Function

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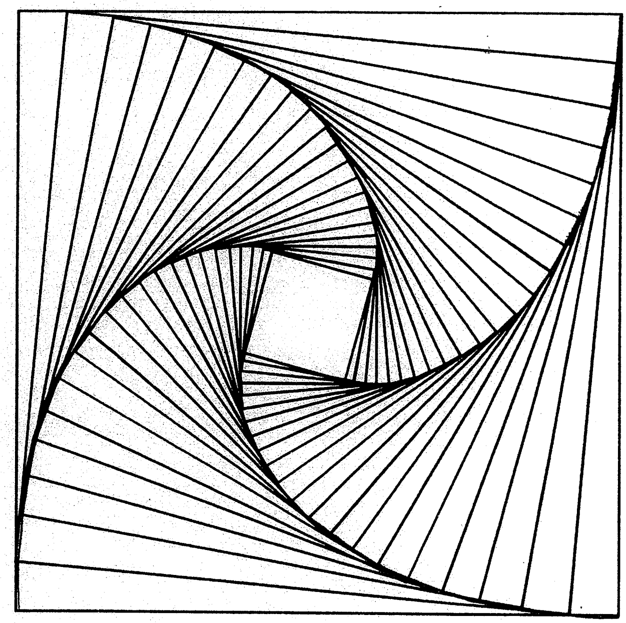

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* \$10 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to our readers to this issue of *Function*!

The diagram on the front cover represents the classic mathematical problem of describing the paths of four turtles initially positioned at the vertices of a square, each moving towards the turtle in front of it. A feature article by Michael Deakin looks at two different ways of solving the problem, using mathematical induction and symmetry. In the *Problem Corner* you will find another argument to determine where the turtles meet; also, the *Computers and Computing* column includes a computer program to draw the paths of the turtles.

Although there is much about turtles in this issue of *Function*, there are also other interesting articles. Bruce Henry presents a geometric problem and shows how this can be solved in a variety of creative ways: using trigonometry, various geometric constructions and proof by contradiction.

If you are a Tattslotto addict, then you should read Malcolm Clark's article which analyses the probability of a run of consecutive numbers when randomly selecting 8 numbers. His conclusion seems to contradict the general belief that randomly selected numbers should not contain clumps or runs of consecutive numbers. Perhaps this article will make you change your strategies.

The *History of Mathematics* column is about the achievements of John Wallis, a contemporary of Isaac Newton. The "Wallis Product" is a formula which expresses the number π as a quotient of two infinite products of positive integers. The article presents the simple methods he used to find what we now know as integrals of quite sophisticated functions which led to the formula.

You will find in the *Problem Corner* solutions and more new problems. If you send your solutions promptly we will publish them in the first issue in 1999.

We hope you enjoy this issue of *Function*.

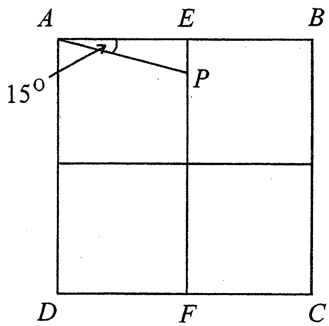
* * * * *

AN INTERESTING PROBLEM AND SOME CREATIVE SOLUTIONS

Bruce Henry

This article is all about the following problem and a variety of solutions to it. Readers may wish to try it themselves before reading on.

ABCD is a square, *EF* bisects both *AB* and *CD*. *P* is a point on *EF* such that $\angle PAE = 15^\circ$. Prove that $DP = DC$.



The problem can be solved by a large variety of methods which use a lot of different ideas in mathematics. Some of these may not be known to the reader, but others certainly are. The "nice" solutions are those which employ an insightful construction that enables elementary Euclidean geometry to solve the problem.

Trigonometry

The first proof is trigonometric.

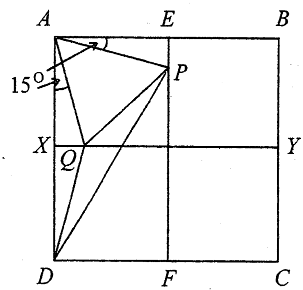
Let $AB = 2$, so that $EF = 2$ and $DF = AE = 1$. (No generality is lost by this; it merely amounts to a convenient choice of units.) Now set $EP = x$. Then $x = \tan 15^\circ$.

But $\tan 30^\circ = \dfrac{1}{\sqrt{3}} = \dfrac{2\tan 15^\circ}{1 - \tan^2 15^\circ} = \dfrac{2x}{1 - x^2}$.

So it follows that $x^2 + 2\sqrt{3}x - 1 = 0$ and $x = \dfrac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = -\sqrt{3} \pm 2$.
But $x > 0$, and so $EP = x = 2 - \sqrt{3}$. But since we had $EF = 2$, then $FP = \sqrt{3}$. But now, since $DF = 1$, we easily find $DP = 2$, by Pythagoras's Theorem.

Creative Construction 1

Let X and Y be the midpoints of AD and BC respectively. Let Q be the point on XY such that $\angle QAD = 15°$. Draw QD, QP and DP.



Triangles AXQ, XQD and AEP are congruent, (assuming symmetry properties of right triangles) and so $AQ = DQ = AP$. Also

$$\angle XAQ = \angle EAP = 15°.$$

So $\angle QAP = (90 - 15 - 15)° = 60°$, and APQ is thus an equilateral triangle and in particular

$$AQ = QP.$$

Now $\angle AQX = 75°$, so that

$$\angle PQY = (180 - 60 - 75)° = 45°.$$

Next $\angle DQY = (180 - 75)° = 105°$, so that $\angle DQP = (45 + 105)° = 150°$. Then the triangles AQD and DQP are congruent (SAS) and $AD = DP$. Then $DP = DC$ as required.

Creative Construction 2

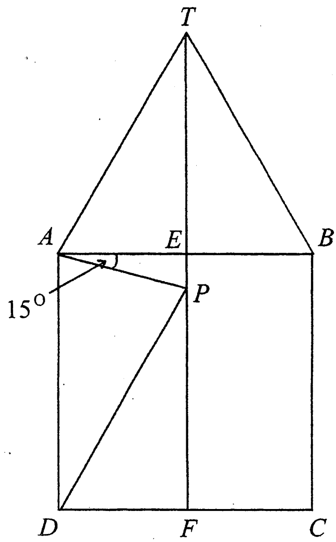
Draw equilateral triangles ATB on AB external to the square as shown. Draw DP.

In triangles TAP and PAD, $\angle TAP = (60 + 15)° = 75° = \angle PAD$.
Also $AD = AT$ (by construction), and AP is common.

So triangles TAP and PAD are congruent (SAS) and

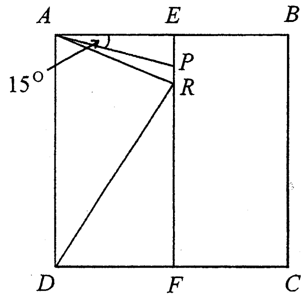
$$\angle ADP = \angle ATP = 30°.$$

Then $\angle PDF = 60°$, triangle PDC is equilateral and so $PD = DC$, as required.



