Function

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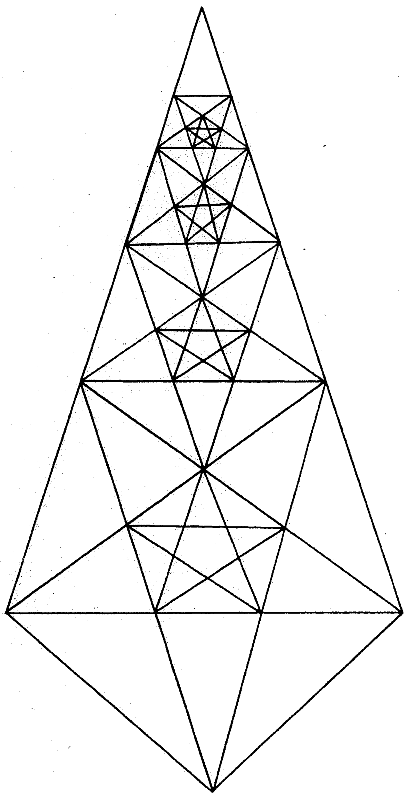

Department of Mathematics & Statistics – Monash University

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Function is a refereed mathematics journal produced by the Department of Mathematics & Statistics at Monash University. The journal was founded in 1977 by Prof G B Preston. *Function* is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

Function deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of *Function* include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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* $10 for *bona fide* secondary or tertiary students.

EDITORIAL

This issue of *Function* includes a variety of articles, letters, columns and problems, with which we hope we will please the mathematical tastes of all our readers.

The figure on the front cover is known as the Lute of Pythagoras; the accompanying article explains its recursive construction, and gives some starting points for you to explore its interesting properties.

We include in this issue of *Function* the work of a very young author, the Year 8 student Mark Nolan, who follows a mathematical argument to make decisions on whether to add milk to his coffee before or after he is called away briefly. We also include an article of our regular author Bert Bolton: this time he presents the equations modelling the oscillation of a pendulum.

The History of Mathematics column is about the mathematical argument known as *proof by induction*, which as stated by its title, is used to prove infinitely many things at once. Michael Deakin explains that although the first explicit statement is found in the work of Blaise Pascal, the principle of proof by induction was used by the early mathematicians. Michael Deakin is also the author of the Computers and Computing column; he gives an account of his race with personal computers to find integrals of a special type of exponential functions. He claims victory in this particular field as he could do by hand what current sophisticated mathematical software could not handle.

We also include in this issue the regular Problem Corner prepared by Peter Grossman with new problems and solutions to the problems published in the February issue, two letters from our readers showing results related to the article on the leaning ladders, and a mathematical analysis of a piece of news that appeared in a Melbourne newspaper.

We hope you enjoy this issue of *Function*!

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THE FRONT COVER

The Lute of Pythagoras

Cristina Varsavsky

The figure on the front cover is known as the Lute of Pythagoras. Its construction follows a recursive process: start with an isosceles triangle ABC; next draw a segment A_1C_1 parallel to the side AC so that $AC_1 = AC$; now draw the segments AC_1 and CA_1 as shown in Figure 1. Repeat the above steps again and again, obtaining the points C_2, A_2, C_3, A_3, ... , and the segments $A_1C_2 = A_1C_1$, $A_2C_3 = A_2C_2$, ... Finally draw a point B_1 so that $AB_1 = B_1C$, and complete the triangles $A_1B_1C_1$, $A_2B_2C_2$, $A_3B_3C_3$, etc. The Lute often includes five pointed stars in the progression of diminishing pentagons.

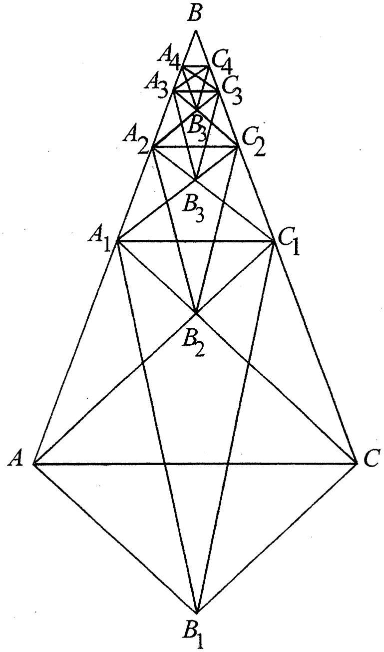

Figure 1

The Lute of Pythagoras is not only a figure pleasing to the eyes, but it also has many interesting properties. Let us find some relationships between lengths of segments. If θ is the angle $A\hat{C}B$ and $AC = 1$, then $CB = \dfrac{1}{2\cos\theta}$; also, by construction, $A\hat{C}C_1 = A\hat{C}_1C = \theta$ and the pairs of triangles ΔCAC_1 and ΔABC, and ΔABC and ΔA_1BC_1 are similar, hence

$$\frac{CC_1}{1} = \frac{1}{CB} \quad \therefore \quad CC_1 = 2\cos\theta$$

$$\frac{A_1C_1}{1} = \frac{CB - CC_1}{CB} \quad \therefore \quad A_1C_1 = 1 - \frac{CC_1}{CB}$$

Using a similar argument, we find that

$$\frac{C_1C_2}{A_1C_1} = \frac{1}{CB} \quad \therefore \quad C_1C_2 = 2\cos\theta\, A_1C_1$$

$$\frac{A_2C_2}{1} = \frac{CB - CC_1 - C_1C_2}{CB} \quad \therefore \quad A_2C_2 = 1 - \frac{CC_1 + C_1C_2}{CB}$$

and then $C_2C_3 = 2\cos\theta\, A_2C_2$ and $A_3C_3 = 1 - \dfrac{CC_1 + C_1C_2 + C_2C_3}{CB}$, and so on.

