally, abolished the "rod, pole or perch" some considerable time before it took the bolder step and introduced the metric system.]

---

[2] The *d* stood for *denarius*, a coin used by the Romans, the *s* stood for *shilling* and the £ for *librum*, a Roman measure of weight, translated as "pound" and here applied to precious metal. The term *£sd* was pronounced as **LSD**, and meant "money". It did not refer to drugs – unless you count money as a drug!

The measurement of area was in part related, but only in part. While 30.25 (i.e. 5.5×5.5) *square yards* made one *square rod, square pole* or *square perch* (abbreviated *sq po*), the table then went on a detour. 40 of these things made one *rood* (*r*) and four *roods* made one *acre* (*ac*). After this 640 *acres* made up one *square mile*.

Practice was the costing of various quantities of material when a unit price was given. It came in two forms, as with multiplication and division. These were "Simple Practice" and "Compound Practice".

Here is an example of Simple Practice. Both it and my next come from the ICS *Arithmetic* referred to earlier.

Problem: Find the cost of 562 articles at £12 3*s* 4*d* each.

Solution:

	£	*s*	*d*	
	562	0	0	= cost at £1 each
			$\times 12$	
	6744	0	0	= cost at £12 each
$3s\ 4d = \frac{1}{6}$ of £1	93	13	4	= cost at 3*s* 4*d* each
	6837	13	4	= total cost.

The total cost was £6837 13*s* 4*d*. This example made use of the then well-known fact that 3*s* 4*d* made up exactly one-sixth of a pound. The School Mathematics classes of my youth made great play with many such special (and in context useful) details. I still remember most of them, but today they have curiosity value only.

Back then, it was possible to exercise great ingenuity and skill in using such "aliquot parts" as they were termed, and many sales assistants even could do so in their heads, combining speed of calculation with accuracy in the result.

Compound Practice combined the complexities of the monetary system with those of one or other of the weights and measures. Here is an example. It concerns the price of a block of land and proceeds by using aliquot parts of the area. It would also have been possible to do it otherwise by (as above) making use of aliquot parts of the price. If the calculation had been set out from this point of view, then one way to get the 5*s* 4*d* into the story would have been to use, as above,

$3s\ 4d = \frac{1}{6}$ of £1 and the further relation $2s\ 0d = \frac{1}{10}$ of £1. But there are also several other ways the calculation could proceed. See if you can sort out the logic of the calculation as it is presented.

Problem: Find the cost of 60*ac* 3*r* 20 *sq po* of land at £6 5*s* 4*d* per *ac*.

Solution:

	£	*s*	*d*		*ac*	*r*	*sq po*
	6	5	4	= cost of	1		
			$\times 60$				
	376	0	0	= cost of	60	0	0
$\frac{1}{2}$ of 1 *ac*	3	2	8	= cost of	0	2	0
$\frac{1}{2}$ of 2 *r*	1	11	4	= cost of	0	1	0
$\frac{1}{2}$ of 1 *r*	0	15	8	= cost of	0	0	20
	381	9	8	= cost of	60	3	20

So the total cost was £381 9*s* 8*d*.

I am sure that after all this, you are grateful for your pocket calculators for Multiplication and Division, and to Decimal Currency and the Metric System for the demise of Practice. How the writer of the dismal little rhyme would have appreciated these modern advantages! But I'm afraid that none of this would have stopped the Rule of three from still puzzling him. Even today we need to understand the basis of the Mathematics we study. Machines may help us but they can't do the understanding for us!

* * * * *

The ultimate inflation

"Infinity rose by 8% last week."

–Magnut Heystek, *Cape Talk 567*

* * * * *

COMPUTERS AND COMPUTING

Programming a Snowflake

Cristina Varsavsky

In the last issue of *Function*, Anthony Sofo presented some very well known and interesting fractal shapes, which included the *von Koch's Snowflake*, also known as *von Koch's Island*. This curve is generated on the perimeter of an equilateral triangle by successive stages of removing the middle third of each side and filling the gaps with the upright sides of equilateral triangles.