The Logician's Solution

Let us instead of the original problem prove its converse:



If P is such that $PD = DC$ $(= AD)$, then

$$\angle PAE = 15°.$$

Since in this case, the triangle PDC is equilateral, $\angle PDC = 60°$, so $\angle PDA = 30°$. Then the triangle PDA is isosceles ($PD = AD$) and the base angles are $75°$. Then $\angle PAE = 15°$ as required.

This in itself is not enough to prove the original proposition, but maybe we can use it to construct a proof by contradiction.

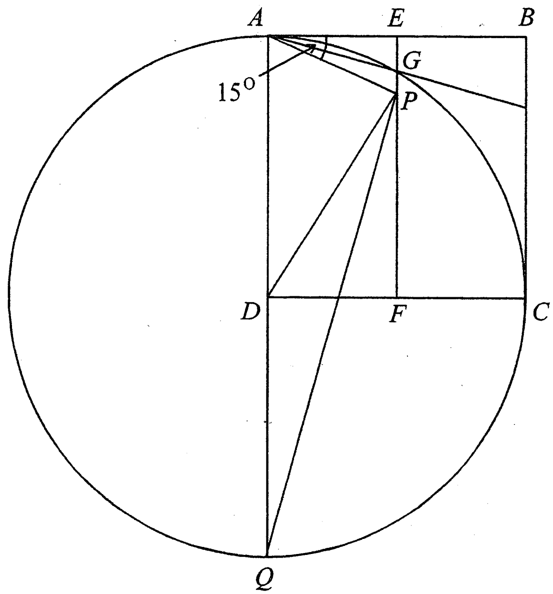
Suppose P is *not* such as to make PD equal to DC. Then there is another point on EF, R say, distinct from P, which is such that $RD = DC$. Then the converse already proved shows that $\angle RAE = 15°$. But we are told that $\angle PAE = 15°$, and now it is not possible for R to be distinct from P. Therefore R coincides with P and $PD = DC$, as required.

Creative Constraints

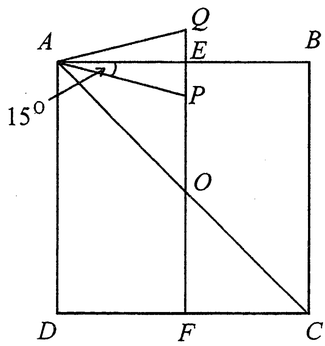
Draw the circle with centre D and radius AD. Let this circle cut EF at G. Extend AD to cut the circle at Q and then join AG, GD and GQ.

Since they are all radii of the circle, $DG = AD = DC$. Also $\angle ADG = 30^\circ$. It follows that $\angle AQG = 15^\circ$. (Because of the theorem that states that the angle formed on the circle itself is half that formed at the centre, when both angles stand on the same arc – in this case, the arc AG. As a result of another standard theorem (angle in the "alternate segment") we deduce that $\angle EAG = 15^\circ$.

But $\angle EAP = 15^\circ$ also, so P and G must coincide. This means that P lies on the circle and so $PD = AD$ as required.



The Sledgehammer



Extend PE to Q so that $PE = EQ$. Join AC. Let AC intersect EF at O. Then the triangles AQE and APE are congruent (SAS). Once again, let $AB = 2$ and $PE = x$. Then $EO = 1$.

In the triangle AOQ, $AQ = \sqrt{1+x^2}$ and $AO = \sqrt{2}, \angle OAP = \angle QAP = 30^\circ$, so PA is the bisector of $\angle OAQ$. Therefore PA divides OQ in the ratio of AO to AQ.

Thus $\frac{OP}{PQ} = \frac{AO}{AQ}$ or in other words $\frac{1-x}{2x} = \frac{\sqrt{2}}{\sqrt{1+x^2}}$. It follows that

$$(1+x^2)(1-x)^2 = 8x^2$$

which expands to

$$x^4 - 2x^3 - 6x^2 - 2x + 1 = 0.$$

The quartic expression factorises (readers may check!) to give

$$(x+1)^2(x-2+\sqrt{3})(x-2-\sqrt{3}) = 0,$$

so that

$$x = -1 \text{ or } x = 2+\sqrt{3} \text{ or } x = 2-\sqrt{3}.$$

However we also have $0 \le x \le 1$ and only one of these three possible solutions satisfies this inequality. So $x = 2-\sqrt{3}$.

Then $PF = \sqrt{3}$ and since $DF = 1$, we have $DP = 2$ by Pythagoras's Theorem and $DP = DC$ as required.

* * * * *

THE FOUR-TURTLES PROBLEM

Michael A B Deakin

Problem 22.2.2 is a hardy perennial that has many interesting features, going beyond the problem itself. Figure 1 shows the problem as posed in *Function* (April 1998) and taken from the collection *The Chicken from Minsk.*

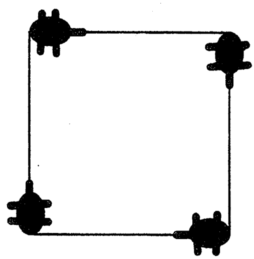


Figure 1

The turtles are trained to move each towards the turtle in front of it and to do so with constant speed. They are initially placed at the vertices of a square. If we place the origin at the centre of this square with axes toward the right and the top, then the top right turtle will have initial coordinates (a,a), where a is half the length of the side of the square. I leave it to readers to supply the initial coordinates for each of the three others. Each turtle is supposed to move with speed V.

We may now ask several questions. What happens to the square? What paths do the turtles follow? When will they meet?

The first result to establish is: *At all subsequent times, the turtles are situated at the vertices of a square.* I would like to give two proofs of this statement.

First, suppose that, instead of moving in a continuous fashion, each turtle moves in a series of finite steps which I will call "jerks". At the outset, the top right turtle moves down (as drawn on the page) for a time τ, in which time it covers a distance $V\tau$. A similar outcome is found for each of the other three turtles. It is then very clear from the diagram that after the first "jerk" the new positions for the turtles also form the vertices of a square. If we now apply a second "jerk", the result will be a smaller square, but quite obviously a square. See Figure 2.

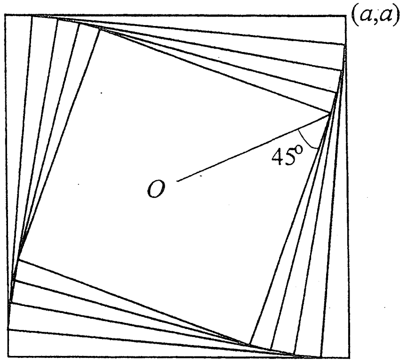


Figure 2

Now because the result of each "jerk" is to produce a square from a previous square, and as the initial positions form a square, then the subsequent positions always form a square. This is as a result of the Principle of Mathematical Induction, described in my History of Mathematics Column in *Function*, *Vol* 22, *Part* 3.