I invite you to use a spreadsheet to check, for several values of θ that the following ratios are constant

$$\frac{CC_1}{C_1C_2} = \frac{C_1C_2}{C_2C_3} = \frac{C_2C_3}{C_3C_4} = \ldots = k$$

$$\frac{AC}{A_1C_1} = \frac{A_1C_1}{A_2C_2} = \frac{A_2C_2}{A_3C_3} = \ldots = k$$

and also

$$\frac{CC_1}{A_1C_1} = \frac{C_1C_2}{A_2C_2} = \frac{C_2C_3}{A_3C_3} = \ldots = m$$

I leave it to you to prove that $k = \dfrac{1}{1 - 4\cos^2\theta}$ and $m = \dfrac{2\cos\theta}{1 - 4\cos^2\theta}$.

An interesting angle corresponds to $\theta = 72^\circ$, that is, the angle at the vertex of a regular pentagon: in this case, $m = 1$ and $k \approx 1.61803$, which corresponds to the *golden ratio* $\phi = \dfrac{1+\sqrt{5}}{2}$. This is the angle that should give the most visually pleasing of all Lutes of Pythagoras!

THE COOLING OF THE COFFEE

Mark Nolan, Year 8, St Leonard's College

This problem appeared on the letters page of *The Age* newspaper last February.

> "I have just poured a cup of coffee and am about to add milk when I am called away briefly. Will the coffee be hotter if I add the milk now or when I return?"

My task is to solve this problem and explain how I got my answer.

To solve this problem, I will use Newton's Law of Cooling and the Mixing Law.

Suppose that when I am called away, the coffee has a temperature of T_c and the milk has a temperature of T_m. The room temperature is called T_r .

Newton's Law of Cooling tells us that after a time t has gone by, the coffee will have a temperature of $T_r + (T_c - T_r)\, e^{-kt}$, where k is a constant and e is the base of the natural logarithms. Similarly the milk will have a temperature of $T_r + (T_m - T_r)\, e^{-kt}$.

If I am called away and do not mix the coffee and the milk before I go, and if I am away for a time t, then these will be the two temperatures when I get back.

On the other hand, if I mixed the coffee and the milk before I left, then the white coffee would have a temperature of T_w at the time I was called away and (again using Newton's Law of Cooling) $T_r + \left(T_w - T_r\right) e^{-kt}$ when I return.

The Mixing Law says that if two liquids with temperatures T_1 and T_2 are mixed in the proportion (α : β), where $\alpha + \beta = 1$, then the temperature of the mixture will be $\alpha T_1 + \beta T_2$.

First think of the case when I mix the coffee and the milk before being called away. Because of the Mixing Law, the temperature of the white coffee when I make it will be

$$T_w = \alpha T_c + \beta T_m .$$

When I come back, this will have cooled down to

$$T_r + (\alpha T_c + \beta T_m - T_r) e^{-kt} . \tag{1}$$

On the other hand, if I waited until I got back and then mixed the coffee and the milk, the temperature of the mixture would be

$$\alpha \, [\, T_r \; + \; (T_c - T_r \,) \, e^{-kt} \;] + \beta \; [\, T_r + (T_m - T_r \,) \, e^{-kt} \;], \tag{2}$$

which turns out to be the same when we replace β by $1-\alpha$ in (1) and (2).

The two expressions are the same!

When the problem was answered in *The Age*, a Mr David Grounds recalled that about 30 years ago, Professor Julius Sumner Miller discussed the problem on TV (in his programme *Why is it so?*). He went on to claim that the coffee would be hotter if the mixing was done first rather than later. His reasons were wrong, because he did not consider the matter fully.

However, his answer can be justified because if he performed an experiment to see which would be hotter, then the coffee mixed first would be. It would actually turn out that the coffee was hotter if the milk had been mixed first.

What Mr Grounds forgot was that the temperature of the cup containing the coffee was affecting his results. Some of the heat energy was absorbed into the cup, and so the coffee ended up being cooler than it should be. Since the coffee would have gone into the cup at a hotter temperature if it was not mixed with milk, more energy would be lost and it would be cooler than the coffee which had been mixed first. Therefore his answer was partly correct, but partly incorrect.

Because he forgot to include the cup in his answer, he left out the very thing that caused the difference, but the answer he gave turned out to be right!

* * * * *

THE MATHEMATICS OF VIBRATIONS

Bert Bolton, University of Melbourne

Vibrations (also known as oscillations) are found everywhere in nature. They can be mechanical: in a gusty wind, the top branches of trees move to and fro, usually in line with the direction of the wind. Or another example: the pendulum of a longcase clock[1] is maintained in its vibrations by small forces arising from the energy stored in a coiled spring that has previously been wound up, or else from the energy stored by a heavy weight that has previously been raised inside the case. Electrical vibrations form the basis of light-waves, radio and television signals and X-rays. A 'quartz watch' has a piece of specially cut quartz crystal which is kept in a state of mechanical vibration by the energy stored in a small electrical battery.

Many of these vibrations can be described in terms of a simple equation which can be derived using calculus. The story starts with Galileo (1564–1642).[2] A story is told of him which may not be true, but which nonetheless bears repeating. It is said that he was attending a service in an Italian cathedral when he noticed that one of the lamps suspended from the ceiling was swinging from side to side, and that each complete swing appeared to take the same time as the last. The cause of the swinging could have been a draught of air blowing through an open door, or perhaps the eddies in an updraught of air from candles burning below. Galileo confirmed that each swing did indeed take the same time as the others by measuring each swing against his pulse-rate.

A normal pulse-rate is about 70 beats per minute. The lamp was probably hanging on a long chain and if the length of the chain was (say) 20 m, then its 'period of vibration' (the time taken for a complete swing) would have been very nearly nine seconds. This is 10.5 pulses of his heart.[3]

Knowing now that the period was constant, Galileo suggested that a pendulum could be used as a control mechanism for a clock, and the result is our first example: the longcase clock.