I am sure that some of our readers with a bent for programming might have asked how could such a curve be drawn on the computer screen to any desired accuracy. As the author showed in the article, it is impossible to draw the *von Koch's* curve accurately; the curve has infinite perimeter and such a task would take for ever. We can only approximate the curve with a polygonal formed with very short line segments; however, this curve does not contain any line segments.

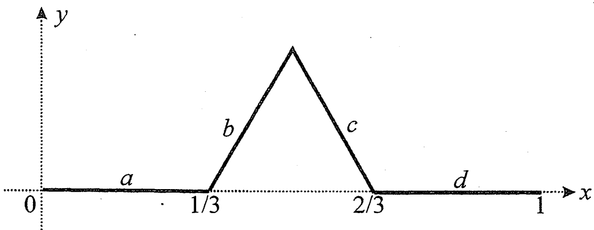


Figure 1

We will need to make some further analysis of the geometric properties of the segments —their length and position— before we set out to write the program. Let us focus only on one side of the triangle. At the first iteration we have a curve made of four segments which we label a, b, c, and d as shown in Figure 1. If we use a set of cartesian axes x and y and we place the starting point at $(0,0)$ and the end point at $(1,0)$, then the line segment a joins the points $(0, 0)$ and $(1/3, 0)$. The line segment b makes an angle of $60°$ with the x–axis, therefore the coordinates of the second endpoint can be obtained by adding to the coordinates of $(1/3, 0)$, one third of $\cos 60^{\circ}$ and $\sin 60^{\circ}$ respectively, that is,

$$(1/3 + 1/3 \cos 60^{\circ}, 0 + 1/3 \sin 60^{\circ}) \approx (0.5, 0.288).$$

Similarly, the direction of the segment c is at $-60°$ with respect to the x–axis, so its endpoint is calculated by adding one third of $\cos(-60^\circ)$ and $\sin(-60^\circ)$ to the previous coordinates to obtain

$$(0.5+\cos(-60^\circ), 0.288+\sin(-60^\circ)) \approx (0.67, 0).$$

Finally, the endpoint of the line segment d is obtained by adding $1/3$ to the first coordinate of this last point. In summary, we start from $(0, 0)$ and we draw the polygonal with segments of length 1/3 at angles 0°, 60°, −60° and 0° respectively with respect to the x–axis.

At the second iteration, the polygonal will consist of 4^2 line segments of length 1/9, and each of them will be of one of the four types a, b, c or d. The 16 segments are shown in Figure 2 numbered from 0 to 15. As before, we draw the polygonal from left to right. In order to determine how to draw each segment we have to see how it was generated. For example, segment 7 is a segment of type d constructed over a segment of type b from the first iteration. To find out the direction in which the segment is drawn we add the angles corresponding to each iteration: 60° in the first iteration, and 0° in the second one, giving and angle of 60°. The segment types a, b, c, and d are associated with the remainders—0, 1, 2, and 3 respectively— of the successive divisions of 7 by 4: the first division by 4 gives 1 with a remainder 3, and so the segment is of type d; next we divide 1 by 4 which gives 0 with a remainder 1, corresponding to type b in the first iteration.

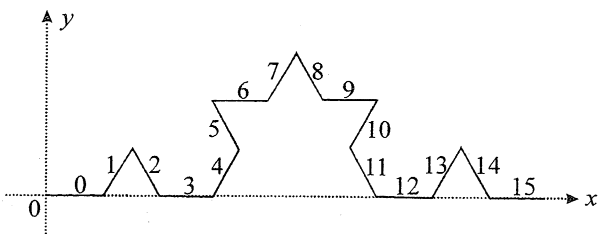


Figure 2

This process of finding the direction in which each segment is drawn is applied to iterations of any order. For example, suppose we have to draw segment 123 (of a total of 256) in the fourth iteration in the construction of *von Koch's* curve. We find the remainders of the successive divisions by 4:

123	remainders
30	3
7	2
1	3
0	1

The two remainders 3 correspond to no vertical displacement, and the remainders 1 and 2 correspond to angles of $60°$ and $-60°$ respectively. Since we have one remainder 1 and one 2, we simply add $60° + (-60°) = 0°$ and so the segment 123 must be drawn parallel to the x–axis, so we simply add $1/3^4$ to the x–coordinate of the endpoint of segment 122 and leave the y–coordinate unchanged.