Now we have not specified the value of τ, the time-interval occupied by each "jerk"; we haven't needed to. It follows that the maintenance of the square pattern is independent of τ, which can be made arbitrarily small. So, as the "jerky" motion approximates the smooth motion better and better as τ gets smaller and smaller, the statement also applies to the smooth motion.

This is interesting as an example of a case of an unusual version of Mathematical Induction. Usually it applies in cases where we are making statements about things we can *count*. (Thus, in the argument above, if the turtles form a square pattern after the nth "jerk", then they will also do so after the $(n+1)$th "jerk".) But our further deduction allowed an extension to a situation where counting is not applicable: the smooth motion. This is the simplest example I know of an inductive argument going beyond the use of the natural numbers.

A second argument uses symmetry. Each turtle is the same as each of the others; they each follow the same programme. Suppose that at some time after the start that the turtles occupied positions that were *not* the vertices of a square. Then they would occupy the vertices of a quadrilateral some of whose sides or some of whose angles (or both) were not equal to other sides or angles (as the case might be). But this would imply that one or two of the turtles had behaved differently from other turtles, and this is not allowed. Thus the initial square pattern must persist.

[We may in fact extend this argument to cover the case in which n turtles set out from the vertices of a regular n-gon. This too will remain a regular n-gon as the motion progresses. In the extreme case as $n \to \infty$, the turtles follow each other round the circumference of a circle for ever. Certain insects can be induced to behave in exactly this fashion! This is possible because of the action of chemical secretions called "trail pheromones".]

Let T be the position of the turtle that sets out from the top right corner and join T to the centre of the pattern by a line OT. This line must now always make an angle of $45°$ with the direction of that turtle's motion, because it is the diagonal of the square. Thus, initially, T is (a,a) and is moving in the "down the page" direction, and the line OT slopes up the page at $45°$, and makes an angle of $45°$ with the direction of T's motion. This pattern persists, as a glance at Figure 2 will show. The initial position is perfectly typical.

This constancy of angle is the defining property of a curve known as the "equiangular spiral", so it follows that the path travelled by T is an equiangular spiral, and similarly for the other turtles.

But now we may answer other questions also. Think back to the "jerky motion". In each time-interval τ, the turtle at T advances a distance $V\tau$, and of this a component $V\tau\cos 45^{\circ} = V\tau / \sqrt{2}$ is directed toward O. Thus the component of the turtle's velocity toward the centre is $V / \sqrt{2}$, a constant. (Again note that this is a property independent of the value of τ and so it applies also to the smooth motion.)

Now, initially the distance OT is $a\sqrt{2}$, and at the end of the motion, when the side of the square has shrunk to zero, it is 0. So, a distance $a\sqrt{2}$ has been covered at an effective speed of $V / \sqrt{2}$. It follows that the time taken is $2a / V$.

Finally we may deduce the total distance each turtle has actually walked. The time is $2a / V$, and the actual speed over the ground is V. Thus the total distance is $2a$.

Further Reading

There are many good accounts of this problem. One (with dogs instead of turtles) is to be found in Chapter 11 of *A Book of Curves*, by E H Lockwood (Cambridge University Press, 1971). This also gives a lot of detail on the equiangular spiral.

PATTERNS IN TATTSLOTTO NUMBERS

Malcolm Clark

The popular Australian lottery Tattslotto is one of the many lottery games around the world in which the winners are determined by the random selection of k balls from n numbered balls. Each player specifies a set of j numbers from 1 to n. Prizes are awarded to players whose choice of j numbers is a subset of the k numbers actually drawn. Depending on the rules of the lottery, selection of the k numbers may be with or without replacement, and the actual order of the winning numbers may or may not be important. In Tattslotto, $j = 6$, $k = 8$, and $n = 45$.

In each Tattslotto draw, the winning numbers are selected by a mechanical randomisation device in which 45 numbered balls are mixed in a spherical container. Eight of these balls are selected without replacement, the first six designated as winning numbers, the last two as supplementary numbers. Those players who happen to select all six winning numbers share the First Division prize pool, winning around $200,000 to $300,000 on average. Lesser prizes are available under less stringent conditions; for example, a player with any 3 of the 6 winning numbers plus either supplementary number wins a 5th Division prize, typically around $20.

The results of a recent Tattslotto draw (No. 1727) were somewhat surprising: the six winning numbers were 2, 3, 7, 20, 21, 22, and the two supplementary numbers were 11 and 23. So amongst the six winning numbers, there was a run of three consecutive numbers, and amongst all eight numbers, there was a run of four consecutive numbers: 20, 21, 22, and 23. This pattern caught my attention, and I asked myself, how unusual is such an outcome? In other words, what is the probability of obtaining *one* run of four consecutive numbers when randomly selecting 8 numbers from 1 to 45?

But this is not the correct probability to calculate if we are concerned about the randomisation or otherwise of the Tattslotto machine. For example, a Tattslotto draw with two separate runs of four consecutive numbers, or a single run of five consecutive numbers, would be even more worrying. What we need to compute is the probability of getting a result equal to or *more extreme* than what actually happened in Draw 1727. In other words, what is the probability of there being *at least one run* of *at least four* consecutive numbers (when selecting at random 8 numbers from 45)?

In principle, the solution is straight-forward: all we need do is count all of the possible selections of 8 numbers from 45 for which the condition is true. In practice, a direct calculation is not so simple: we must ensure that each possible selection for which the condition holds is counted once and only once. This is quite difficult.

Instead, let us look at a simpler problem where it is possible to obtain a direct algebraic solution. Let's look at the occurrence of runs of *any length,* i.e., pairs, triples, and so on. So we ask the simpler question: what is the probability of *at least one run* of *at least two* consecutive numbers when k numbers are selected from n?

We can answer this by considering the complementary event: that there are *no runs* of any kind in the k numbers selected from 1 to n. To derive the solution, we represent each possible draw of k numbers in a particular format. Unlike theTattslotto machine which has n numbered balls, we imagine that we have n cells numbered from 1 to n, and k identical but un-numbered balls. Each time the Tattslotto machine draws a numbered ball, we put one of our k balls in the corresponding cell. When the draw is completed, k cells will be occupied and $n-k$ cells will be empty.

We can represent each selection of k numbers from n by a line of k stars and $n-k$ circles, the stars indicating occupied cells (or equivalently, drawn numbers) and the circles indicating empty cells. To be definite, suppose $k = 3$ and $n = 7$. Then the selection {2,3,5} would be represented schematically by

$$\circ \ * \ * \ \circ \ * \ \circ \ \circ$$

Clearly every possible arrangement of k stars $n-k$ circles corresponds to a possible selection of k numbers out of n, and hence the total number of such selections is $\binom{n}{k}$.