For a simple pendulum the period can be calculated. Figure 1 shows a simple pendulum. It consists of a strong light string of length l metres hanging from a

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[1] More commonly known as 'grandfather' and 'grandmother' clocks.
[2] This is his given name; his surname was Galilei. It is usual to refer to scientists by their surnames, but an exception is made in this case.
[3] You may care to try a similar experiment for yourself.

point of attachment A and held taut by a compact weight or 'bob' of mass M kilograms at the other end B of the string. We suppose the string to make an angle θ (measured in *radians*) with the downward vertical.

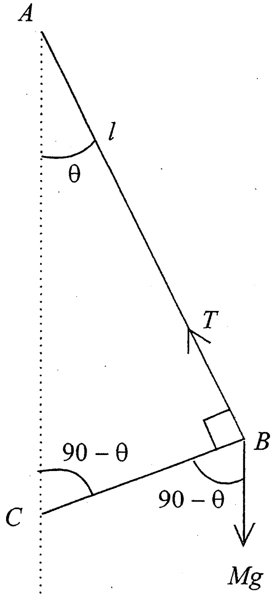


Figure 1

Let C be a point vertically below A and such that $\angle BAC = \theta$ is a right angle; also call $\angle BAC = \theta$.

Newton's second law of motion states that the time-rate of change of momentum is equal to the total force applied. Now the bob moves under the combined influence of two forces: its weight and the tension in the string. The weight is Mg newtons (the newton is the basic *SI* unit of force), where g is the acceleration due to gravity (about $9.81\,ms^{-2}$). Take the tension in the string to be T newtons.

Now the tension in the string may be resolved into two components: $T\cos\theta$ acting vertically and $T\sin\theta$ acting horizontally. The actual motion of B takes place along a circular arc lying below the line *BC*, but if θ is very small we may take this circular arc and the straight line to be for all practical purposes the same.

Thus the horizontal component of T gives rise to the motion and this horizontal component is $T\sin\theta$.
Thus

$$l\frac{d^2}{dt^2}=(ML\sin\theta) = -T\sin\theta \tag{1}$$

(the – sign arising because T *acts to decrease* θ).

Now in the vertical direction, there is very little motion and so the vertical component of T balances the weight of the bob.

$$T\cos\theta = Mg \tag{2}$$

However, when θ is very small, we have $\sin\theta \approx \theta$ (remember, we're working in *radians* and $\cos\theta \approx 1$). Putting all this together reduces Equations (1), (2) to the single equation:

$$l\frac{d^2\theta}{dt^2}=-g\theta \tag{3}$$

It is usual to write $\frac{g}{l}=\omega^2$ because, as we shall see, doing so simplifies the subsequent equations. With this change of notation, Equation (3) becomes:

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0 \tag{4}$$

Equation (4) is an example of a *differential equation*. We have followed its derivation for the case of the simple pendulum, but in fact it holds for all small vibrations. It is one of the most fundamental differential equations in all of physics. Its *general solution* is known to be

$$\theta = a\cos\omega t + b\sin\omega t \tag{5}$$

where a and b are some constants. By this is meant that *whatever values a and b may take*, we may satisfy Equation (4) by using the formula of Equation (5). This, the reader should readily verify. We also mean that there are no solutions of Equation (4) which are *not* of the form given by Equation (5), but this is something much harder to show.

So now let us finish by going back to Galileo and his swinging lamp. Suppose he timed a swing in which the lamp started at its far left position, moved through the central position, went all the way to the right, back through the centre and finished up once again at the far left. The time taken to do this is the *period* and it is denoted by τ. The lamp will be in its far left position when $t = 0$, at which time $\theta = a$. The next time $\theta = a$, we must have $\omega t = 2\pi$, and since $t = \tau$ at this time, we have

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}} \qquad (6)$$

(where we have put back the original quantities in place of ω).

Equation (6) shows how the pendulum can control a clock. If the period ω is too long and the clock on that account runs slow, then its length has to be shortened. Careful observation and adjustment can make the clock run at exactly the correct rate to show the correct time.[4]

The story given above is not the full story; it can't be because in real life all mechanical motion is slowed down by the action of friction. In the case of the pendulum, there is friction from the air moving past the bob and the string and also a little inner friction at A. Galileo would surely have noted that the lamp stopped swinging some time after the door was closed, the candles extinguished and other such stimuli removed.

Finally an exercise for the reader. Note that Equation (6) contains and connects two measurable quantities: l (the length of the pendulum) and τ (the period). Square both sides of Equation (6) and rearrange the result, to find

$$g = 4\pi^2 \frac{l}{\tau^2}$$

Now make a simple pendulum up to a metre long and with anything reasonably heavy as a bob. You may vary the length by lengthening or shortening the string and for each value of l, we measure the value of τ (most conveniently by using a watch with a second hand or its digital equivalent) for some reasonably large number, 20 say, of periods. If you now plot l against τ^2, the result should be a straight line passing through the origin. Multiplying the value of this line's gradient by $4\pi^2$ will now produce a value for g, the measure of the acceleration due to gravity. However be warned: it takes a lot of skill and care to get a value as accurate as 9.8.

[4] Clocks may be tested against the time-signals broadcast on (e.g.) the ABC radio.

LETTERS TO THE EDITOR

Dear Editor,

I found Rik King's article on 'Leaning Ladders' (*Function* Vol 21. Part 4. *pages* 117–121) very absorbing reading.

May I suggest a shorter way of tacking this problem.

Here is the problem.

Two ladders AD (of length 3 m) and BE (of length 4 m) lean across a narrow path by making angles with vertical walls on each side. Their crossing point F is 1 m above BD. Find the width of the path.

