Function

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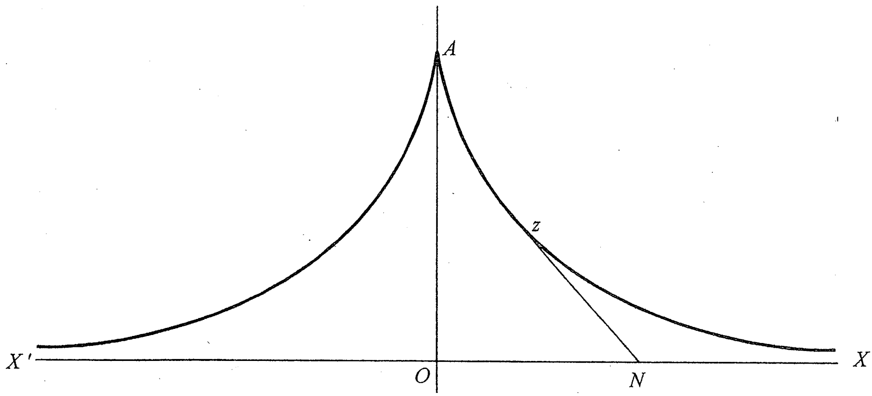

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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EDITORIAL

The curve on the front cover of this last 1998 issue of *Function* is known as the tractrix; over this page you will find a bit more about how this curve is constructed.

There are two feature articles in this issue. The first one is a thorough mathematical analysis of a very practical problem encountered by two beach goers who wanted to avoid a parking fine; I am sure you will find Lionel and LinMei's beach adventure very entertaining. The second article is an extension to an important result proved by the ancient Greek mathematician Hippocrates which involves curved areas formed with a triangle and semicircles; the article also discusses Hippocrates' advanced logical thinking for proving theorems.

The *History of Mathematics* column is about another great contributor to mathematics: the Dutch mathematician Simon Stevin. He is best remembered for his work in the area of mechanics; our editor gives a detailed account of his famous and ingenious proof of the equilibrium of forces when an object is moved vertically using a pulley and a weight.

The article in the *Computers and Computing* column gives a short introduction of the description of curves using polynomial functions; an approach which is only possible with a powerful computing tool at hand.

The *Problem Corner* includes, as usual, solutions to past problems and several new problems. The Australian team at this year's International Mathematical Olympiad has performed very well, receiving 2 silvers and positioning themselves within the 10 top countries in the world. The problems they had to solve also appear in this issue – a challenge for you for the long summer break ahead.

I take this opportunity to thank all who contributed to another successful year of *Function*; this was only possible thanks to the continuous support of editors, authors, and readers.

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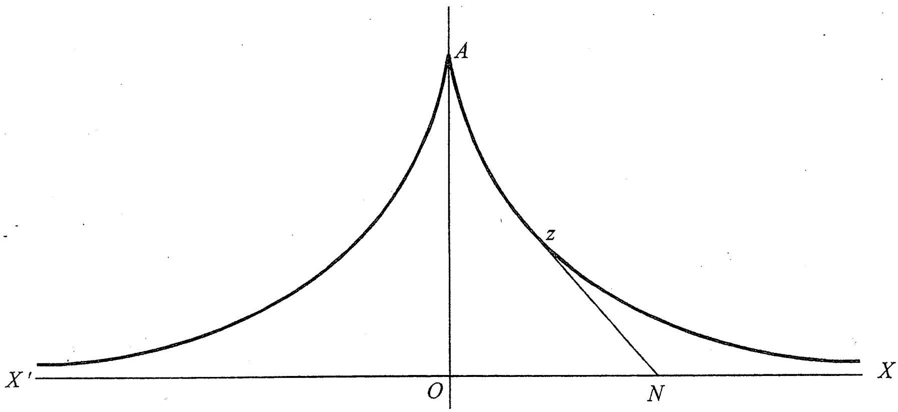
THE FRONT COVER

This issue's Front Cover (which for convenience is reproduced below) shows a curve known as the **tractrix**. The easiest way to think of the tractrix is to imagine someone dragging an object along the ground by means of a string. Suppose the person set out from the point O and walked towards X . Suppose the weight was initially at A. When the person reaches the point N, the object will have reached the point Z. The distance NZ remains constant and equal to OA, as it is assumed that the string does not stretch. The path traced out by the object (i.e. the point Z) is the tractrix. It is usual to complete it by also including the mirror image formed if the weight was dragged to the left (toward X') instead of to the right.



The curve was first studied by Leibniz (1646–1716), who with Newton founded the calculus. The problem of the path of a weight being dragged along was put to him by a French doctor. However, it was Leibniz's contemporary Huygens (1629–1695) who first accurately described the curve.

There is no simple equation for the tractrix. In terms of so-called "hyperbolic functions" we may write x as a (very complicated) function of y, but we will spare our readers the details. Furthermore, there is no straightforward way at all of giving y as a function of x. In giving the equations for the curve, other methods are used. Even these are complicated.

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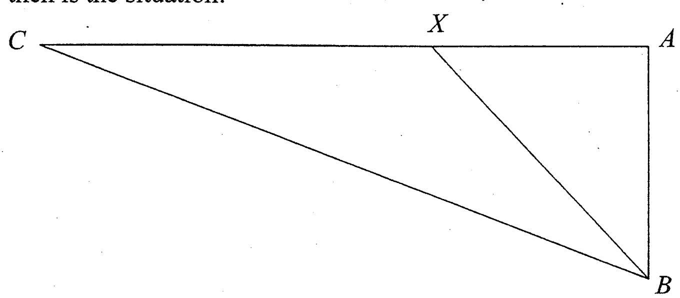
LIONEL AND LINMEI'S BEACH ADVENTURE

Michael A B Deakin, Monash University

1. How it all began

Lionel and LinMei decided that it was just the perfect day to go to the beach. When they got there, they parked the Car at C, walked along the concrete walkway between the carpark and the sand until they reached an Appropriate point A, then headed toward the water till they took up a place on the Beach at B. They were enjoying themselves so much that they rather lost track of the time and suddenly were aghast to see, very near their car and working his way toward it, a parking inspector. They realised that it was an absolute necessity to get back and move the car, and to do so as quickly as possible, hoping to reach it before the inspector got there.

Here then is the situation:



Lionel and LinMei must make it back from B to C in the shortest possible time. Now they can make good speed along the concrete path AC. Let's say their speed over the concrete is v. Over the sand, they can only achieve the (generally speaking) slower speed u. So we have $v \geq u$. Also let us write a for the distance BA, and b for the distance AC.

2. The Mathematics

Now our pair have various possible strategies to minimise the travel time from B to C.

Strategy 1 is to take the shortest route and head directly for C. This involves a distance of $\sqrt{a^2 + b^2}$ taken at speed u for a travel time of

$$T_1 = \frac{\sqrt{a^2 + b^2}}{u}$$

This however makes their entire journey take place over the sand where the going is slow. So let us look at other possibilities.

Strategy 2 is to get the sand behind them as quickly as possible by heading directly for A, and then making good time along the concrete path. This scenario takes them a time

$$T_2 = \frac{a}{u} + \frac{b}{v}$$

and it may well be that this is quicker. However, there is yet another possibility.