We now need to count how many of those arrangements contain no runs, ie, no pairs (such as {3,4}), no triples (such as {3,4,5}) and so on. If there are no runs, then in terms of our "balls-in-cells" equivalence, each occupied cell must have at least one empty cell on either side of it. What will the corresponding line of stars and circles look like?

Suppose we are about to watch a draw of k numbers from 1 to n in which there are no runs. Initially we don't know which numbers will be drawn, or which of our cells will be occupied or empty. We can indicate this unknown configuration by a line of n dots, e.g.

$$\bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet \ \bullet$$

Once the draw (with *no runs*) is completed, we may move along this array of dots from left to right, changing them to circles or stars indicating empty or occupied cells as before. But this time, for the first $k-1$ stars, the next symbol *must* be a circle. For example, the partially completed array might look like:

$$\circ \quad \circ \quad * \quad \circ \quad \bullet \quad \bullet \quad \bullet$$

When we get to the last star, the next symbol on the right (if there is one) will automatically be a circle. This is because when the last ball is allocated to a cell, all the remaining cells (if any) must be empty.

The important point is that for the first $k-1$ stars, we have no choice regarding the next symbol: it *must* be a circle. So this "star-followed-by-circle" should be regarded as a single entity, which we may denote by a different symbol, say a plus sign. In this process, $k-1$ of the original $n-k$ circles disappear. The last star may also be replaced by a plus sign, provided we interpret it as a single occupied cell.

With this interpretation of plus signs, each selection of k numbers out of n with no runs corresponds to an arrangement of k plus signs and $n-2k+1$ circles (a total of $n-k+1$ symbols), and conversely. Hence the corresponding number of arrangements is $\binom{n-k+1}{k}$.

So the probability of no runs in the selection of k numbers out of n is

$$Q(k,n) = \frac{\binom{n-k+1}{k}}{\binom{n}{k}} = \frac{(n-k+1)!(n-k)!}{(n-2k+1)!n!}$$

Finally, the probability of at least one run of at least two consecutive numbers is

$$P(k,n) = 1 - Q(k,n)$$

The numerical consequences of this formula are surprising and counter-intuitive. For Tattslotto ($n = 45$), the probability of at least one run in the first six winning numbers is 0.5287, while the corresponding probability for all eight numbers is 0.7731.

Most people believe that randomly-selected numbers should not contain clumps or runs of consecutive numbers. The above calculation shows that such clumping is not at all unusual in Tattslotto. Therefore a good strategy in choosing your Tattslotto numbers is to select numbers which are clumped together to some extent. This is because most other players will *not* make such a choice. Whatever choice of numbers you make, you still have the same small probability of winning. But, in the unlikely event that your selected numbers *are* the winning numbers, you will share the First Division prize with fewer other people.

Essentially the same procedure can be used to find the probability of at least one run of *at least three* consecutive numbers. Once again, we consider the complementary event: no runs longer than two consecutive numbers. In this case, it's more complicated because of the various ways this complementary event could occur. For example, if $k = 6$, there could be six "single" numbers, or four singles and one pair, or two singles and two pairs, or three pairs. We need to consider these four possibilities separately. The balls-in-cells idea represented by a line of symbols still works, but this time we need different symbols to distinguish between "singles" and "pairs". We leave this calculation as a challenge to our more adventurous readers.

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When we learn to drive a car
we are able to "go places"
easily and pleasantly
instead of walking to them
with a great deal of effort.
And so you will see that
the more Mathematics we know
the EASIER life becomes
for it is a TOOL with which
we can accomplish things
that we could not do at all
with our bare hands.
Thus Mathematics helps
our brains and hands and feet,
and can make
a race of supermen out of us.

Lillian Lieber in *The Education of T C Mits*

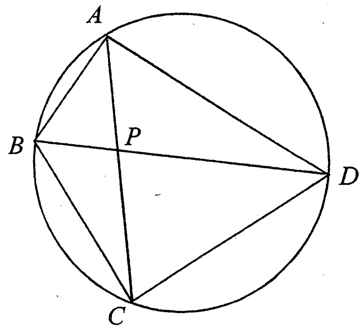
LETTER TO THE EDITOR

Dear Editor,

An article in Part 2 of Vol 22 reminds me of the following proof of a special case of Ptolemy's Theorem.

Let $ABCD$ be a cyclic quadrilateral with $AB = AD$. Let AC cut BD at P. Then

$$\angle APD = \angle ABP + \angle BAP$$
$$= \angle ADP + \angle CDP = \angle ADC$$



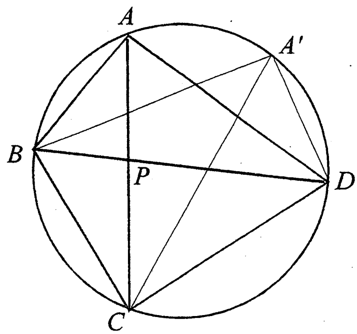
Hence

$$\frac{1}{2}\ AC \cdot BD \,\text{sum}\, ADC = \frac{1}{2}\ AC \cdot BD\ \sin APD = \text{Area}\,(ABCD)$$
$$= \text{Area}\,(ACD) + \text{Area}\,(ACB)$$
$$= \frac{1}{2}\ AD \cdot CD \sin ADC + \frac{1}{2}\ AB \cdot BC \sin ABC$$
$$= \frac{1}{2}\ AB \cdot CD \sin ADC + \frac{1}{2}\ AD \cdot BC \sin ADC$$

It follows that $AC \cdot BD = AB \cdot CD + AD \cdot BC$.

Andrei Storozhev of the Australian Mathematics Trust showed me the generalisation of the above argument.

Let $ABCD$ be a cyclic quadrilateral. Reflect A across the perpendicular bisector of BD to A'. Let AC cut BD at P. Then



$$\angle APD = \angle ABP + \angle BAP$$
$$= \angle A'DP + \angle CDP = \angle A'DC$$

Hence $\frac{1}{2}\ AC \cdot BD \sin A'DC = \frac{1}{2} AC \cdot BD\ \sin APD$

$$= \text{Area}\,(ABCD) = \text{Area}\,(A'BCD) = \text{Area}\,(A'DC) + \text{Area}\,(A'BC)$$

$$= \frac{1}{2}\,A'D \cdot CD \sin A'DC + \frac{1}{2}\,A'B \cdot BC \sin A'BC$$

$$= \frac{1}{2}\,AB \cdot CD \sin A'DC + \frac{1}{2}\,AD \cdot BC \sin A'DC$$

It follows that $AC \cdot BD = AB \cdot CD + AD \cdot BC.$

Note that the special case is not used to prove the general case, just to motivate it.

Yours sincerely

Andy Liu
University of Alberta

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Number Crunching Software

Our reader, Daniel Corbier, has released *UCalc Fast Math Parser*, a piece of software that may be of interest to *Function* readers. This is a component for software developers which allows programs to evaluate algebraic expressions defined at runtime. It is particularly designed to run very fast inside loops, making this math engine ideal for heavy-duty number crunching.