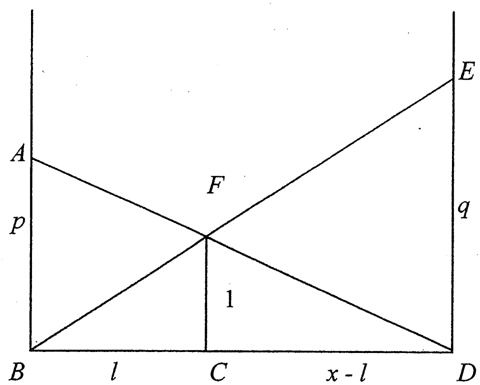


Figure 1

Let circumstances be as shown above.

By data $AD = 3$ m, $BE = 4$ m, $FC = 1$ m. Next let $BC = l$, $BD = x$, $AB = p$ and $ED = q$.

In triangle ABD we find via Pythagoras: $x^2 + p^2 = 9$. (1)

In triangle BED we find via Pythagoras: $x^2 + q^2 = 16$. (2)

Next, triangles ABD and FCD are similar, so it follows that

$$\frac{x}{p} = \frac{x-l}{1} \tag{3}$$

Again, triangles BED and BFC are similar, so we find

$$\frac{x}{q} = \frac{l}{1} \tag{4}$$

Now, if the introduction of square roots is avoidable do so, for this move makes for needless extra algebra and extraneous roots. Here this is possible.

Eliminating x from (3) and (4) we find that $p = q/(q-1)$. (5)

Eliminating p from (1) and (5) we obtain $x^2 + \left[\frac{q}{q-1}\right]^2 = 9$ (6)

From (2) we have that $x^2 = 16 - q^2$. (7)

So on eliminating x^2 from (6) and (7) we arrive at

$$q^4 - 2q^3 - 7q^2 + 14q - 7 = 0. \tag{8}$$

Solving this quartic gives

$$\begin{aligned} q_1 &= 0.854246 + 0.331410i, \\ q_2 &= 0.854246 - 0.3314140i, \\ q_3 &= -2.745414, \\ q_4 &= 3.036922. \end{aligned} \tag{9}$$

Now, the first three roots are inadmissible in our problem. Hence using (2) gives $x^2 = 6.777105$. (10)

Ignoring the inadmissible negative root gives at last

$$x = 2.603287.$$

So $BD = 2.60\ m$.

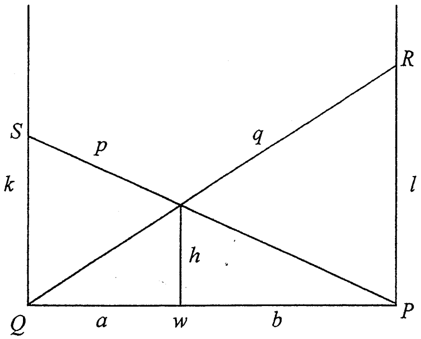
Julius Guest

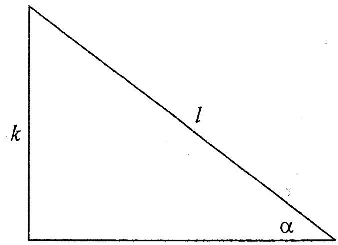
Dear Editor,

Further to *Leaning Ladders* in *Function* August 1997, one can reduce it to a quartic, at least, or better still solve it by simply looking up a trigonometric table and checking the cosine and cotangent columns simultaneously to see if they add up to $\sqrt{7}$.

Apart from the difficulty of solving an octic, where exactness is eschewed for dependency on numerical approximation to a nominated number of decimal places, there is a further problem: when one squares to eliminate the square root signs to obtain the octic, one doubles the number of 'solutions', half of which must be inadmissible. Here is an alternative method that concentrates on a trigonometric ratio.

Consider the ladders $QR = q$ and $PS = p$ which cross at height h above ground. Let their bases be distant w apart, and let the height above the wall be l and k respectively. Let $w = a + b$ as in the diagram.





By similar triangles we have

$$\frac{l}{w} = \frac{h}{a} \therefore\ l = \frac{wh}{a} \quad \text{and also} \quad \frac{k}{w} = \frac{h}{b} \therefore\ k = \frac{wh}{b}$$

Hence

$$l + k = wh\left(\frac{1}{a} + \frac{1}{b}\right) = \frac{w^2 h}{ab}$$

and

$$kl = \frac{w^2 h^2}{ab} = h(l + k) \qquad \therefore \qquad h = \frac{kl}{k + l}$$

Also,

$$w^2 = q^2 - l^2 = p^2 - k^2 \quad \therefore \quad l^2 - k^2 = q^2 - p^2$$

Now draw a right triangle where the hypotenuse is l and the height is k, leaving the base as $\sqrt{q^2 - p^2}$. We have

$$\cos\alpha + \cot\alpha = \frac{\sqrt{q^2 - p^2}}{l} + \frac{\sqrt{q^2 - p^2}}{k}$$

$$= \sqrt{q^2 - p^2}\,\frac{k + l}{k\,l} = \frac{\sqrt{q^2 - p^2}}{h} = R$$

If $q = 4, p = 3$, and $h = 1$, then $\cos\alpha + \cot\alpha = \sqrt{7}$. This is where we use the trigonometric table to find that $\alpha = 0.513163$ radians.

Now,

$$\cos\alpha + \cot\alpha = \cos\alpha\left(1 + \frac{1}{\sin\alpha}\right) = \sqrt{1 - \sin^2\alpha}\left(\frac{\sin\alpha + 1}{\sin\alpha}\right) = R$$

If we let $s = \sin\alpha$, and square the above equation, we obtain

$$(1 - s^2)(1 + 2s + s^2) = R^2 s^2$$

Expanding and replacing R^2 with 7 we obtain the quartic

$$s^4 + 2s^3 + 7s^2 - 2s - 1 = 0$$

This quartic has 2 complex solutions and two real solutions, namely, $s = -0.266992$ and $s = 0.490937$. We discard the negative solution because this corresponds to an angle greater than 180°. Therefore we obtain

$$k = \sqrt{7}\,\tan\alpha = \frac{\sqrt{7}\,s}{\sqrt{1 - s^2}} = 1.49094$$

$$l = \sqrt{7}\sec\alpha = \frac{\sqrt{7}}{\sqrt{1 - s^2}} = 3.03692$$

$$w = \sqrt{q^2 - l^2} = 2.60329$$

The *Derive* software also produced the exact solutions of the quartic, but these look rather cumbersome.