Strategy 3 is a compromise between the previous strategies. It picks an intermediate point X lying between A and C and such that the distance $AX = x$. Working out the time T for this case gives us

$$T = \frac{\sqrt{a^2 + x^2}}{u} + \frac{b - x}{v}$$

In fact this is a general formula.[1] If we put $x = b$, we find that $T = T_1$, and if on the other hand we put $x = 0$, we find that $T = T_2$. Thus we can formulate the problem facing Lionel and LinMei as choosing that value of x that minimises T.

The obvious thing to do is to differentiate T with respect to x. This is a little tricky, but it can be done, and the result is

$$\frac{dT}{dx} = \frac{x}{u\sqrt{a^2 + x^2}} - \frac{1}{v}$$

[1] Clearly the derivation suggests that x should lie in the range from 0 to b. However, this is not all that there is to be said on the matter. See Section 3 below.

This expression may be re-organised. We find

$$\frac{dT}{dx}=\frac{va-u\sqrt{a^2+x^2}}{u\sqrt{a^2+x^2}}\times\frac{vx+u\sqrt{a^2+x^2}}{vx+u\sqrt{a^2+x^2}}=\frac{\left(v^2-u^2\right)x^2-u^2a^2}{u\sqrt{a^2+x^2}\left(vx+u\sqrt{a^2+x^2}\right)}$$

We must now set this equal to zero and solve for x, and clearly the result is

$$x=\frac{au}{\sqrt{v^2-u^2}}$$

This gives us a positive value of x as we would expect – remember $v>u$. So now we can calculate the time T achieved by this value of x. We do this by substituting this value into the formula for T. This gives (calling the value found T_3):

$$\begin{aligned}
T_3&=\frac{1}{u}\sqrt{a^2+\frac{a^2u^2}{v^2-u^2}}+\frac{1}{v}\left(b-\frac{au}{\sqrt{v^2-u^2}}\right)\\
&=\frac{av}{u\sqrt{v^2-u^2}}+\frac{b}{v}-\frac{au}{v\sqrt{v^2-u^2}}\\
&=\frac{b}{v}+\frac{a}{\sqrt{v^2-u^2}}\left(\frac{v}{u}-\frac{u}{v}\right)\\
&=\frac{b}{v}+\frac{a\left(v^2-u^2\right)}{uv\sqrt{v^2-u^2}}\\
&=\frac{b}{v}+\frac{a}{uv}\sqrt{v^2-u^2}
\end{aligned}$$

After all that heavy algebra, we take stock. The differentiation suggests that the minimum time they can take is

$$T_3=\frac{b}{v}+\frac{a}{uv}\sqrt{v^2-u^2}\ .$$

Strategy 1 gives a time

$$T_1 = \frac{\sqrt{a^2 + b^2}}{u}$$

and **Strategy 2** gives a time

$$T_2 = \frac{a}{u} + \frac{b}{v}$$

Now we must work out which of these times is the least.

The first point to make is that in all circumstances $T_3 < T_2$. The proof of this fact I leave to the reader. However note that if the ratio $\frac{v}{u}$ is very large, then x is very small and T_3 approximates T_2, although it always remains slightly smaller than T_2 no matter how large $\frac{v}{u}$ may be. So we need only consider T_1 and T_3.

Let us work out the conditions under which **Strategy 1** might be the best. To do this, first look at the conditions under which $T_1 = T_3$. In this case, we have:

$$\frac{1}{u}\sqrt{a^2 + b^2} = \frac{b}{v} + \frac{a}{uv}\sqrt{v^2 - u^2}$$

$$v\sqrt{a^2 + b^2} = ub + a\sqrt{v^2 - u^2}$$

And continuing to solve for v we find

$$v^2\left(a^2 + b^2\right) = u^2b^2 + a^2\left(v^2 - u^2\right) + 2abu\sqrt{v^2 - u^2}$$

$$\left(v^2 - u^2\right)b^2 + a^2u^2 = 2abu\sqrt{v^2 - u^2}$$

$$\left(v^2-u^2\right)^2 b^4 + a^4u^4 + 2a^2b^2u^2\left(v^2-u^2\right) = 4a^2b^2u^2\left(v^2-u^2\right)$$

$$\left(v^2-u^2\right)^2 b^4 + a^4u^4 - 2a^2b^2u^2\left(v^2-u^2\right) = 0$$

$$\left[\left(v^2-u^2\right)b^2 - a^2u^2\right]^2 = 0$$

$$v^2-u^2 = \frac{a^2}{b^2}u^2$$

So

$$v = u\frac{\sqrt{a^2+b^2}}{b}$$

is the condition that the two times are equal.

We now see that there is a critical value of v given by the right-hand side of this equation. If we consider that in the special case $v = u$, then **Strategy 1** is clearly the best, then we see that if

$$v < u\frac{\sqrt{a^2+b^2}}{b}$$

then **Strategy 1** should be followed; otherwise follow **Strategy 3**.

This makes sense: if it's not very much slower over the sand, then it may still be worthwhile not to deviate from the straight line which gives the shortest route; only when the speed penalty becomes more severe is it worth going by the longer way.

3. More Mathematics and a Puzzle

It will be remembered that when we considered **Strategy 3** we deduced a value for x the distance AX. This turned out to be

$$x = \frac{au}{\sqrt{v^2 - u^2}}$$

If we set this expression equal to b, we find a condition that is exactly equivalent to the condition for $T_1 = T_3$ worked out above. This too makes sense. If $x > b$, it would surely be folly to follow **Strategy 3**. We would be running (over sand!) beyond the point we wanted to reach, and clearly this is inefficient.

But now think of the special case in which $a = b$ and $v = \frac{5u}{4}$. Our criterion tells us in this instance to follow **Strategy 1** if $v < u\sqrt{2}$ and clearly this is the case here, so **Strategy 1** is the way to go. In this case, we find from the formula just above that $x = \frac{4b}{3}$. In other words, **Strategy 3** has $x > b$, which clearly is impractical, and so we would obviously follow **Strategy 1**.

However, in this case

$$T_3 = \frac{b}{v} + \frac{b}{uv}\sqrt{v^2 - u^2} = \frac{4b}{5u} + \frac{4b}{5u^2}\sqrt{\left(\frac{5u}{4}\right)^2 - u^2} = \frac{b}{u}\left(\frac{4}{5} + \frac{3}{5}\right) = \frac{7b}{5u}$$

while on the other hand

$$T_1 = \frac{\sqrt{a^2 + b^2}}{u} = \sqrt{2}\,\frac{b}{u}$$

But $\sqrt{2} > \frac{7}{5}$ and so $T_1 > T_3$, when clearly it should be less!

Something has gone drastically wrong! Can't we trust mathematics?

I leave this for the reader to think about.

4. How it all worked out

But what about Lionel and LinMei? What happened to them?

Well, actually there wasn't time to do all this mathematics there and then. But guided by instinct they raced toward the car, finding a path which wasn't quite the mathematically ideal one, but actually was quite close to it. They reached the car at exactly the same time as the parking inspector.

Whether it was LinMei's delicate beauty or Lionel's muscular physique that influenced him has never been made clear; but he chose not to issue a ticket. Our pair moved the car. It was only then that they realised that in their haste they had left their Esky on the beach. There was another quick dash, and once again they were successful. The Esky was still there; their Coke was still inside. And it was still cold!

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Murphy's Law

Murphy's Law, also known as the Law of Universal Perversity, manifests itself in many forms. One form is frequently observed in supermarkets: The Other Queue Moves Faster.

There is a rational explanation for this observation. When you are in a long supermarket queue, with queues on your left and right, there is only a one in three chance that your queue will move faster than the other two. So two times out of three your queue will not be the fastest.