This description of each line segment of an iteration of the *von Koch's* curve construction forms the basis of the program included below which was written in QuickBasic and you could easily adapt to any other programming language.

```
SCREEN 9: WINDOW (-.1, -.2)-(1.1, .7)
order = 5
DIM remainder(order): length = (1 / 3) ^ order

PSET (0, 0)

FOR n = 0 TO 4 ^ order - 1
  aux = n
  FOR L = 0 TO order - 1
    remainder(L) = aux MOD 4: aux = aux \ 4
  NEXT L
  angle = 0
  FOR k = 0 TO order - 1
   IF remainder(k) = 1 THEN angle = angle + 1
   IF remainder(k) = 2 THEN angle = angle - 1
  NEXT k

  x = x + COS(angle * 1.047198) * length
  y = y + SIN(angle * 1.047198) * length

  LINE -(x, y)
NEXT n
```

It should not be hard to understand the role of each variable. The order of iteration is set to 5, but you can change it to any number. The array `remainder` keeps the remainders of the division of the number of the segment to be drawn next by successive divisions by 4, and the variable `aux` stores the integer part of the division at each stage. The loop using the counter `k` determines the direction in which the segment should be drawn, by adding 1 for each remainder 1 and subtracting 1 for each remainder 2. Also, the number 1.047198 is the representation of $60°$ in radians. Figure 3 shows the output of this program.

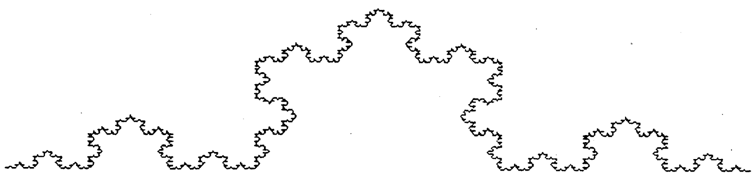


Figure 3

This gives only one third of the *von Koch's Snowflake* and I will leave it to you to complete the remaining two sides of the triangle. You may also try other variations to the construction presented here. For example, you could construct the *Anti-Snowflake Curve* shown on page 44 of *Function* 22, *Part* 2 or you could explore similar constructions with other shapes such as squares instead of triangles.

* * * * *

US President discuss Mathematics

"The investigation of mathematical truths accustoms the mind to method and correctness in reasoning, and is an employment peculiarly worthy of rational beings ... . From the high ground of mathematical and philosophical demonstration, we are insensibly led to far nobler and sublime meditations."

–George Washington

from William Dunham's *The Mathematical Universe*, New York: Wiley, 1994

* * * * *

LETTER TO THE EDITOR

Dear Editor,

I was delighted to read Dr Fwls' adventures in trigonometry [*Vol* 23, *Part* 2, page 53]. By golly, it almost made sense … . Goodness me, Pythagoras' theorem in tatters, together with geometry and algebra! Perhaps Dr Fwls slipped up somewhere?

Well, let's see.

I must agree with

$$b \cos C = a - c \cos B \tag{1}$$

and

$$a \cos C = b - c \cos A \tag{2}$$

But from here on we part company.

If now $a > b$ and $\cos C$ lies in (0, 1) then our doctor and me agree with each other. Under those conditions

$$a^2 + b^2 > c^2 \tag{3}$$

However, if $a > b$ and $\cos C$ lies in (–1, 0) then

$$a \cos C \leq b \cos C \tag{4}$$

So, using (1) and (2) gives

$$b - c \cos A \leq a - c \cos B \tag{5}$$

Under these circumstances

$$a^2 + b^2 \leq c^2 \tag{6}$$

Ah, thank goodness. Pythagoras' theorem still holds!

I can breathe again … .

Julius Guest, East Bentleigh

* * * * *

If one be bird-witted, that is easily distracted and unable to keep his attention as long as he should, mathematics provides a remedy; for in them if the mind be caught away but a moment, the demonstration has to be commenced anew.