Yours sincerely,

David Halprin
North Balwyn, Victoria

Mathematics in The Press

In an article in *The Age* (30/12/97), regular columnist Gerard Henderson wrote:

"Constitutional Convention candidate P P McGuinness says that 'anyone who votes for me will be more sophisticated than the average voter'. He got 5856 votes. What does that say about Australian sophistication?"

Well, what *does* it say? Since 5856 is only a small proportion of the total number of votes cast, Gerard Henderson's rhetorical question was probably meant to lead us to conclude that Australians are not very sophisticated. If we analyse the reasoning carefully, however, we discover that there are two fallacies, one logical and one statistical.

Let's suppose, first of all, that P P McGuinness had instead made the following statement:

> "Anyone who is sophisticated will vote for me."

McGuinness would be claiming that *all* sophisticated voters will vote for him, without ruling out the possibility that some unsophisticated voters might vote for him also. It would then follow that *at most* 5856 voters are sophisticated, and so we could conclude that Australian voters on the whole are a pretty unsophisticated bunch.

Now let's change the statement to something closer to what McGuinness actually said:

> "Anyone who votes for me will be sophisticated."

This statement is the logical *converse* of the earlier statement. While it now rules out the possibility that some of the people voting for McGuinness are unsophisticated, it also allows that some sophisticated voters might *not* vote for him. In this case, McGuinness's low level of voter support can't be used as a basis for arguing that Australians are not sophisticated; maybe they are, it's just that many of them chose to vote for someone else.

Now, what is the effect of replacing "sophisticated" with "more sophisticated than the average voter"? Isn't the latter expression just a more precise way of saying "sophisticated"? It might appear to be so, but the situation is actually rather more subtle. To avoid confusing the issue with the logical fallacy which we have already analysed, we'll suppose that McGuinness made the following statement:

> "Anyone who is more sophisticated than the average voter will vote for me."

We know that only a small number of voters voted for McGuinness, so not many voters are more sophisticated than the average voter. This says nothing about the overall level of sophistication of voters, but it does say something rather unusual about the way sophistication is distributed in the population. However "sophistication" might be measured, we would probably expect that roughly half the population would have above-average sophistication and half would have below-average sophistication; this would be the case if the distribution is symmetric. If much less than half the population is above the average, then the distribution of sophistication is highly *skewed*: a bar graph showing the number of people at each level of sophistication would have a long "tail" extending to the right. It's unlikely that this is what Gerard Henderson had in mind when he made his comment!

* * * * *

A Mathematical Surprise

A mathematician wanted to surprise his family with a Christmas present. Asked for a hint, he wrote

$$\int \frac{d\,(\text{cabin})}{\text{cabin}}$$

As he explained on Christmas morning,

$$\begin{aligned}\int \frac{d\,(\text{cabin})}{\text{cabin}} &= \log(\text{cabin}) + C \quad (\text{sea})\\ &= \text{cabin cruise}\end{aligned}$$

* * * * *

What is the strength of mathematics? What makes mathematics possible? It is symbolic reasoning. It is like 'canned thought'. You have understood something once. You encode it, and then you go on using it without each time having to think about it. Without symbolic reasoning you cannot make a mathematical argument.

– Lipman Bers

HISTORY OF MATHEMATICS

Proving Infinitely Many Things All at Once

Michael A B Deakin

The word "induction" normally signifies an argument from particular instances to a general rule. It is usually contrasted with *deduction*, which is argument from the general to the particular. Thus for example, if we know the general principle

$$1+2+3+4+\ldots\ldots\ldots+n = \frac{n(n+1)}{2}$$

then we can apply this to (say) the case $n = 100$ and so find that the sum of the first 100 positive integers is 5050. That is deduction.

If however we were to find by experiment that

$$1 = \frac{1\times 2}{2}$$

$$1+2 = \frac{2\times 3}{2}$$

$$1+2+3 = \frac{3\times 4}{2}$$

Etc. up to (say)

$$1+2+3+4+\ldots\ldots\ldots+100 = \frac{100\times 101}{2}$$

we might well be tempted to say that we had discovered a general rule. We would have excellent evidence, and the argument adopted would be an example of induction.

However, such arguments, persuasive as they may be, are not watertight. They can break down. One very nice demonstration of this principle is to take n points randomly chosen around the circumference of a circle and then join each point to all the others. As a general rule (i.e. ignoring certain rare special cases), we find that this will divide the interior of the circle into $N(n)$ distinct regions, where $N(n)$ is given by the following table:

n	1	2	3	4	5
$N(n)$	1	2	4	8	16

so that the general rule $N(n)=2^{n-1}$ is strongly suggested. However, appearances are deceptive. $N(6)=31$, not 32.

[As an exercise, can you determine the correct formula for $N(n)$?]

If we had *only a finite number* of statements to check, then we could (in principle at least) come up with a valid proof that some general statement or other was always true. For example consider the following theorem.

> Take any 3-digit number of which the first digit is greater than the third and the third is greater than zero (eg. 763.) Reverse the order of the digits (to get 367 in the example), and now subtract the two numbers (396). Reverse the order of the digits in this answer (693) and add the last two numbers. However we chose the initial number, the result is always 1089.

A way to do this would be simply to check every possible case. This would be a very inefficient approach for a human, but a computer could be programmed to get through the work quite quickly. Such a proof is referred to as *complete induction*. It gives a full proof that the statement in question is always true.