The explanation also shows how you can improve the odds in the unceasing battle against Murphy. When your supermarket trolley is full, push it to the queue at the end. Then there is only one neighbouring queue, and the chance that your queue moves faster is increased to fifty-fifty.

– From *Mathematical Digest*, No 111

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A "GREATNESS" OF A GREAT THEOREM

K R S Sastry, Bangalore, India

In that delightful book *Journey Through Genius* [1], the author William Dunham considers several "great" theorems of mathematics. The first great theorem he considers concerns a fifth century BC discovery by the ancient Greek mathematician Hippocrates of Chios. Hippocrates took a right angled isosceles triangle ABC, angle BAC being the right angle, and described semicircles on and above the sides $\overline{AB}$, $\overline{AC}$ and $\overline{BC}$ as shown in Figure 1.

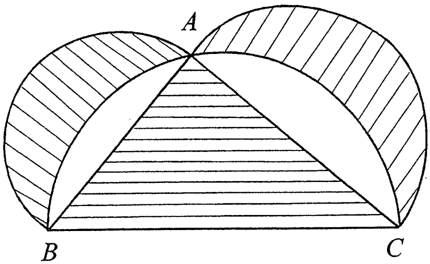

Figure 1

Hippocrates then proved that the sum of the areas of the two two-sided curved figures, called *lunes*, on $\overline{AB}$ and $\overline{AC}$, equals the area of the triangle ABC. Dunham advances the following justification to explain why he considers this to be the first great theorem of mathematics.

1. About a hundred and fifty years before Hippocrates, the Greek mathematician Thales was the first to insist that geometric propositions must be proved. This tradition was continued by Pythagoras and others, but their original proofs are not with us. The proof by Hippocrates of the theorem mentioned earlier has reached us, albeit indirectly.

2. A problem of great interest to the early Greek mathematicians was to construct a square equal in area to another given figure, using only the simple instruments, compasses and (unmarked) straightedge. Their success in this regard was confined mostly to "squaring" polygonal figures. They devoted much time and effort to trying to construct a square of the same area as a given circle, but did not succeed. Today we know this is impossible using just the instruments mentioned. Hippocrates' lune is the first figure with a curved boundary to be squared.

Let me take this opportunity to tell you that Hippocrates commands our admiration for two other important reasons. Firstly, he is the first known mathematician to compose geometry with a logical structure imposed on it. That is, he took postulates and axioms as given statements and from them developed theorems. He did this before Euclid. We know today that Euclid surpassed everyone before him in this regard. Secondly, and very importantly, Hippocrates is the first known mathematician to advocate the principle of "reduction" in solving mathematical problems; see reference [2]. This means that if the problem you are trying to solve turns out to be hard, then find similar but simpler problems and solve them. Take the insight thereby gained with you when you go back to solving the original problem.

The "greatness" of this great theorem

What "greatness" of this theorem of Hippocrates are we considering? You know that an angle inscribed in a semicircle is a right angle. The converse is: if angle BAC is a right angle then the semicircle on $\overline{BC}$ on the same side as A passes through A. You may wish to prove this as an exercise. If we analyse the construction of lunes by Hippocrates, it transpires that he simply described segments of circles in which right angles may be inscribed (i.e. semicircles) on and above the sides $\overline{AB}$, $\overline{AC}$, $\overline{BC}$ of a right angled isosceles triangle. *We* may regard what he did as his solution to the "reduced" instance of a more general problem. Take a triangle ABC. Construct arcs on and above its sides $\overline{AB}$, $\overline{AC}$, $\overline{BC}$ so that, in each of the resulting segments of circles, angles of the same magnitude as angle BAC may be inscribed. (How? This is possible, and will be described shortly.) This construction also produces lunes AB and AC. Now the problem is:

Determine the triangle(s) ABC in which the sum of the areas of the lunes AB and AC equals the area of triangle ABC.

We will provide a solution to this problem. As we shall see, it turns out that ABC must be a right angled triangle, but it need not be isosceles.

Background mathematics

Let us begin with the construction I mentioned in the previous section.

On a given line segment, describe an arc containing a given angle.

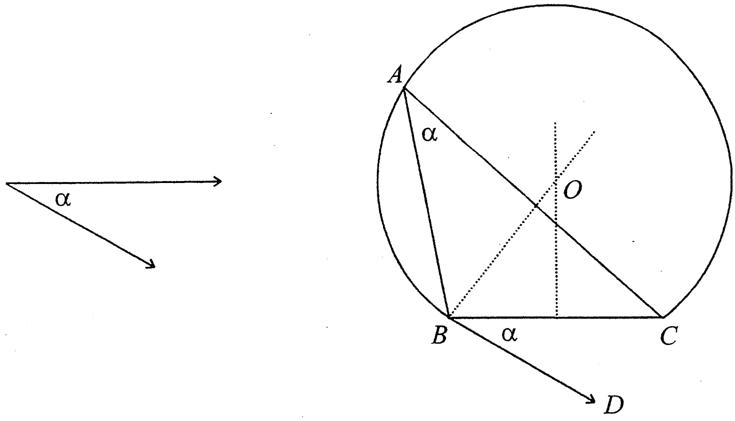

Figure 2

Here we assume that you know how to copy a given angle, construct the perpendicular bisector of a given line segment, and construct a line perpendicular to a given line segment at a specified point on it – all using the permitted simple instruments. Suppose α is the given angle and $\overline{BC}$ the given line segment. (See Figure 2.) Copy the angle α by drawing the ray $\overrightarrow{BD}$ below $\overline{BC}$ so that angle *CBD* equals α. Construct the perpendicular bisector of $\overline{BC}$. Construct the ray through the point *B* and perpendicular to $\overrightarrow{BD}$. Call *O* the intersection point of this ray and the perpendicular bisector of $\overline{BC}$. With centre *O* and radius *OB*, describe the arc on and above $\overline{BC}$. Notice that $\overrightarrow{BD}$ is a tangent and $\overline{BC}$ is a chord of this arc. A well-known theorem in circle geometry says: if *A* is any point on the arc then $\angle BAC = \angle CBD = \alpha$.

The main result

To establish our main result, we will also use ideas developed after Hippocrates.

Theorem. An arc, in which an angle of magnitude equal to angle *BAC* may be inscribed, is described on each side of and above a non-degenerate triangle *ABC*. Then the sum of the areas of the lunes *AB* and *AC* equals the area of the triangle if and only if angle *BAC* is a right angle.

Proof. The hypothesis of our theorem assures us that the circle segments formed by the arcs are similar to one another. This is because they contain equal inscribed angles. A well-known theorem on similar figures says: the areas of similar figures are proportional to the squares of the corresponding side lengths. Let us use the standard notation: $BC = a$, $AC = b$, $AB = c$, area of triangle $ABC = \Delta$. So, for some non-zero constant, λ, we have the areas of the circle segments on $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively equal to λa^2, λb^2, λc^2. See Figure 3.