PROBLEM CORNER

SOLUTIONS

Editors Note: The problems in edition 23 were incorrectly numbered 22.1.1 to 22.1.5 instead of 23.1.1 to 23.1.5.

PROBLEM 23.1.1 (Adapted from New Scientist # 1597, submitted by Greg Sheehan, Montrose, Vic)

A person has 2 pockets in their trousers and only carries amounts of money such that the sum of money in both pockets is to equal the product of the money in the right and left pockets (measured in dollars).

For example the person could carry a total of $6.25 since it is possible to put $5 in one pocket and $1.25 in the other as $5 + 1.25 = 5 \times 1.25$.

Given that the smallest unit of currency is one cent and that 100 cents equals $1, how many different sums can be carried?

SOLUTION (Carlos Victor, Rio de Janeiro, Brazil)

Let the amount in dollars in the pockets be x, y. Then $x + y = x.y$, and if let $x = 0.01X$, $y = 0.01Y$ we obtain

$$0.01X + 0.01Y = 0.0001XY$$

which can be written as

$$(X - 100)(Y - 100) = 10^4$$

This last equation is to have positive integer solutions. The number 10^4 has 25 divisors, so there are 25 pairs (X, Y). For instance the pair of divisors (2, 5000) corresponds to the solution $x = \$1.02$, $y = \$51.00$. The table below gives a list of all such solutions and the corresponding 13 sums.

X	y	**Sum**
1.01	101.00	102.01
1.02	51.00	52.02
1.04	26.00	27.04
1.05	21.00	22.05
1.08	13.50	14.58
1.10	11.00	12.10
1.16	7.25	8.41
1.20	6.00	7.20
1.25	5.00	6.25
1.40	3.50	4.90
1.50	3.00	4.50
1.80	2.25	4.05
2.00	2.00	4.00

A solution was also received from Julius Guest and a partial solution from Ian Preston.

PROBLEM 23.1.2 (from Parabola)

In a family of 6 children, the five eldest children are respectively 2, 6, 8, 12 and 14 years older than the youngest. The age of each child is a prime number.

How old are they? Show that their ages will never again be all prime numbers (even if they live indefinitely).

SOLUTION (D. Garson, Leichhardt, NSW)

Let the age of the youngest child at any time be y, so that the six ages are

$$y,\ y+2,\ y+6,\ y+8,\ y+12,\ y+14. \qquad \text{(i)}$$

Now, modulo 5 these ages are congruent to y, $y+2$, $y+1$, $y+3$, $y+2$, $y+4$. But the numbers y, $y+1$, $y+2$, $y+3$, $y+4$ are 5 successive integers, so exactly one of these numbers is divisible by 5. If $5|y+1$ then $5|y+6$ so that not all the ages in (i) will be prime. If $5|y+2$ then $5|y+12$. Similarly if $5|y+3$ then $5|y+8$ and if $5|y+4$ then $5|y+14$. The only possibility left for all ages in (i) to be prime is $5|y$, in which case the solution is

5, 17, 11, 13, 17, 19.

A solution was also received from Julius Guest and Carlos Victor.

PROBLEM 23.1.3 (Julius Guest, East Bentleigh, vic.)

Solve the system

$$\begin{aligned} x+y+z &= 9 \\ x^2+y^2+z^2 &= 29 \\ x^3+y^3+z^3 &= 99 \end{aligned}$$

SOLUTION (Colin Wilson, Rye, Vic.)

Let $x = X+3$, $y = Y+3$ and $z = Z+3$ so that the first equation gives

$$X+Y+Z = 0 \qquad \text{(i)}$$

The second equation together with (i) gives

$$X^2+Y^2+Z^2 = 2 \qquad \text{(ii)}$$

and then the third equation gives

$$X^3+Y^3+Z^3 = 0 \qquad \text{(iii)}$$

Substitution from (i) into (ii) and (iii) gives

$$X^2+XY+Y^2 = 1 \qquad \text{(iv)}$$

$$X^2Y+XY^2 = 0 \qquad \text{(v)}$$

and hence

$$X(XY+Y^2) = X(1-X^2) = 0$$

from the last equation we see that $X = -1, 0, 1$.