However such an approach is unavailable if the number of cases is *infinite*. This is the situation with our formula

$$1+2+3+4+\ldots\ldots..+n = \frac{n(n+1)}{2}$$

where there is no limit to the value of n. In such a case we have, for each value of n, a statement about n. Call this statement $S(n)$. So we have:

$$S(1): \quad 1 = \frac{1\times 2}{2}$$

$$S(2): \quad 1+2 = \frac{2\times 3}{2}$$

$$S(3): \quad 1+2+3 = \frac{3\times 4}{2}$$

and so on with the general case being represented by

$$S(n): \qquad 1+2+3+4+\ldots\ldots\ldots+n = \frac{n(n+1)}{2}$$

We cannot hope to prove *each individual case* separately because there are infinitely many such cases, but there is a clever method that allows up to prove the general result with certainty. That is to say we can after all show that the formula *is true in each individual case.*

This technique is called *mathematical induction,* and it reduces the problem to proving two things:

1. For *all values of n*, $S(n) \Rightarrow S(n+1)$
2. $S(1)$ is true.

Then we may argue as follows: In the first of the above conditions, put $n = 1$. Then $S(1) \Rightarrow S(2)$, and since $S(1)$ is true, then $S(2)$ is also true. Now go back to the first condition and put $n = 2$, and repeat the process. This assures us of the truth of $S(3)$, ad so we may proceed as far as we wish. The process, once started, continues for ever.

Let us now see how we might use this approach to prove our formula $S(n)$:

$$1+2+3+4+\ldots\ldots\ldots+n = \frac{n(n+1)}{2}$$

First we write down $S(n+1)$. This is the statement that results if we replace the n in $S(n)$ by $n+1$. So $S(n+1)$ is the statement

$$1+2+3+4+\ldots\ldots\ldots+n+(n+1) = \frac{(n+1)(n+2)}{2}$$

What we now need to do is to show that $S(n) \Rightarrow S(n+1)$, that is to say that we assume $S(n)$ to be true and show that *if this is so,* then $S(n+1)$ must also be true. Let us do this. Start with the left-hand side of $S(n+1)$. We have

$$1+2+3+4+\ldots\ldots\ldots+n+(n+1) = \frac{n(n+1)}{2}+(n+1)$$

because we are assuming $S(n)$ to be true. But now it is very easy to show that the right-hand side just produced is equal to the right-hand side of $S(n+1)$.

We have now shown that the first of our conditions is satisfied. All that now remains is to show that $S(1)$ is true, and this is extremely simple to do. This completes the proof.

This extremely elegant technique of proof has been around for a long time, and various versions are given of its history.

The earliest use of a form of argument based on this logic is probably from the work of Pappus of Alexandria, who lived in the 4th century of our era. Pappus wrote a sequence of books known collectively as *The Collection*. It is pretty much what it says: a collection of various mathematical exercises.[1] In this work there is a discussion of a geometric problem in which a chain of circles is formed inside the space between three mutually tangent semi-circles. See Figure 1, which shows the three given semi-circles all resting on the diameter of the largest one.

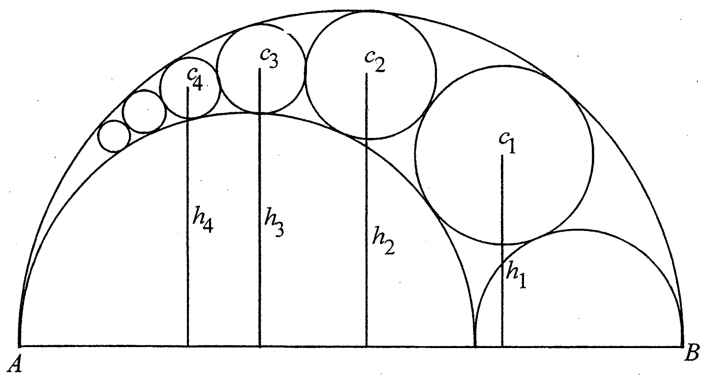


Figure 1

For each of the circles c_1, c_2, c_3, ..., we have a diameter d_1, d_2, d_3, ...and a vertical co-ordinate h_1, h_2, h_3, ... respectively. What Pappus was concerned to show was that $h_n = nd_n$. It has been shown by a Melbourne-based Mathematics

[1] See *Function*, *Vol* 16, *Part* 3, p 87 for more on this work.

teacher (Mr Hussein Tahir) that Pappus employed an argument that is, in essence, one based on mathematical induction.[2]

The idea later appeared in the Islamic world. A mathematician named al-Karaji is next on the scene.[3] He is credited with the discovery of the relation

$$1^3+2^3+3^3+......+n^3=\left(1+2+3+......+n\right)^2$$

In fact he stated explicitly only the case $n = 10$, but the proof he gave is readily extendible to other cases, and it is clear that he meant to show the general rule by means of a typical example. [Can you use the technique of mathematical induction to prove this result?]

al-Karaji's work was extended by an Egyptian contemporary: Abu 'Ali al-Hasan ibn al-Hasan ibn al-Haytham. Ibn al-Haytham gave formulae for the sum of the first n postive integers, for the sum of their squares, for the sum of their cubes and for the sum of their fourth powers. His technique was to work via induction, and it would seem that he was capable of working out the formulae for higher powers than the fourth, although he did not actually do this.

The principle of mathematical induction also appears in the work of the later Jewish mathematician Levi ben Gerson (1288–1344). Levi considered matters related to those that the Islamic mathematicians had already examined, and indeed he may well have known of their work. But he also considered basic questions and used induction to answer these. Thus, for example, he proved that if a, b, c are positive integers, then $a(bc)=b(ac)=c(ab)$. He also generalised this result to four and then to any number of factors.