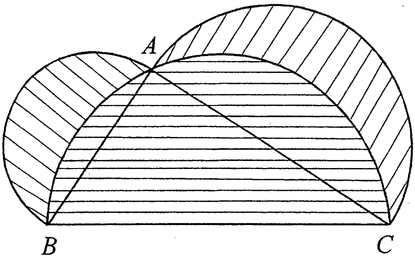

Figure 3a

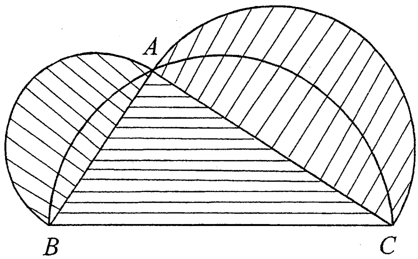

Figure 3b

The sum of the areas of the lunes *AB* and *AC*, plus the area of the circle segment on $\overline{BC}$, equals the area of the circle segment on $\overline{AB}$, plus the area of the circle segment on $\overline{AC}$, plus the triangle area, Δ. Hence the sum of the areas of the lunes *AB* and *AC* equals the triangle area, Δ, if and only if the area of the circle segment on $\overline{BC}$ equals the sum of the areas of the circle segments on $\overline{AB}$ and $\overline{AC}$. This is so, if and only if $\lambda a^2 = \lambda b^2 + \lambda c^2$, $\lambda \neq 0$, i.e. if and only if $a^2 = b^2 + c^2$. By the Pythagorean theorem coupled with its converse, the assertion of the theorem follows.

References

1. W Dunham, *Journey Through Genius*, Penguin (1991), pp 1–26.

2. D E Smith, *History of Mathematics*, Vol I, Dover (1958), pp 82–83.

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Ah! why, ye Gods, should two and two make four?

–Pope, Alexander in *The Dunciad*

LETTER TO THE EDITOR

Dear Editor

I always enjoy reading M Deakin's historical articles and his most recent one on "John Wallis and his Wonderful Product" proves no exception.

As a mathematical historian Deakin has certainly done his homework very well and I agree with all he said. But what he omitted to say also deserves some comment.

Wallis's product for π is very slowly convergent and turns out to be quite useless for computational purposes.

Having said all that, it may be of some interest to find the computing time of Wallis's product had he attempted to get an accuracy of, say, 6 significant figures (without using any modern technology of course).

I wrote a short little program for this task in Qbasic. My main loop consisted of 500000 cycles, and each cycle contained 2 multiplications and 2 divisions. Let us now assume that Wallis would take an average of 3 minutes for either multiplication or division. He would then spend about $12\times 500{,}000 = 6{,}000{,}000$ minutes i.e. 100,000 hours for this task. If now Wallis stuck to a regular 48 hour working week, he would spend 2496 hours on his self-imposed task in one year. Hence he would complete his little exercise in $100{,}000/2496 = 40.06$ years.

So much needless effort for so little return!

The moral of my little story is "Admire Wallis's achievement by all means, but please don't use it for computing π".

Sincerely yours

Julius Guest

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Rich in the past, dynamic in the present, prodigious for the future, replete with simple and yet profound ideas and methods, surely mathematics can give something to anyone's culture.

– Langer, R E in *The Things I Should Have Done, I Did Not Do*
Amer Math Monthly 59

HISTORY OF MATHEMATICS

Simon Stevin and his Non-Magical Magic

Michael A B Deakin

Simon Stevin (or Stevinus) lived from 1548–1620. He was born in Bruges, now in Belgium but then part of Holland. Following an early career in business, he wrote extensively on mathematical topics and involved himself with engineering. His first book sprang directly from his business studies, being on the topic of compound interest. It is his later works however that hold more "interest" for us! An incomplete edition of his writings shows nonetheless that he was extremely prolific: it runs to five large volumes.

Perhaps he is best remembered today for his work in mechanics. It was Stevin who made the first advances in areas of this discipline since Archimedes about a millennium and a half before.

In this article, I shall trace the best-known of his contributions, which is today seen as showing that force is a vector. This is not of course how Stevin would have put it. Vector algebra lay some 300 years in the future when he wrote. The work in which Stevin explored the laws of force was *De Beginselen der Weeghconst*, which is today translated (somewhat ponderously!) as *The Elements of the Art of Weighing*. It's actually rather hard to express the feel of the original title in good idiomatic English; *Fundamentals of Pondometry* is the best I have been able to find, and "Pondometry" is hardly an everyday English word!

The work is concerned with the science of statics, that is to say the study of the equilibrium of forces. It is a quite practical study, and related to his investigations into engineering. Figure 1 (taken from his work) shows the basis of the enterprise. The object is to use a smaller weight P, applied *vertically* via a pulley, to raise a heavier object D (a boat or a laden cart) up an incline AB.

We, today, would take the vertical component of the weight D and balance this against the full weight P to find the equilibrium between the two objects. So, if θ is taken to represent the angle between AB and the vertical (equal to the angle OCE in Stevin's diagram), then we would write

$$P = D\cos\theta$$

as the condition that the two forces balance, which is to say that equilibrium is achieved. (And so a slightly larger value of P would suffice to raise the boat or the cart.)

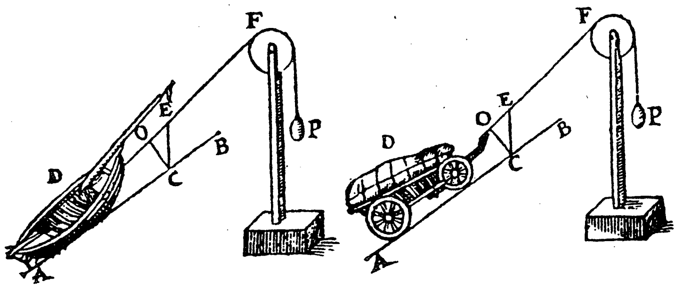

Figure 1

It is this condition that Stevin discovered. His proof is most ingenious and very convincing. It is widely known, but can withstand another retelling.

Look at Figure 2. It depicts a weighted chain suspended on a triangular form.

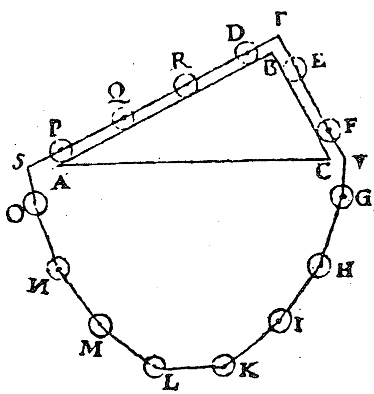

Figure 2

Now let Stevin himself take up the story.[1]

SUPPOSITION. Let *ABC* be a triangle, whose plane shall be at right angles to the horizon, and base *AC* parallel to the horizon, and on the side *AB*, which shall be double of the side *BC*, let there lie a sphere *D*, and on the side *BC* a sphere *E*, of equal weight and equal size to the sphere *D*. WHAT IS REQUIRED TO PROVE. We have to prove that as the side *AB* (2) is to *BC* (1), so is the apparent weight of the sphere *E* to the apparent weight of the sphere *D*. PRELIMINARY. Let us make about the triangle *ABC* a wreath of fourteen spheres, of equal size and equal weight, and equidistant from one another, as *E, F, G, H, I, K, L, M, N, O, P, Q, R, D*, all of them strung on a line passing through their centres, in such a way that they can revolve about those centres; let there also fit two spheres on the side *BC* and four on *BA*, i.e. as line to line, so spheres to spheres. Let there also be fixed points at *S, T, V*, over which the line or the string of the spheres can slide, in such a way that the two parts of the string above the triangle shall be parallel to the sides *AB, BC*, so that if the wreath is pulled down on one side or the other, the spheres roll on the lines *AB, BC*. PROOF. If the apparent weight of the four spheres *D, R, Q, P* were not equal to the apparent weight of the two spheres *E, F*, either one or the other will be the heavier. Let us suppose (if this were possible) this to be the one of the four spheres *D, R, Q, P*, But the four spheres *O, N, M, L* are of equal weight to the four spheres *G, H, I, K*. The side therefore of the eight spheres *D, R, Q, P, O, N, M, L* is heavier in appearance than the side of the six spheres *E, F, G, H, I, K*. But because that which is heavier always predominates over that which is lighter, the eight spheres will roll downwards and the other six will rise. Let this be so, and let *D* have fallen where *O* is now, then *E, F, G, H* will be where *P, Q, R, D* are now, and *I, K* where *E, F* are now. But this being so the wreath of spheres will have the same appearance as before, and on this account the eight spheres on the left side will again roll down and the other six will rise. This descent on the one and ascent on the other side will continue for ever, because the cause is always the same, and the spheres will automatically perform a perpetual motion, which is absurd. The part of the wreath *D, R, Q, P, O, N, M, L* therefore is of equal apparent weight to the part *E, F, G, H, I, K*. But if from such equal weights there are subtracted equal weights, the remainders will have equal weight. Let us therefore subtract from the former part the four spheres *O, N, M, L*, and from