To find out more about this math component, and to download a copy, visit `http://www.ucalc.com/dll`.

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A good student is one who will teach you something.

– Irving Kaplansky

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HISTORY OF MATHEMATICS

John Wallis and his Wonderful Product

Michael A B Deakin

John Wallis (1616–1703) is much less well-known than his younger contemporary, Isaac Newton. Nonetheless he achieved much in the direction of early development of the ideas that we now collectively refer to as "Calculus".

He was born in the English county of Kent, and was some 26 years older than Newton. Those were turbulent days in England as it was the time of civil war between the monarchy and the high church on the one hand and the forces of parliament and low church puritanism on the other. He was the son of a parson, but the family sympathies lay with the puritan side. He followed an ecclesiastical career for some time, indeed being consecrated bishop of Winchester in 1640.

In 1649, and to the surprise of many, he was appointed to the prestigious Savilian chair of Mathematics at Oxford. It was a political appointment, as Wallis had little reputation in the area of Mathematics at this time. Indeed he replaced a man whose royalist sympathies had led to his being removed when the other side was in the ascendant. However, Wallis managed to rise to the challenge of his new appointment and also to steer a middle course politically, so that with the restoration of the monarchy he did not suffer the fate of his predecessor.

The story I will tell here is of his method for finding areas and some of the consequences he derived from it. It should be noted that he lived in an age when standards of mathematical proof were less rigorous than they became later and remain today. Many of his results were not fully proved, but they were correct and may be proved quite easily by modern methods. However by diligent work and great ingenuity he was able to discover them from quite elementary considerations.

Suppose we want to work out the area under a parabola, and suppose for simplicity that the parabola in question is $y = x^2$. If we seek the area bounded by this curve, the x-axis and the lines $x = 0$ and $x = n$ (n being a positive integer), then in modern terms we may write this area as

$$\int_0^n x^2 \, dx$$

and it is not difficult to evaluate the result, which is $\frac{1}{3}n^3$.

Indeed we can now go further and say that if we look for a rectangle on a base of n units with the same area as we have just been discussing, then the height of this rectangle will be $\frac{1}{3}n^2$. Nowadays we speak of this value's being the average, or mean, value of x^2 over the interval $0 \le x \le n$.

Wallis looked at precisely this problem but he came from a different direction. He considered the value of x^2 at each of the values $x = 0, 1, 2, 3, \ldots\ldots, n$ and he averaged these. The result is

$$\frac{0^2 + 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2}{n + 1}.$$

Now this is an expression whose value is easily worked out if we can sum the numerator. This was already known when Wallis worked. It was known much much earlier, to Archimedes. The relevant formula is

$$0^2 + 1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \tfrac{1}{6}n(n + 1)(2n + 1),$$

and it may be proved by the method of induction discussed in my last column. So now we know that the average value we seek is $\frac{1}{6}n(2n+1)$. If we now express this as a fraction of the final (and largest) value n^2, we find that the average value is

$$\left[\frac{1}{3} + \frac{1}{6n}\right]n^2$$

and as n gets larger and larger, the factor in the brackets tends towards the value $\frac{1}{3}$. This much would have been familiar ground, for again, Archimedes had got this far.

But now Wallis asked about more complicated functions than $y = x^2$. What about $y = x^3$? He studied a lot of special cases and came to the conclusion that in this case, we could proceed similarly and reach a factor of $\frac{1}{4}+\frac{1}{4n}$, which, as $n \to \infty$, approached the value $\frac{1}{4}$.

Proceeding now to higher cases $y = x^k$, where $k > 3$, Wallis found a very simple formula for the limit, which works out to be $\frac{1}{k+1}$ for every positive integral value of k. Of course, we today find what is essentially this result very cheaply using integral calculus, but Wallis didn't have that advantage.

So by modern standards, Wallis lacked a full proof. However, the formula he discovered was true.

His next step was even more daring. He looked at the case in which k was not necessarily integral. Suppose we put $k = \frac{p}{q}$. In this case the limiting ratio will be $\frac{1}{\frac{p}{q}+1}$ or in other words $\frac{q}{p+q}$.

Wallis's word for the value of k was "index" and we still use it today. It was during the course of this investigation that Wallis discovered the relation between roots and fractional powers: $\sqrt[q]{x} = x^{1/q}$.

Thus, in essence, Wallis was computing integrals, even though the precise concept of "integral" did not at that time exist. The results he obtained so laboriously are now routine and his formula can readily be written

$$\int_0^1 x^k dx = \frac{1}{k+1}$$

However, we do not properly appreciate the greatness of Wallis's work if we fail to understand it in its own terms.

This much would have been quite an achievement in itself, but Wallis went further. He next considered the area of a quarter-circle. In our terms, he set out to investigate the integral (with its known value)

$$\int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$$

The left-hand side is an example of a formula of the type

$$\int_0^1 (1-x^{1/p})^q\,dx,$$

where, in the case we want, $p = q = \frac{1}{2}$. Notice again that, in order to present the problem to a modern audience, I have used modern notation; Wallis did not have that advantage.

In order to get a handle on the problem, Wallis looked at the expression that we could write like this:

$$f(p,q) = \frac{1}{\int_0^1 (1-x^{1/p})^q\,dx}$$

and tabulated the result of calculations for integral values of p and q for integral values between 0 and 10. This was to be an intermediate step in getting to where he wanted to go. What he was after was the formula $f(\frac{1}{2},\frac{1}{2}) = \frac{4}{\pi}$ but to get there he started with simpler problems. Table 1 gives some of his results.

	q						
p	0	1	2	3	4	....	10
0	1	1	1	1	1	....	1
1	1	2	3	4	5	....	11
2	1	3	6	10	15	....	66
3	1	4	10	20	35	....	286
4	1	5	15	35	70	....	1001
...	...	...	...	...	...	...	...
10	1	11	66	286	1001	....	184756

Table 1: Values of $f(p,q)$

There are several features that will strike the reader about the numbers generated in Table 1. It is symmetric about its main diagonal, and all of the numbers are numbers that arise in the construction of Pascal's triangle. (The binomial theorem was not known at this time; it was later to be discovered by Newton. However, Pascal's triangle was available and Wallis knew about it.) You may care to look for other features of interest; several more are mentioned below.

If we follow the rows across, we see first simply a row of 1s. Below this simply a list of the natural numbers. The next one would also have been familiar to Wallis. The numbers involved are the "triangular numbers". Each entry is the sum of the numbers in the row above and to the left. Indeed this is a feature of every entry in the table. In this instance, we find: 1 = 1, 3 = 1 + 2, 6 = 1 + 2 + 3, etc. The entries are the successive sums of the simple arithmetic progression in the line above. We know a formula for these and so did Wallis. It goes

$$f(2,q)=\tfrac{1}{2}(q+1)(q+2).$$

His next idea was to use this formula even when q was not integral. Thus he wrote $f(2,\tfrac{1}{2}) = \tfrac{1}{2}(\tfrac{1}{2}+1)(\tfrac{1}{2}+2) = \tfrac{15}{8}$ and so on. Similarly, he was able to calculate many other values of $f(p,\tfrac{n}{2})$. For example, he knew the formulae

$$f(1,\tfrac{n}{2}) = \frac{n+1}{2}$$

and

$$f(3,\tfrac{n}{2}) = \tfrac{1}{6}\left(\tfrac{n}{2}+1\right)\left(\tfrac{n}{2}+2\right)\left(\tfrac{n}{2}+3\right)$$

and the pattern such formulae set up led him to guess others. He also made use of the symmetry of the table and so set up an extended version. This is shown in Table 2.