None of these early mathematicians actually stated the principle in exactly its modern form, although the idea is clearly there. The name most usually associated with the emergence of mathematical induction is that of Francesco Maurolico[4] (1494–1575). While several authors mention him in connection with mathematical induction, none of those I have read say anything at all about what his claims to the technique may be. He was instrumental in making the work of Pappus better

[2] This work apeared in *The Australian Mathematical Society Gazette*, *Vol* 22, *Part* 4 (October 1995), pp 166-167. It formed part of the subject of a letter I wrote to *Function Vol* 20, *Part* 5, pp. 161-162). Figure 1 is taken from Mr Tahir's article.

[3] Little is known of al-Karaji. He worked in Baghdad in the years around 1000 AD and died in 1019.

[4] Also spelt Maurolyco and in various other ways.

known and so he may well have found the same passage that Mr Tahir also found, and it may be that he too recognised its significance.

The first really explicit statement of the principle is to be found in the work of Blaise Pascal (1623-1662). In his *Treatise on the Arithmetical Triangle* (the work in which he introduced what we still know as "Pascal's Triangle"), the principle is fully stated and the generating formula is proved. If we denote by $\binom{n}{k}$ the number of ways we can choose k items from a given set of n items, then this formula may be written

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

The name "mathematical induction" came into English only last century. Mr Tahir mentions a use by the English mathematician Augustus De Morgan in 1838. The *Shorter Oxford Dictionary* dates it to the mid-19th Century, gives a quotation that is probably the De Morgan one, but then also quotes a very clear statement of the principle from the 7th edition of Todhunter's *Algebra* (1875).

What is perhaps the most striking use of the principle occurs in the work of Giuseppe Peano (1858–1932). Peano is now famous for his set of axioms that put the basic properties of the natural numbers (ie. the positive integers and, on some accounts, also zero) onto a firm basis. He could perhaps be seen as continuing the tradition of Levi ben Gerson, but his work is much more developed and systematic.

A modern version of Peano's axioms runs as follows (where I have used the simple term "number" as a shorthand for "natural number"). The fifth is the induction principle.

1. There exists a number 0,
2. Every number n has a successor n',
3. There is no number n such that $0 = n'$,
4. If $m' = n'$, then $m = n$,
5. If a set of numbers contains n' whenever it contains n and if it also contains 0, then it contains all numbers.

These five axioms he then used to investigate all the laws of arithmetic, and to prove them. For example, we *define* the sum of two numbers by stipulating that $n+0=n$ and $n+m'=(n+m)'$. Let us see how this works.

By the first of the equations, we can add 0 to any number. This will merely give us back the original number, n say. 1 is the successor of 0, so the second equation tells us that $n+1$ is the successor of n, that is to say n'. To add 2 (the successor of 1) we form the successor of n'. And so on. Notice how the very *definition* of addition uses the principle of mathematical induction!

Then we have, as our first theorem, $(m+n)+p=m+(n+p)$. To prove this, we first note that it holds for $p=0$. $(m+n)+0=m+n=m+(n+0)$. Now suppose that the rule holds for some value of p. We prove that it must then hold for p'. But

$$(m+n)+p'=((m+n)+p)'=(m+(n+p))'=m+(n+p)'=m+(n+p').$$

[Work through this argument noting how every step follows from one of the axioms or else from the definition of addition.]

In the same way, Peano was able to show the validity of all the basic laws of arithmetic as consequences of his five axioms.

Peano's work is now seen as absolutely fundamental to the understanding of what Mathematics is all about. With that work, the principle of mathematical induction really came into its own.

Further Reading

Apart from the references already given, I have used Victor Katz's *A History of Mathematics* (New York: Harper Collins, 1992), especially for the sections on the Islamic mathematicians, on Levi ben Gerson and on Pascal. There is also valuable material in the *Dictionary of Scientific Biography*. The account of the Peano axioms relies mainly on *The Number-System* by H A Thurston (London: Blackie, 1956).

* * * * *

Mathematics is a world created by the mind of man, and mathematicians are people who devote their lives to what seems to me a wonderful kind of play!

– Constance Reid

COMPUTERS AND COMPUTING

Keeping Ahead of The Machines: Integrals of Runaway Functions

Michael A B Deakin

In an earlier article in *Function*,[1] I told the story of how I beat a computer in a race to solve a mathematical problem. Alas, that feat could not now be repeated. Computers have got better. The version of the *Mathematica* package that laboured in vain for 48 hours in 1990 has long ago been supplanted, and even back then there were computer algebra packages around that could have beaten me.[2]

However, at about this same time, I also had another race with a, not exactly computer, but rather a, for its time, very sophisticated calculator. This race I also won. It occurred to me the other day that in this area as well, perhaps nowadays computers had reached the point where it is quite hopeless to try to compete. But this isn't entirely the case. They have improved, but not to as great a degree as I imagined.

The problem is concerned with the integration of what I call "runaway functions" (a term I apply somewhat loosely to functions that increase extremely rapidly). I first encountered a "runaway function" about ten years ago when, in a technical context, I came across a function known as the "Dawson Integral". I will get to the details of this later, but it depends on the "runaway function" e^{x^2}. The function e^x is the exponential function, and it is almost legendary for its rapid rate of increase. It is now a cliché to speak of something's "increasing exponentially" when what we mean is that it is increasing very rapidly.

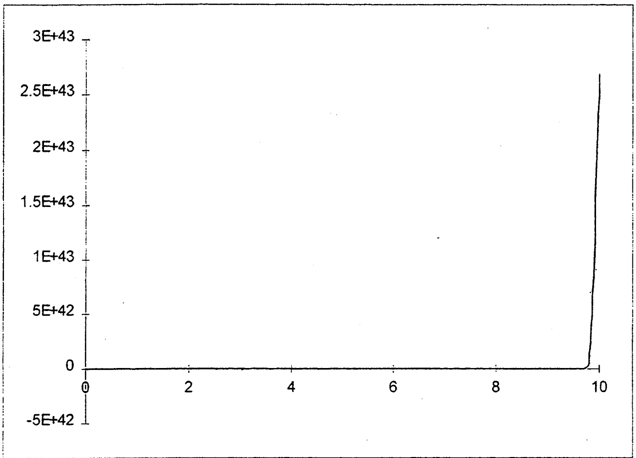
Exponential increase can very easily be explored by means of a simple calculator or a spreadsheet on a computer. I have just pressed a few buttons on my calculator, and these tell me that if I start at $x = 0$ and then increase the value of x, then the exponential function begins with $e^0 = 1$, continues on to $e^1 = 2.718281\ldots$, and so on through to $e^5 = 148.41\ldots$ and later to $e^{10} = 22026.465\ldots$ and beyond. This is rather spectacular increase.