[1] As translated by E J Dijksterhuis. Stevin wrote in Dutch and made a point of doing so. At the time, it was expected that works of scholarship would be written in Latin. *De Beginselen der Weeghconst* begins with a dedication to Emperor Rudolph II to whom the work is addressed and then proceeds to a lengthy *Discourse on the Worth of the Dutch Language*. There is another point also: the work was intended to be of practical use and to be accessible to practical engineers who would not necessarily know Latin.

the latter part the four spheres G, H, I, K (which are equal to the aforesaid O, N, M, L); then the remainders D, R, Q, P and E, F will be of equal apparent weight. But the two latter being of equal apparent weight to the four former, E will have twice the apparent weight of D. As therefore the line BA (2) is to the line BC (1), so is the apparent weight of the sphere E to the apparent weight of the sphere D. CONCLUSION. Given a triangle, whose plane, etc.

The proof has become famous as the "ring (or wreath) of spheres proof". It is one of several physically-based arguments that rely for their thrust upon the impossibility of perpetual motion. It should also be noted that the method of argument can obviously be extended to cases in which the ratio is not necessarily 2:1 but is more general. Indeed Stevin goes on to deduce five corollaries. The third is shown in Figure 3.

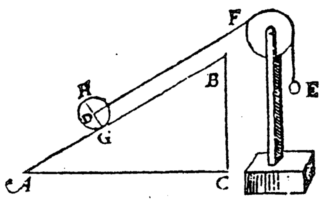

Figure 3

This is exactly the result needed to solve the practical problem of the boat or the laden cart (Figure 1).

More recently, it has been pointed out that the corollary can be deduced directly. Figure 4 is redrawn from a more recent discussion by the Nobel Prize winning physicist Richard Feynman. Here the three spheres lying on the vertical side AB of the triangle ABC balance the four on the oblique side BC. Because we have here a (3, 4, 5)-triangle, the angle A is a right angle, and so side AC is horizontal, as the proof requires. This argument too can be generalised, because the cosine of any angle can be approximated to arbitrary accuracy by the ratio of a side to the hypotenuse of a pythagorean triangle with integral sides.

The ring of spheres proof provides a diagram that is now widely seen. In particular, the reference work *The Dictionary of Scientific Biography* adopts it as a logo. So too does the 5-volume set of Stevin's works. Stevin himself used it as a frontispiece to a later (1608) work, *Hypomnemata Mathematica* (Mathematical Writings), a Latin translation of a collection of his Mathematical papers. There he supplied the accompanying motto "*Wonder en is gheen wonder*" – usually translated as "The magic is not magical". In other words, it is perfectly natural that four spheres can balance two if the slopes on which they rest are suitably adjusted. Both the diagram (Figure 2) and the motto are sometimes referred to as *Stevinus' Epitaph.*

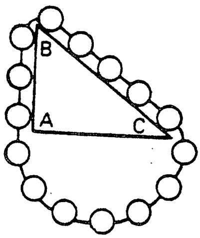

Figure 4

The Belgian Mathematical Society's Bulletin is titled "Simon Stevin".

Further Reading

There is a good article on Stevin in Volume 13 of *The Dictionary of Scientific Biography* and there is a lot more in *The Principal Works of Simon Stevin.* E J Dijksterhuis provides a useful account of Stevin's life in Volume 1 of this work, and it is also in this volume that *De Beginselen der Weeghconst* is reprinted and translated. Feynman's account is to be found in Volume 1 of *The Feynman Lectures on Physics* by R Feynman, R Leighton and M Sands. I have also relied on a pair of articles I co-authored over 20 years ago (M A B Deakin and G J Troup in the *International Journal of Mathematical Education in Science and Technology*, Vol 7 (1976), pp 271-276 and 493-494.)

* * * * *

No other field can offer, to such an extent as mathematics, the joy of discovery, which is perhaps the greatest human joy.

– Rósza Péter in *Mathematics is Beautiful*
Math Intell 12, p 62

COMPUTERS AND COMPUTING

About Cubics and Kangaroos

Cristina Varsavsky

You are already familiar with a wide range of functions. Linear, quadratic and cubic functions are probably the ones you dealt with more often; these are part of a larger family of functions called *polynomials*, that is, functions of the form

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots + a_n x^n$$

where $a_1, a_2, a_3, \ldots, a_n$ are real numbers known as *coefficients* of the polynomial, and n is the *degree* of the polynomial function p. So a linear function is a polynomial of degree 1, a quadratic function is a polynomial of degree 2, and a cubic function is a polynomial of degree 3.

Polynomial functions are often used to interpolate points. Given two points you can uniquely determine a line through them: you need two points to determine the two coefficients – the slope and the y–intercept. Similarly, given three points, there is only one parabola that contains them. In general, given a set of n points in the plane, there will be a unique polynomial of degree $n-1$ whose graph contains the n points; in other words we need n points to determine the n coefficients of a polynomial of degree $n-1$.

Let us now look at a different, but related problem: suppose we are to describe any given curve as a graph of a function. Take for example, the curves involved in the sketch of the kangaroo which appears in the following page, and focus our attention on the curve corresponding to the back of the kangaroo (we can treat all the others in a similar way).

A simple approach is to first pick several points on the curve an then find the polynomial through them. We put the curve on a grid as shown in Figure 2, and carefully select a few points on the curve, taking points closer together where there are more changes on the curve. We choose eight points:

(0, 1)	(2.2, 0)	(4.6, 0.7)	(6.2, 1.6)
(7.5, 1.82)	(9.5, 1.1)	(10,1)	(10.5, 1.4)