What Wallis wanted to do now was to fill in the "blanks" in this table; in particular the "blank" value of $f(\tfrac{1}{2},\tfrac{1}{2})$ that I have indicated by the question mark. By studying Table 1, and using his knowledge of Pascal's triangle, Wallis was now able to find the formula:

$$f(p,q) = \frac{p+q}{q} f(p,q-1).$$

				q			
p	0	1/2	1	3/2	2	5/2	3
0	1	1	1	1	1	1	1
1/2	1	?	3/2		15/8		105/48
1	1	3/2	2	5/2	3	7/2	4
3/2	1		5/2		35/8		315/48
2	1	15/8	3	35/8	6	63/8	10
5/2	1		7/2		63/8		693/48
3	1	105/48	4	315/48	10	693/48	20

Table 2: More values of $f(p,q)$

He now applied this formula to Table 2, and in particular to its second row, that for which $p = \frac{1}{2}$ and which contains the entry I designated by ?. If we apply this formula to this row, we find first that

$$f(\tfrac{1}{2},\tfrac{3}{2}) = \frac{\frac{1}{2}+\frac{3}{2}}{\frac{3}{2}}? = \frac{4?}{3}$$

and proceeding in this manner, we can fill in the entire row in terms of the one unknown, namely ?.

We find as a general formula $f(\frac{1}{2},q) = \dfrac{2q+1}{q} f(\frac{1}{2},q-1)$ and applying this successively, we reach the row entries:

$$\frac{1}{1} \quad \frac{?}{1} \quad \frac{3}{2} \quad \frac{4?}{3} \quad \frac{15}{8} \quad \frac{8?}{5} \quad \frac{105}{48} \quad \frac{128?}{35} \quad \frac{945}{192} \quad \text{etc.}$$

Still he needed some further handle on the problem in order to determine the value of ?. This he found by looking at the *ratios* of the various entries along each row. If we look at the second row in Table 1 for example, we find the ratios $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\ldots$. Each is smaller than its predecessor and as we go further and further out, we get closer and closer to 1. The same is true for all the other rows in Table 1.

So Wallis decided that this pattern must likewise hold for the rows of Table 2. Applying this insight to the row displayed above, we find successively

$$\frac{?}{1} > \frac{3/2}{?}$$
$$\frac{3/2}{?} > \frac{4?/3}{3/2}$$
$$\frac{4?/3}{3/2} > \frac{15/8}{4?/3}$$
$$\frac{15/8}{4?/3} > \frac{8?/5}{15/8}$$

and so on.

From the first inequality we learn that $? > \sqrt{3/2}$, from the second that $? < \frac{3}{2}\sqrt{4/3}$, from the third that $? > \frac{3}{2}.\frac{3}{4}\sqrt{5/4}$, from the fourth that $? < \frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\sqrt{6/5}$, and so the pattern goes. There are successive *upper bounds* (overestimates) for ? of $\frac{3}{2}\sqrt{4/3}$, $\frac{3}{2}\frac{3}{4}\frac{5}{4}\frac{5}{6}\sqrt{6/5}$, and so on. The square root aspect of these gets closer and closer to 1, while all the others terms bring the overestimate *down*. Similarly if we look at the *underestimates*, we see a pattern in which these come *up*.

Thus Wallis concluded that in the limit the value was

$$\frac{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \ldots}{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \ldots}$$

and remembering that this has the value $\frac{4}{\pi}$ we may now turn the fractions upside down and so reach

$$\frac{\pi}{4} = \frac{2 \times 4 \times 4 \times 6 \times 6 \times 8 \times \ldots}{3 \times 3 \times 5 \times 5 \times 7 \times 7 \times \ldots}$$

This formula is today known as "Wallis's Product". There are easier ways to reach it, but they need more advanced Mathematics than Wallis had at his command. Nowadays too we are able to prove the validity of the many steps that

Wallis merely guessed. But Wallis got there first, and the formula he discovered is quite remarkable.

Further Reading

See Victor J Katz, *A History of Mathematics*, pp 443-446, C HEdwards, *The Historical Development of the Calculus*, pp 113-117, 170-176 and the article on Wallis in the *Dictionary of Scientific Biography*.

* * * * *

Digesting mathematics

I was at the mathematical school, where the master taught his pupils, after a method, scarce imaginable to us in Europe. The propositions, and demonstrations, were fairly written on a thin wafer, with ink composed of a cephalic tincture. This, the student was to swallow upon a fasting stomach, and for three days following, eat nothing but bread and water. As the wafer digested, the tincture mounted to his brain, bearing the proposition along with it. But the success has not hitherto been answerable, partly by some error in the *quantum* or composition, and partly by the perverseness of lads; to whom this bolus is so nauseous, that they generally steal aside, and discharge it upwards, before it can operate; neither have they been yet persuaded to use so long an abstinence as the prescription requires.

– Swift, Jonathan in *Gulliver's Travels; A Voyage to Laputa*

* * * * *

So highly did the ancients esteem the power of figures and numbers, that Democritus ascribed to the figures of atoms the first principles of the variety of things; and Pythagoras asserted that the nature of things consisted of numbers.

– Lord Bacon

* * * * *

COMPUTERS AND COMPUTING

The Turtles' Paths

Cristina Varsavsky

For the readers interested in computers and computing, here is a program that displays the spiral paths of the turtles following each other as described in the article *The Four-Turtles Problem.*

```
REM Setting the cartesian axes and scale definition
  SCREEN 9: WINDOW (-4 / 3, -1)-(4 / 3, 1)

REM Definitions
   turtles = 4
   DIM a(turtles), b(turtles)
   Pi = 3.1416
   alpha = .1
   beta = (1 - 2 / turtles) * Pi
   factor = SIN(beta) / (SIN(alpha + beta) + SIN(alpha))

REM Coordinates of first polygon
  FOR s = 0 TO turtles
    c = Pi * (2 * s + 1) / turtles
    a(s) = SIN(c): b(s) = COS(c)
  NEXT s

REM Drawing all polygons
  FOR m = 1 TO 70
   PSET (a(0), b(0))
    FOR s = 1 TO turtles
      LINE -(a(s), b(s))
    NEXT s

   REM Definition of next vertices
   FOR n = 0 TO turtles
    aux = a(n)
    a(n) = factor * (a(n) * COS(alpha) + b(n) * SIN(alpha))
    b(n) = factor * (-aux * SIN(alpha) + b(n) * COS(alpha))
   NEXT n
  NEXT m
```

The initial distance from the centre of the square to each turtle is 1. The coordinates of the position of the turtles are stored in the arrays `a` and `b`. At each step the square is rotated clockwise through the angle `alpha`, and then reduced so that the vertices are on the previous square. The variable `turtles` is initialised to 4, but you can change it to any number. Figures 1 and 2 show the screen outputs corresponding to `turtle = 3` and `turtle = 6` respectively.