Then substitution into (iv) and (i) then gives the solutions for (X, Y, Z) as

$(-1, 0, 1)$, $(-1, 1, 0)$, $(0, 1, -1)$, $(0, -1, 1)$, $(1, 0, -1)$ and $(1, -1, 0)$.

Hence the solutions (x, y, z) are $(2, 3, 4)$, $(2, 4, 3)$, $(3, 4, 2)$, $(3, 2, 4)$, $(4, 3, 2)$ and $(4, 2, 3)$.

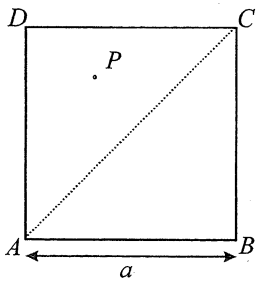
A solution was also received from the proposer, and a partial solution was received from Carlos Victor.

PROBLEM 23.1.4 (K.R.S. Sastry, Bangalore, India)

Let P be a point inside a square $ABCD$.

Prove that P lies on the diagonal AC if and only if PA^2, PB^2, PC^2 are in arithmetical progression in that order.

SOLUTION (Julius Guest, East Bentleigh)



Choose A as the origin of cartesian axes along the direction of AB and AD.

(i) Let $P(x, x)$ lie on AC. Then $PA^2 = 2x^2$, $PB^2 = x^2 + y(a-x)^2$, $PC^2 = 2(a-x)^2$ and $PA^2 + PC^2 = 2PB^2$ which establishes that PA^2, PB^2 and PC^2 are in arithmetic progression.

(ii) Let $P(x, y)$ be any point inside the square. Then $PA^2 = x^2 + y^2$, $PB^2 = y^2 + (a-x)^2$, $PC^2 = (a-x)^2 + (a-y)^2$. If $PA^2 + PC^2 = 2B^2$, then $x^2 + y^2 + (a-x)^2 + (a-y)^2 = 2(y^2 + (a-x)^2)$ from which it follows that $x = y$ and hence P is on the diagonal AC.

PROBLEM 23.1.5 (from Crux Mathematicorum with Math. Mayhem)

Suppose that a, b, c are positive real numbers such that

$$abc = (a+b-c)(b+c-a)(c+a-b)$$

Clearly $a = b = c$ is a solution. Determine all others.

SOLUTION (Colin Wilson, Rye, Vic.)

We show that for all positive a, b, c we have

$$abc \geq (a+b-c)(b+c-a)(c+a-b)$$

with equality holding, if and only if $a = b = c$. Without loss of generality we may assume $a \geq b \geq c > 0$ as the inequality is symmetric in a, b, c.

$$\begin{aligned}\text{Then } & abc-(a+b+c)(b+c-a)(c+a-b)\\ &= abc-(a+b-c)[c^2-(a-b)^2]\\ &= abc-(a+b-c)c^2+(a+b-c)(a-b)^2\\ &= (ab-ac-bc+c^2)c+(a+b-c)(a-b)^2\\ &= (a-c)(b-c)c+(a+b-c)(a-b)^2\\ &\geq 0\end{aligned}$$

with equality holding if and only if $a = b = c$. Thus the only solution to the equation is $a = b = c$.

A solution was also received from Carlos Victor, and an incomplete solution was received from Julius Guest.

Editors note on problem 22.5.6

The problem was to give a plausible explanation of how a student arrives at a 'solution' $x = 2.88539$ of the equation $\ln x = 2$. The published solution although plausible is not the most plausible explanation. The proposer of the problem gives the following explanation based on classroom experience. The student divides both sides of the equation $\ln x = 2$ by 'ln' to give $x = 2/\ln$. Then, by entering the key sequence "2/ln =", the value 2.88539 is obtained. In reality the calculator has evaluated 2/ln2, but the student is unaware of this fact in treating ln just as a number!