Figure 1

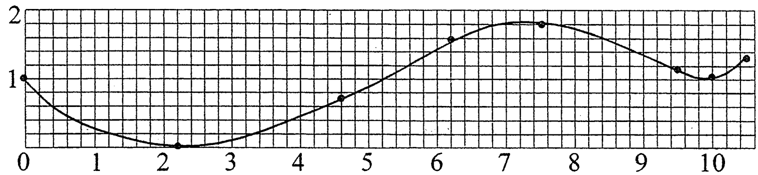

Figure 2

Let $p(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7$ be the polynomial of degree 7 through these 8 points. We find the coefficients by solving the system of 8 equations with 8 unknowns:

$$p(0) = a = 1$$

$$p(2.2) = a + b\,2.2 + c\,2.2^2 + d\,2.2^3 + e\,2.2^4 + f\,2.2^5 + g\,2.2^6 + h\,2.2^7 = 0$$

$$p(4.6) = a + b\,4.6 + c\,4.6^2 + d\,4.6^3 + e\,4.6^4 + f\,4.6^5 + g\,4.6^6 + h\,4.6^7 = 0.7$$

$$p(6.2) = a + b\,6.2 + c\,6.2^2 + d\,6.2^3 + e\,6.2^4 + f\,6.2^5 + g\,6.2^6 + h\,6.2^7 = 1.6$$

$$p(7.5) = a + b\,7.5 + c\,7.5^2 + d\,7.5^3 + e\,7.5^4 + f\,7.5^5 + g\,7.5^6 + h\,7.5^7 = 1.82$$

$$p(9.5) = a + b\,9.5 + c\,9.5^2 + d\,9.5^3 + e\,9.5^4 + f\,9.5^5 + g\,9.5^6 + h\,9.5^7 = 1.1$$

$$p(10) = a + b\,10 + c\,10^2 + d\,10^3 + e\,10^4 + f\,10^5 + g\,10^6 + h\,10^7 = 1$$

$$p(10.5) = a + b\,10.5 + c\,10.5^2 + d\,10.5^3 + e\,10.5^4 + f\,10.5^5 + g\,10.5^6 + h\,10.5^7 = 1.4$$

Fortunately we do not have to solve this system by hand, computer packages such as *MAPLE* can assist us in this task. This is the solution:

$a = 1$	$b = 3.09$	$c = -4.35$	$d = 1.99$
$e = -4.38$	$f = 0.52$	$g = -0.031$	$h = 0.00008$

The graph of the resulting polynomial is shown in Figure 3. The polynomial we found goes through the 8 points, but its graph does not follow the kangaroo back curve (dotted line); the major difference appears in the tail area. This is because the only information we used is the 8 points that belong to the curve. If we want to "improve" the curve, then we have to include more information such as the slope of the curve and how the curve is bent at each of those points.

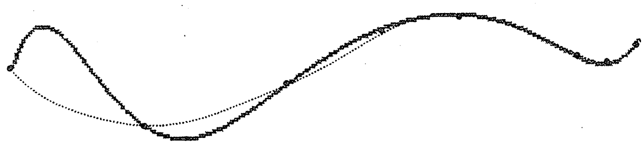

Figure 3

So instead of trying to find *the* polynomial through the points we rather find a polynomial for each portion of the curve between two consecutive points. Since we have 4 pieces of information for each such portion (two end-points, and slope and curvature at one end) the polynomial should be of degree 3. Therefore we will have 7 cubics which we write as

$$p_i(x) = a_i + b_i x + c_i x^2 + d_i x^3 \quad 1 \le i \le 7$$

We have to construct them so that at each point their graphs join smoothly and they remain bent in the same way. For each cubic we have 4 equations; for example, for the cubic p_2 joining the points $(2.2, 0)$ and $(4.6, 0.7)$ we have:

1. The point $(2.2, 0)$ belongs to the cubic, then

$$a_2 + b_2\, 2.2 + c_2\, 2.2^2 + d_2\, 2.2^3 = 0$$

2. The point $(4.6, 0.7)$ belongs to the cubic, then

$$a_2 + b_2\,4.6 + c_2\,4.6^2 + d_2\,4.6^3 = 0.7$$

3. The slope at the point $(4.6, 0.7)$ is the same as the slope of p_3; this will ensure that the two consecutive cubics join smoothly. The slope is given by the first derivative[1], therefore

$$b_2 + 2\,c_2\,4.6 + 3\,d_2\,4.6^2 = b_3 + 2\,c_3\,4.6 + 3\,d_3\,4.6^2$$

4. p_2 and p_3 are bent the same way at $(4.6, 0.7)$, that is, the second derivatives[2] must coincide at 4.6:

$$2\,c_2 + 6\,d_2\,4.6 = 2\,c_3 + 6\,d_3\,4.6\ .$$

We will have similar equations for all other curve segments, but the first and last cubics will differ in one of the equations to state the slope at the first and last points on the kangaroo curve, which according to the diagram can be set to -1 and 1 respectively. So we have

$$a_1 = -1 \text{ and } b_7 + 2c_7\,10.5 + 3\,d_7\,10.5^2 = 1$$

Therefore we have to find the solution to a system of 28 linear equations to determine the 28 coefficients. Computers are good at that; I used *MAPLE* to find the 28 coefficients and to plot the piece-wise defined function which merges the 7 cubics found. We should expect to obtain a better result than before, that is, a curve that matches more closely the kangaroo back line. How much better is it? The graph of the function appears in Figure 4; you can judge for yourself!

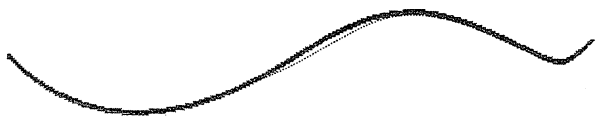

Figure 4

* * * * *

[1] The first derivative of $a + b\,x + c\,x^2 + d\,x^3$ is $b + 2\,c\,x + 3\,d\,x^2$.

[2] The second derivative of $a + b\,x + c\,x^2 + d\,x^3$ is $2\,c + 6\,d\,x$.

PROBLEM CORNER

SOLUTIONS

PROBLEM 22.3.1 (Julius Guest, East Bentleigh, Vic)

Let $ABCD$ be a square, and let P be a point inside it such that $AP = 40$, $BP = 30$, and $CP = 50$. Find the side length of the square.

SOLUTION by Julius Guest

Let the side length of the square be l, and let the vertices be $A(0, 0)$, $B(l, 0)$, $C(l, l)$, and $D(0, l)$. Let $P(p, q)$ be the interior point. See Figure 1.

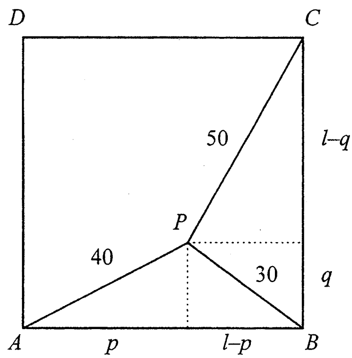

Figure 1

As $AP = 40$:

$$p^2 + q^2 = 1600 \tag{1}$$

Next, $BP = 30$, so:

$$(l - p)^2 + q^2 = 900 \tag{2}$$

Lastly, $CP = 50$, thus:

$$(l-p)^2+(l-q)^2=2500 \tag{3}$$

Eliminating q from (1) and (2) gives $l^2-2pl+700=0$. Hence:

$$p=(l^2+700)/2l \tag{4}$$

From (1), (3) and (4) we find that:

$$q=(l^2-1600)/2l \tag{5}$$

We now use (4) and (5) in (1) which leads after a little algebra to:

$$l^4-4100l^2+1525000=0$$

This is a quadratic equation in l^2. Its solutions are:

$$l^2=2050\pm150\sqrt{119}$$

Hence either $l=60.71$ or $l=20.34$. But from our data we know that $\sqrt{2}l$ must exceed 50, so there is only one acceptable solution here: $l=60.71$.

Also solved by Carlos Victor (Nilópolis, Brazil), Keith Anker (Glen Waverley, Vic), and J A Deakin (Shepparton, Vic).

PROBLEM 22.3.2 (K R S Sastry, Bangalore, India)

Prove that, in any triangle, the inradius equals one third of an altitude if and only if the side lengths of the triangle are in arithmetic progression.