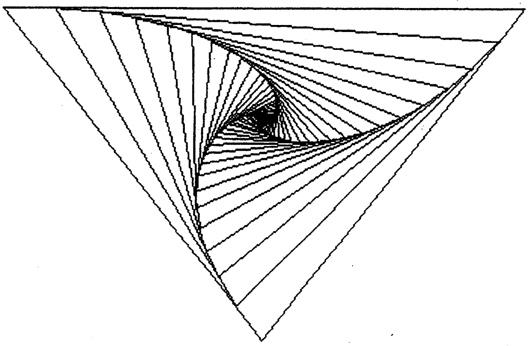

Figure 1

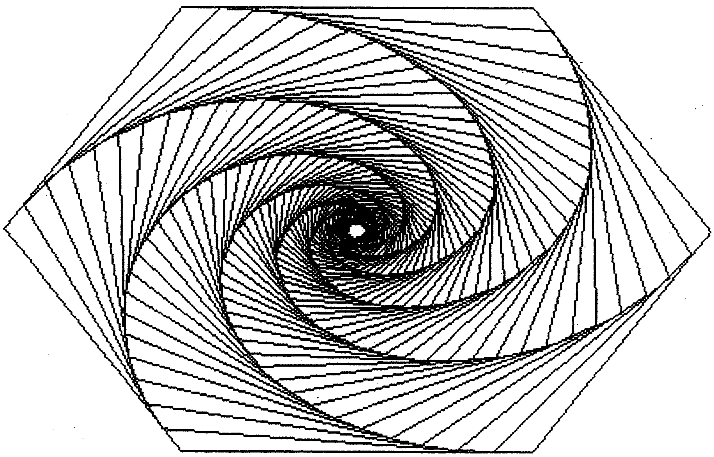

Figure 2

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 22.2.1 (from *The Chicken from Minsk*; submitted by Lachlan Harris)

Old Man Mazay rows for his vodka

Old Man Mazay (the alcoholic) is rowing down a river. The current is 2 miles/hour. Just as Mazay is passing under a bridge, he takes a drink, but instead of returning the bottle to the stern of the boat, he drops it into the river! Mazay continues rowing downstream for half an hour, until he realises he is thirsty once again. Mazay rows at 3 miles/hour, but aided by the current, he goes at 5 miles/hour. How long will Mazay take to retrieve the bottle, and how far from the bridge will he be at that time?

SOLUTION

The simplest way to solve the problem is to measure the speeds relative to the flowing water. In this frame of reference, the floating bottle is stationary, and Mazay moves away from the bottle for half an hour, then towards the bottle at the same speed, so it takes Mazay another half an hour to return and retrieve the bottle. During the one hour it has taken him altogether, the bottle has moved at 2 miles/hour relative to the bridge, so it must be 2 miles from the bridge.Mazay's rowing speed is not needed to solve the problem.

Solutions were received from Carlos Alberto da Silva Victor (Nilópolis, Brazil), Julius Guest (East Bentleigh, Vic), Keith Anker (Glen Waverley, Vic), and Lachlan Harris (Gisborne South, Vic).

PROBLEM 22.2.2 (from *The Chicken from Minsk*; submitted by Lachlan Harris)

Masha's mathematical turtles

Masha has trained her four turtles to always follow each other. She arranges them at the corners of a square, as shown in the diagram [Figure 1], with each turtle facing its clockwise neighbour. The turtles move at one constant speed, V. What will happen to the square [with a vertex at each turtle]? When will the turtles meet?

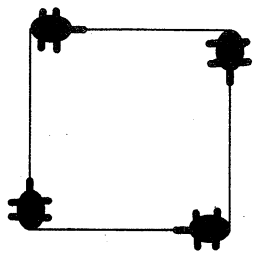

Figure 1

SOLUTION by Julius Guest

We realise that the four turtles form the corners of a uniformly shrinking square which rotates clockwise as the turtles move toward the centre where they must all meet eventually. Suppose the turtles are arranged in clockwise order: A, B, C and D. Turtle A would then eventually meet turtle B at the centre, turtle B would likewise meet turtle C there, and the same would apply to C meeting D and D meeting A. The length of each path must then be the length of the side of the given square. (To see this, note that the velocity of turtle B is always perpendicular to that of turtle A, so turtle B is not moving toward or away from turtle A; hence turtle A will reach turtle B in the same time as it would have if turtle B had stayed in its original position.) If now the side length of the original square is a units, and their constant common speed is V, then the four turtles must meet at the centre after a / V units of time have elapsed.

Also solved by Carlos Victor, Keith Anker, and Lachlan Harris.

A further analysis of this problem can be found in Michael Deakin's article *The Four–Turtles Problem*, which appears elsewhere in this issue.

PROBLEM 22.2.3 (based on a problem in *New Scientist*)

Three churches A, B and C are equally spaced from one another, ie., they lie at the vertices of an equilateral triangle. George is standing at a point which is 8km from A, 5 km from B and 3 km from C.

(a) Show that George must be outside the triangle.
(b) How far apart are the churches?

SOLUTION

(a) Let P be the point where George is standing. Using the triangle inequality applied to the triangle BCP, we deduce that $BC < 8$. Since George is further from A than the side length of the triangle ABC, George must be outside the triangle.

(b) There are various ways of approaching the problem. The one we present here is based on the solution submitted by Julius Guest.

Let the side length of ABC be x km, let the coordinates of the vertices be $A(0,\ 0)$, $B(x,\ 0)$, and $C(x/2,\ x\sqrt{3}/2)$, and let $P(p,\ q)$ be George's position. Using the three distances given, we obtain the following equations:

$$p^2 + q^2 = 64 \tag{1}$$

$$(x-p)^2 + q^2 = 25 \tag{2}$$

$$(p - x/2)^2 + (q - x\sqrt{3}/2)^2 = 9 \tag{3}$$

Eliminate q from (1) and (2) to obtain

$$p = (x^2 + 39)/2x \tag{4}$$

Next, use (1), (3) and (4) to yield

$$q = (x^2 + 71)/2x\sqrt{3} \tag{5}$$

Now, use (1), (4) and (5) to arrive at

$$(x^2 - 49)^2 = 0 \tag{6}$$

Since only the positive solution of (6) is meaningful here, $x = 7$.