PROBLEMS

PROBLEM 23.3.1 (from Australian Mathematics Olympiad 1999)

Points P, Q, R and S lie, in that order, on a circle such that PQ is parallel to SR and $QR = SR$. Point T lies in the same plane as the circle such that QT is a tangent of the circle and the angle RQT is acute. Prove that

(a) $PS = QR$,

(b) angle PQT is trisected by QR and QS.

PROBLEM 23.3.2 (from Mathematical Spectrum)

Show that no prime number can be written as the sum of two squares in two different ways.

PROBLEM 23.3.3 (from Crux Mathematicorum with Math. Mayhem)

Find all real solutions of the equation

$$\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2} .$$

PROBLEM 23.3.4 (from Crux Mathematicorum with Math. Mayhem)

$ABCD$ is a square with incircle Γ. Let l be the tangent to Γ and let A', B', C', D' be points on l such that AA', BB', CC', DD' are all perpendicular to l. Prove that

$$AA' \cdot CC' = BB' \cdot DD' .$$

PROBLEM 23.3.5 (K.R.S. Sastry, Bangalore, India)

Determine conditions on the integers b and c so that the three quadratic polynomials $x^2 + bx + c$, $x^2 + bx + c + 1$, $x^2 + (b+1)x + c$ factor over the integers.

* * * * *

OLYMPIAD NEWS

The 1999 Australian Mathematical Olympiad

The contest was held in Australian schools on February 10 and 11. On either day 135 students in years 8 to 12 sat a paper consisting of four problems, for which they were given four hours. As a result, 28 students were invited to represent Australia at the Eleventh Asian Pacific Mathematics Olympiad, (APMO), a major international competition for students from about twenty countries on the Pacific Rim and Argentina, South Africa and Trinidad & Tobago. These are the two papers.

First Day

Wednesday, 10th February, 1999

Time allowed: 4 hours
NO calculators are to be used
Each question is worth seven points

1. Points P, Q, R and S lie, in that order, on a circle such that PQ is parallel to SR and $QR = SR$. Point T lies in the same plane as the circle such that QT is a tangent of the circle and the angle RQT is acute. Prove that

   (a) $PS = QR$;

   (b) angle PQT is trisected by QR and QS.

2. A town has 99 clubs $C_1, C_2, \ldots, C_{99}$, each of which has at least one member and no two of which have exactly the same members. Determine the smallest positive integer n such that one can be certain there is a set S of n people with the property; whenever C_i and C_j, $1 \leq i, j \leq 99$, are different clubs in the town, then there is either a person in S who belongs to C_i but not to C_j, or there is a person in S who belongs to C_j but not to C_i.

3. (a) Find positive integers a_1, a_2, a_3 and d_3 such that

   (i) $a_k - a_{k-1} = d_3$ *for* $k = 2, 3$, and

(ii) there are integers $m_i > 1$ and b_i such that $a_i = b_i^{m_i}$ for $i = 1, 2, 3$.

(b) Show that for each integer $n > 1$ there exist positive integers $a_1, a_2, \ldots, a_n$ and d_n such that

(i) $a_k - a_{k-1} = d_n$ for $k = 2, 3, \ldots, n$, and

(ii) there are integers $m_i > 1$ and b_i such that $a_i = b_i^{m_i}$ for $i = 1, 2, \ldots, n$.

4. In triangle Δ the radius of the incircle is r. Prove that the sum of the lengths of the altitudes of Δ is at least $9r$.

Second Day

Thursday, 11th February, 1999

Time Allowed: 4 hours
NO *calculators are to be used*
Each question is worth seven points

5. Let $x > 1$ be a real number and $n > 1$ an integer. Prove that

$$1 + \frac{x-1}{nx} < \sqrt[n]{x} < 1 + \frac{x-1}{n}$$

6. Let ABC be a triangle and D, E, F points in its exterior such that ΔABD, ΔBCE and ΔCAF are equilateral. The sides of these triangles are extended to produce the following intersections; BE and AF intersect in K, DB and FC intersect in L, and DA and EC intersect in M. Prove that DK, EL and FM are parallel.

7. Let n be an integral and p a prime number such that $1 + np$ is a perfect square. Prove that $n + 1$ is the sum of p perfect squares.