SOLUTION by Keith Anker

Denote the side lengths of the triangle ABC by a, b, and c, where $a\le b\le c$. Let I be the incentre. Subdivide ABC into three smaller triangles, IAB, IBC, and ICA, with bases c, a and b respectively, and common altitude r, the inradius. See Figure 2.

Now express the area of the triangle in two different forms:

$$Area=\frac{1}{2}ar+\frac{1}{2}br+\frac{1}{2}cr=\frac{1}{2}bh$$

where h is an altitude of the triangle ABC. Now, if a, b, c is an arithmetic progression, then $a+c=2b$, so $r=\dfrac{bh}{a+b+c}=\dfrac{h}{3}$. Conversely, if $r=\dfrac{h}{3}$, then $(a+b+c)\dfrac{h}{3}=bh$, so $a+c=2b$.

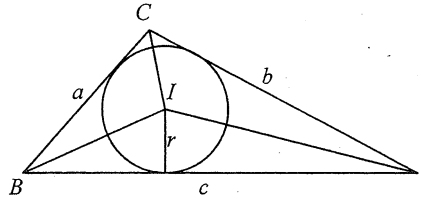

Figure 2

Also solved by Julius Guest, Carlos Victor, J A Deakin, and the proposer.

PROBLEM 22.3.3 (from *Mathematical Spectrum*)

Solve the simultaneous equations

$$ax+by=2 \tag{1}$$

$$ax^2+by^2=20 \tag{2}$$

$$ax^3+by^3=56 \tag{3}$$

$$ax^4+by^4=272 \tag{4}$$

SOLUTION by Bill Tetley (Cowes, Vic)

Multiply each of Equations (1), (2) and (3) in turn by x and subtract from the equation below it:

$$by(y-x)=20-2x \tag{5}$$

$$by^2(y-x)=56-20x \tag{6}$$

$$by^3(y-x) = 272 - 56x \tag{7}$$

Then divide each pair in turn (i.e., (6) by (5) and (7) by (6)) to obtain:

$$y = \frac{56-20x}{20-2x} = \frac{272-56x}{56-20x}$$

Now solve for x: $(56-20x)^2 = (20-2x)(272-56x)$, which, after expanding and simplifying, gives $x^2 - 2x - 8 = 0$, so $x = -2$ or $x = 4$. If $x = -2$ then $y = 4$, while if $x = 4$ then $y = -2$, because of the symmetry between x and y in the equations. Substituting these values of x and y into (1) and (2) gives $a = b = 1$.

This problem turned out to be very popular. It was also solved by John Barton (Carlton North, Vic), Carlos Victor, Etiene Silva Aguera Ramos (São Paulo, Brazil), Ron Adlem (Buxton, Vic), Keith Anker, and J A Deakin.

In connection with this problem, John Barton noted that a set of m polynomial equations of degrees p, q, r, …, in m unknowns, has $pqr\ldots$ solutions in general. Some of these solutions can involve complex numbers, and there can be special cases involving coincident solutions or solutions at infinity. In the present problem, there are four polynomial equations in four unknowns (a,b, x and y), with degrees 2, 3, 4 and 5, so there should be 120 solutions in this general sense. John Barton writes: "For the equations of 22.2.3 there are two (only) finite solutions, yet it seems to me less than full value for the money to leave the answer thus, no matter after how many manipulations, without saying something meaningful about the other (what I assume to be) 118 solutions." Perhaps some of our readers will explore the problem further.

PROBLEM 22.3.4 (from *Alpha*)

Let A, B, C, D be four points in three-dimensional space that do not all lie in the same plane. Let the midpoint of $\overline{AB}$ be M, and let the midpoint of $\overline{CD}$ be N. Prove that $\frac{1}{2}(AD + BC) > MN$.

We received several different solutions to this problem. The following solution, by Carlos Victor, is especially neat.

SOLUTION by Carlos Victor

Let the midpoint of $\overline{AC}$ be P. Then $\overline{PN} || \overline{AD}$ with $PN = \dfrac{AD}{2}$, and $\overline{PM} || \overline{BC}$ with $PM = \dfrac{BC}{2}$. See Figure 3.

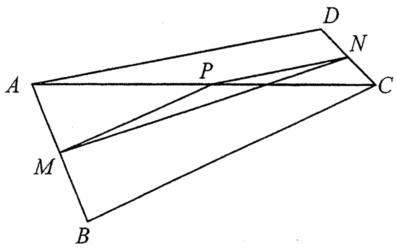

Figure 3

In the triangle PMN, we have $MN < PM + PN$, so $MN < \dfrac{BC}{2} + \dfrac{AD}{2}$. Therefore $\frac{1}{2}(AD + BC) > MN$.

Also solved by John Barton, Keith Anker, and J A Deakin.

PROBLEM 22.3.5 (from *Alpha*)

Prove, for any natural number n, that the inequalities

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}$$

are satisfied.

SOLUTION by J A Deakin

$$\begin{aligned} 2\sqrt{n+1} - 2\sqrt{n} &= 2(\sqrt{n+1} - \sqrt{n}) \\ &= \frac{2(n+1-n)}{\sqrt{n+1} + \sqrt{n}} \quad \text{(rationalising the numerator)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{n+1}+\sqrt{n}} \\
&< \frac{2}{2\sqrt{n}} \\
&= \frac{1}{\sqrt{n}}
\end{aligned}
$$

Also:

$$
\begin{aligned}
2\sqrt{n}-2\sqrt{n-1} &= 2(\sqrt{n}-\sqrt{n-1}) \\
&= \frac{2[n-(n-1)]}{\sqrt{n}+\sqrt{n-1}} \\
&= \frac{2}{\sqrt{n}+\sqrt{n-1}} \\
&> \frac{2}{2\sqrt{n}} \\
&= \frac{1}{\sqrt{n}}
\end{aligned}
$$

Hence, $2\sqrt{n+1}-2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n}-2\sqrt{n-1}$.

Also solved by Julius Guest, John Barton, Carlos Victor, Ron Adlem, and Keith Anker.

Correction

In the previous (August) issue, several words were inadvertently left out just prior to Problem 22.4.1. In case readers are still confused, it was supposed to say that you are invited in Problem 22.4.1 to extend the balls-in-cells idea presented in Malcolm Clark's article "Patterns in Tattslotto Numbers" in that issue. We apologise to our readers for the error.

A web page for problems

Mark Bowron of MathPro Press writes:

"Readers of *Function* will be pleased to know that many of its problems can now be searched electronically (at no charge) on the World Wide Web at

http://problems.math.umr.edu

Over 22000 problems from 42 journals and 22 contests are referenced by the site, which was developed by Stanley Rabinowitz's MathPro Press. Ample hosting space for the site was generously provided by the Department of Mathematics and Statistics at the University of Missouri – Rolla, through Leon M Hall, Chair.

Problem statements are included in most cases, along with proposers, solvers (whose solutions were published) and other relevant bibliographic information. Difficulty and subject matter vary widely; almost any mathematical topic can be found."

Check it out – it's well worth a look!

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 1 March 1999 will be acknowledged in the April 1999 issue, and the best solutions will be published.

PROBLEM 22.5.1 (K R S Sastry, Bangalore, India)

Let I be the incentre of triangle ABC. Let E and F be points on $\overline{AC}$ such that $\overline{BE}$ bisects $\angle ABC$ and $\overline{BF}$ bisects $\angle EBC$.