The problem can also be solved using the cosine rule for triangles, but the proof is still rather complicated. The fact that the answer is an integer suggests that it should be possible to find a simpler proof, perhaps a purely Euclidean proof (with no coordinate geometry or trigonometry). The problem offers some tantalising hints in this direction. It turns out that P is on the circumcircle of the triangle; this follows from $AP = BP + CP$, using the converse of a particular case of Ptolemy's theorem that readers were recently invited to prove (Problem 22.1.2). Also, the angles *APB* and *APC* both have measure 60°. Perhaps one of our readers can construct a Euclidean proof that incorporates these observations in some way.

Solutions were received from Julius Guest, Carlos Victor, and Keith Anker.

PROBLEM 22.2.4 (Garnet J Greenbury, Brisbane, Qld)

Let k be the positive solution of the equation $k^2 + k - 1 = 0$. Prove that:

$$k^n = (-1)^{n+1}(a_n k - a_{n-1}) \quad (n \geq 2)$$

where a_n is the nth term of the Fibonacci sequence 1, 1, 2, 3, 5, 8, … .

SOLUTION

The proof proceeds by induction on n, starting with $n = 2$. When $n = 2$, the equation to be proved becomes $k^2 = -(a_2 k - a_1)$, i.e. $k^2 = -(k-1)$, which is true by the definition of k. Now suppose the equation is true for a fixed value of n, say n_0. We need to deduce that it is also true when $n = n_0 + 1$. We have:

$$\begin{aligned}
k^{n_0+1} &= k^{n_0} k \\
&= (-1)^{n_0+1}(a_{n_0} k - a_{n_0-1})k \\
&= (-1)^{n_0+1}(a_{n_0} k^2 - a_{n_0-1}k) \\
&= (-1)^{n_0+1}[a_{n_0}(1-k) - a_{n_0-1}k] \quad \text{since } k^2 + k - 1 = 0 \\
&= (-1)^{n_0+1}[a_{n_0} - (a_{n_0} + a_{n_0-1})k] \\
&= (-1)^{n_0+1}(a_{n_0} - a_{n_0+1}k) \\
&= (-1)^{(n_0+1)+1}(a_{n_0+1}k - a_{n_0})
\end{aligned}$$

By induction, the result is true for n = 2, 3, 4, 5, … .

Solutions were received from Carlos Victor, Julius Guest, and Keith Anker.

PROBLEM 22.2.5 (from *Crux Mathematicorum with Mathematical Mayhem*)

Lucy and Anna play a game where they try to form a ten-digit number. Lucy begins by writing any digit other than zero in the first place, then Anna selects a different digit and writes it down in the second place, and they take turns, adding one digit at a time to the number. In each turn, the digit selected must be different from all previous digits chosen, and the number formed by the first$n$ digits must be divisible by n. For example, 3, 2, 1 can be the first three moves of a game, since 3 is divisible by 1, 32 is divisible by 2, and 321 is divisible by 3. If a player cannot make a legitimate move, she loses. If the game lasts ten moves, a draw is declared.

(a) Show that the game can end up a draw.

(b) Show that Lucy has a winning strategy and describe it.

SOLUTION

(a) The games ends up a draw with the number 3816547290. (This was the answer to Problem 20.5.5 from the October 1996 issue.)

(b) There are many winning strategies for Lucy. One strategy is to start with 2. Anna must then respond with 0, 4, 6 or 8. We consider each of these responses in turn.

If Anna chooses 0, Lucy then chooses 4. Anna must now write down 8, and Lucy wins by choosing 5.

If Anna chooses 4, Lucy then chooses 0. As before, Anna must now write down 8, so again Lucy wins with 5.

If Anna chooses 6, Lucy chooses 1 and wins.

If Anna chooses 8, Lucy then chooses 5. Anna must now write down 6, and Lucy wins with 0.

Solutions were received from Carlos Victor and Keith Anker.

PROBLEM 22.2.6 (Mathematical Team Contest "Baltic Way – 92")

Find all integers satisfying the equation $2^x(4-x)=2x+4$.

SOLUTION by Carlos Victor

Since $x = 4$ is not a solution, we have:

$$2^{x-1} = \frac{x+2}{4-x} \qquad (7)$$

Observe that $2^{x-1} > 0$ for all $x \in \mathbf{R}$, so we have $-2 < x < 4$. The integers in this range are −1, 0, 1, 2, 3. Substituting each of these values in turn into (7), we find that $x = 0, 1, 2$ are the only solutions.

Solutions were also received from Julius Guest and Keith Anker.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 7 December 1998 will be acknowledged in the February 1999 issue, and Clark's article "Patterns in Tattslotto Numbers" in this issue.

PROBLEM 22.4.1

Use a "balls in cells" approach to find the probability that there are no runs longer than two consecutive numbers when drawing 6 numbers from 1, 2, … , *n*.

PROBLEM 22.4.2 (J A Deakin, Shepparton, Vic)

Find all 2×2 matrices that commute with the matrix $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

PROBLEM 22.4.3 (Julius Guest, East Bentleigh, Vic)

Given that $x + y + z + u = 0$, prove that

$$x^3 + y^3 + z^3 + u^3 + 3(x+y)(y+z)(z+x) = 0.$$

PROBLEM 22.4.4 (from *Mathematical Spectrum*)

A piece of wire of length l is bent into the shape of a sector of a circle. Find the maximum area of the sector.

PROBLEM 22.4.5 (from the Memorial University Undergraduate Mathematics Competition, 1997)

Prove that $1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2}<2$.

(Note: Advanced methods can be used to show that $1+\frac{1}{4}+\frac{1}{9}+\ldots+\frac{1}{n^2}+\ldots=\frac{\pi^2}{6}$, from which of course the inequality follows immediately. However, the inequality can be proved using elementary methods, without appealing to this result.)

PROBLEM 22.4.6 (from *Crux Mathematicorum with Mathematical Mayhem*)

In how many ways can the 12 vertices of a regular icosahedron be partitioned into four classes of three vertices, such that the vertices in each class belong to the same face?

* * * * *

The Pythagoreans and Platonists were carried further by this love of simplicity. Pythagoras, by his skill in mathematics, discovered that there can be no more than five regular solid figures, terminated by plane surfaces, which are all similar and equal; to wit, the tetrahedron, the cube, the octahedron, the dodecahedron, and the eicosihedron. as nature works in the most simple and regular way, he thought that all elementary bodies must have one or other of those regular figures; and that the discovery of the properties and relations of the regular solids must be a key to open the mysteries of nature.

This notion of the Pythagoreans and Platonists has undoubtedly great beauty and simplicity. Accordingly it prevailed, at least to the time of Euclid. He was a Platonic philosopher, and is said to have wrote all the books of his Elements, in order to discover the properties and relations of the five regular solids. The ancient tradition of the intention of Euclid in writing his elements, is countenanced by the work itself. For the last book of the elements treats of the regular solids, and all the preceding are subservient to the last.

– Thomas Reid in *Essence of the Powers of the Human Mind*

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