8. (a) Find one sequence $\{a_1, a_2, a_3, \ldots\}$ of integers with the following properties:

   (i) $a_n = 1$ or -1 for each n;
   (ii) $a_{mn} = a_m a_n$ for all m and all n;
   (iii) for no value of n does $a_n = a_{n+1} = a_{n+2}$ hold.

   (b) Determine all sequences $\{a_1, a_2, a_3, \ldots\}$ of integers with properties (i), (ii) and (iii).

The Eleventh Asian Pacific Mathematics Olympiad

On the basis of the AMO results, 28 students were invited to represent Australia at the Eleventh Asian Pacific Mathematics Olympiad (APMO). It was held on 9 March. The APMO, an annual competition, was started in 1989 by Australia, Canada, Hong Kong and Singapore. Since then the APMO has grown into a major international competition for students from about twenty countries on the Pacific Rim as well as from Argentina, South Africa and Trinidad and Tobago.

XI APMO

MARCH, 1999

Problem 1

Find the smallest positive integer n with the following property. There does not exist an arithmetic progression of 1999 terms of real numbers containing exactly n integers.

Problem 2

Let $a_1, a_2 \ldots$. be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all i, $j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \ldots. + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

Problem 3

Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent, closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Problem 4

Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Problem 5

Let S be a set of $2n + 1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n - 1$ points in its interior and $n - 1$ in its exterior. Prove that the number of good circles has the same parity as n.

Time: 4 hours
Each problem is worth 7 points

The performance of students at the APMO as well as at the AMO was used in selecting 25 students (12 senior and 13 junior students) for further training. These students participated in the ten-day Team Selection School of the Australian Mathematical Olympiad Committee. Following a tradition, the School was held in Sydney. Participants were offered a day-and-evening-filling program consisting of tests and examinations, problem sessions and lectures given by mathematicians. After two selection tests the team was selected that is to represent Australia at the fortieth International Mathematical Olympiad (IMO) in July. Its venue will be Bucharest (Romania). Romania is the country where in 1959 the First IMO was held.

The Australian IMO Team Members are:

Andrew Cheeseman (year 12) Mentone Grammar School, Vic
Geoffrey Chu (11), Scotch College, Vic
Peter McNamara (10), Hale School, WA

Allan Sly (11), Radford College, ACT
Kevin Sun (12), James Ruse Agricultural High School, NSW
Kester Tong (11), Lake Ginnindera High School, ACT
Team reserve: Thomas Sewell (11), James Ruse Agricultural High School, NSW.

Congratulations to all!

Taipei (Republic of China-Taiwan) is the venue of the XXXIX IMO scheduled for July. There the Australian team, consisting of six members, will have to contend with six problems during 9 hours spread equally over two days in succession. The following students have qualified for team membership and team candidature respectively:

* * * * *

Yet Another Proof

The History of Mathematics column for *Volume* 22, *Part* 2 considered a number of different proofs of the irrationality of $\sqrt{2}$. William Dunham's *The Mathematical Universe* (New York: Wiley, 1994) gives yet another. If $\sqrt{2}$ is rational, then

$$p^2 = 2q^2,$$

where p, q are positive integers. Decompose p, q into their prime factors. Then there will be an even number of such factors on the left (because each factor necessarily occurs twice in p^2) but an odd number on the left (because of the extra 2). Thus if $\sqrt{2}$ is to be rational, then an even number would have to equal an odd number (to paraphrase Aristotle).

Note that this proof, like the second one given in the column, depends on the unique factorisation theorem. But unlike that version, it does not require us to express the ratio p/q in its lowest terms. If, however, we do this, we must have a 2 among the prime factors of p (to agree with the 2 on the right). But then there are necessarily two of these on the left and so there must be at least one more on the right (among the factors of q). But because p/q is in its lowest terms, q is odd. We are back to the first of the proofs given before.

In Dunham's version, we have an odd number of 2's on the right and an even number on the left. This amounts to a slight extra piece of precision in the statement of Paragraph 1 above.

Published by Department of Mathematics & Statistics, Monash University