(a) Prove that $\overline{FI}$ is parallel to $\overline{BC}$ if and only if $\angle ABC = 2\angle ACB$.

(b) Let G be the point where $\overline{FI}$ extended meets $\overline{AB}$. Prove that $\overline{GE}$ is parallel to $\overline{BF}$ if and only if $\angle ABC = 2\angle ACB$.

PROBLEM 22.5.2 (K R S Sastry, Bangalore, India)

Let P be an interior point of the square $ABCD$. Prove that P lies on the diagonal $\overline{AC}$ if and only if, in that order, PA^2, PB^2, PC^2 are in arithmetic progression.

PROBLEM 22.5.3 (Juan-Bosco Romero Márquez, Valladolid, Spain)

Find all integer solutions of the equation $(x^2+y^2)^2=(x+y)^3$ with $x\geq 0$ and $y\geq 0$.

PROBLEM 22.5.4 (Republic of Slovenia 38th Mathematics Competition for Secondary School Students, April 1994)

Prove that every number of the sequence 49, 4489, 444889, 44448889, ... is a perfect square (in every number there are n fours, $n-1$ eights and a nine).

PROBLEM 22.5.5

Find a function $f:\mathbf{R}\to\mathbf{R}$ such that:

(i) $f'(x)$ exists, and $[f(x)]^2+[f'(x)]^2=1$, for all $x\in\mathbf{R}$;

(ii) $f(x)>0$ for all $x>0$, and $f(x)<0$ for all $x<0$.

(There is exactly one such function.)

Our final problem for 1998 is just as much a question of psychology as it is of mathematics!

PROBLEM 22.5.6

A not very bright student, using a calculator, obtained the following incorrect result in an examination:

"$\ln x=2$, so $x=2.88539$ (to 5 d.p.)"

What erroneous but plausible reasoning led the student to obtain this answer?

*****

OLYMPIAD NEWS

The XXXIX International Mathematical Olympiad.

Taipei (Taiwan) was this year's venue for the IMO. Teams, usually having six members, from 76 countries had to contend with six problems during nine hours spread equally over two days in succession.

Here are the two papers:

39th International Mathematical Olympiad
First Day – Taipei – July 15, 1998

Problem 1

In the convex quadrilateral *ABDC,* the diagonals, *AC* and *BD* are perpendicular and the opposite sides *AB* and *DC* are not parallel. Suppose that the point *P*, where the perpendicular bisectors of *AB* and *DC* meet, is inside *ABCD.* Prove that *ABCD* is a cyclic quadrilateral if and only if the triangles *ABP* and *CDP* have equal areas.

Problem 2

In a competition, there are a contestants and b judges, where $b \geq 3$; 3 is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}$$

Problem 3

For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

Time allowed: 4[1/2] hours.
Each problem is worth 7 points.

Second Day – Taipei – July 16, 1998

Problem 4

Determine all pairs (a,b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Problem 5

Let l be the incentre of triangle ABC. Let the incircle of ABC touch the sides BD, CA and AB at K, L and M respectively. The line through B parallel to MK meets the lines LM and LK at R and S respectively. Prove that RIS is acute.

Problem 6

Consider all functions f from the set N of all positive integers into itself satisfying

$$f(t^2 f(s)) = s(f(t))^2 ,$$

for all s and t in N. Determine the least possible value of $f(1998)$.

Time Allowed: 4[1/2] hours.
Each problem is worth 7 points.

Australia came thirteenth with 146 points out of possible 252 as its score and won two silver and four bronze medals. Ahead of Australia were the teams from Iran, (211), Bulgaria (195) Hungary and the United States of America (186 each), Taiwan (184), Russia (175), India (174), Ukraine (166), Vietnam (158), Yugoslavia (156), Romania (155) and Korea (154).

The other countries of the Asian Pacific Mathematics Olympiad scored as follows: Canada (113), Singapore (110), Hong Kong (102), South Africa (98), Argentina (97), Colombia (66), Thailand (65), Mexico (62), Peru (60), New Zealand (50), Trinidad & Tobago (36), Malaysia (32), Indonesia (16), the Philippines (11), Sri Lanka (5).

The medal distribution for the Australian team was

Hiroshi Miyazaki, Victoria, Silver
Justin Ghan, South Australia, Silver
Geoffrey Chu, Victoria, Bronze
Justin Koonin, New South Wales, Bronze
Stephen Farrar, New South Wales, Bronze
Andrew Cheeseman, Victoria, Bronze.

Congratulations to our excellent team!

The 1988 Senior Contest of the Australian Mathematical Olympiad Committee (AMOC)

The AMOC Senior Contest is the first hurdle for mathematically talented Australian students who wish to qualify themselves for membership of the team that represents Australia in the following year's International Mathematical Olympiad. This year that four-hour competition took place on 12 August.

These are the questions:

The 1998 AMOC Contest

Senior Paper

Wednesday, 12th August, 1998

Time allowed: 4 hours
NO calculators are to be used
Each question is worth seven points

1. Triangle Δ has side lengths a, b, c, triangle Δ_a has side lengths a', b, c, triangle Δ_b has side lengths a, b', c, triangle Δ_c has side lengths a, b, c', and each of the four triangles has area 1. Furthermore $a \neq a'$, $b \neq b'$, $c \neq c'$.

   (a) Prove that there exists a triangle, Δ' say, with side lengths a', b', c'.
   (b) Determine the area of triangle Δ'.

2. Determine all positive integers n that satisfy

$$\sqrt{\frac{1+\frac{1}{2^{n-1}}}{2}} < 1 - \frac{2}{n}.$$

3. Let f be a function defined for all real values such that the following conditions are satisfied for every real number x:

   (i) $f(999 + x) = f(999 - x)$;
   (ii) $f(1998 + x) = -f(1998 - x)$.

   Prove that f has the following two properties:

   (a) $f(-x) = -f(x)$ for all real numbers x.
   (b) there exists a real number T such that $f(x + T) = f(x)$ for all real numbers x.

4. Let $ABCD$ be a cyclic quadrilateral with the property that its diagonals AC and BD intersect, at M say, in a right angle. Let N be the midpoint of AV and P be the point on CD such that NP and CD are perpendicular. Prove that the points M, N and P are collinear.

5. Let n be a positive integer. Prove:

   (a) If there is a positive integer a, $a < n$, such that the line defined by the equation $\frac{x}{a} + \frac{y}{n-a} = 1$ contains a point both of whose coordinates are positive integers, then n is not a prime number.

   (b) If n is not a prime number, then there is a positive integer a, $a < n$, such that the line defined by the equation $\frac{x}{a} + \frac{y}{n-a} = 1$ contains a point both of whose coordinates are positive integers.

* * * * *

Archimedes ... had stated that given the force, any given weight might be moved, and even boasted, we are told, relying on the strength of demonstration, that if there were another earth, by going into it he could remove this. Hiero being struck with amazement at this, and entreating him to make good this problem by actual experiment, and show some great weight moved by a small engine, he fixed accordingly upon a ship of burden out of the king's arsenal, which could not be drawn out of the dock without great labor and many men; and, loading her with many passengers and a full freight, sitting himself the while far off with no great endeavour, but only holding the head of the pulley in his hands and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea. The king, astonished at this, and convinced of the power of the art, prevailed upon Archimedes to make him engines accommodated to all the purposes, offensive and defensive, of a siege. ... the apparatus was, in most opportune time, ready at hand for the Syracusans, and with it, also the engineer himself.

– Plutarch in *Life of Marcellus [Dryden]*

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