[1] See *Vol* 15, *Part* 1, p 8.

[2] As subsequent correspondence in *Function* showed.

However it pales into insignificance when we compare it with the increase in e^{x^2}. This starts out with the value 1 when $x = 0$, just as above, and continues on to $e^1 = 2.718281\ldots$, again just as above. However when $x = 5$, this function has the value $7.200489934 \times 10^{10}$ (that is to say 11 figures before the decimal point); and when we reach $x = 10$, the value has risen to a massive $2.688117142 \times 10^{43}$, that is to say a number with 44 digits before the decimal point.

Functions like this are often called "superexponential", but I prefer the name "runaway". You can try drawing a graph of such a function. I used the *Excel* spreadsheet to do this and I recommend readers to do this also. If you generate values of e^{x^2} from (say) $x = 0$ to $x = 10$, their graph will be nothing much to look at. In fact, here it is. Notice how, on the scale the computer generates, it looks almost flat until very late in the day, when all of a sudden it hares up almost vertically.



As I say, a rather uninteresting-looking graph, but not without some noteworthy features. Look first at the vertical scale. The first number to register is 5×10^{42}, and if we take *this* as our benchmark, then the graph "doesn't get off the ground" until about $x = 9.7$. Now if we calculate $e^{9.7^2}$, we find 7.29×10^{40}, an enormous number, but still less than 1.5% of our benchmark number. On scales such as this, most of the graph is seen as little more than zero (enormous as the *actual* numbers might be).

Now I said that the question that concerned me was the integration of such "runaway functions". In other words, we are interested in the areas under curves like the one in the graph reproduced above. And now once more, we find that the graph provides an interesting insight. Suppose that we compared the area under the curve with the final value reached by the curve. This final value is also the area of a rectangle of height $2.688117142 \times 10^{43}$ (the final value itself) and standing on a base of 1, which we may take as the sub-interval between $x = 9$ and $x = 10$.

The area under the curve is clearly *less than* the area of this rectangle. That is to say that $\int_0^{10} e^{x^2}\, dx < e^{10^2}$. In fact, it is found that this is a general rule. If we integrate from 0 to X, say, then $\int_0^X e^{x^2}\, dx < e^{X^2}$. H G Dawson, a minor mathematician working 100 years ago, was the first to study the integral of e^{x^2} and so now the function

$$F(X) = e^{-X^2} \int_0^X e^{x^2}\, dx, \qquad (1)$$

is referred to as "Dawson's Integral" in his honour[3] (although he did not in fact consider it). The initial factor, however, is a very rapidly *decreasing* function, which "tames" the second, which is rapidly *increasing*.

Now our example with $X = 10$ has shown us that at that value the Dawson Integral is actually quite small. Indeed, a little experimentation will show us that $F(X)$ actually *decreases* as X increases. In fact, it may be proved that $\lim_{X\to\infty} F(X) = 0$, although the details are somewhat advanced for *Function.* However, once we know this, we have $\lim_{X\to\infty} F'(X) = 0$, and we can use this piece of information to estimate the value of $F(X)$.

[3] The function is sometimes denoted by daw X, but this notation can be confusing so it will not be used here.

If we differentiate $F(X)$, we use the product rule, the chain rule on e^{-X^2}, and the rule $\frac{d}{dX}\int_0^X e^{x^2}\,dx = e^{X^2}$. With only a little work, then, we find that

$$F'(X) = 1 - 2X\,F(X)$$

and as the left-hand side approaches 0 as X gets large, we must have

$$F(X) \approx \frac{1}{2X}.$$

More precise analysis gives the better formula

$$F(X) \approx \frac{1}{2X} + \frac{1}{4X^3} + \frac{3}{8X^5} + \frac{15}{16X^7} + \frac{105}{32X^9} + \ldots .$$

[Can you work out the pattern here?] Putting $X = 10$ into this formula, we find

$$F(10) = 0.0502538,$$

a value that agrees with that in a table I have[4] (although that gives three more figures of accuracy).

But now, knowing this figure, we are able to work out the area of the region under the graph. From (1) $\int_0^X e^{x^2}\,dx = e^{X^2}$, so the area will be $F(10)\,e^{10^2}$, which is to say $F(10)\,e^{100}$. All we now need do is calculate e^{100}. This may also be readily done. We have $e^{100} = 10^{100\log e}$. Now $\log e = 0.434294481$, and so $100\log e = 43.4294481$, from which it follows that

$$10^{100\log e} = 10^{43.4294481} = 10^{43}\times 10^{0.4294481} = 2.688117134\times 10^{43}.$$

Multiplying this now by $F(10)$, we find 1.35088×10^{42} as the value of the area.

If we are content with a somewhat less accurate estimate, we can almost do this calculation in our heads. The value of $\log e$ is quite well-known and I memorised the first four figures of it years ago. This gives as a rough estimate

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[4] M Abramowitz and I Stegun's *Handbook of Mathematical Functions*, p. 319.

$e^{100} \approx 10^{43.43}$. We now need an estimate of $10^{0.43}$, and back when logarithms were taught in much more detail, I learned that $\log 2 = 0.3010$ and $\log 3 = 0.4771$. So 0.43 is the logarithm of a number between 2 and 3, probably somewhat closer to 3 than to 2, about 2.7, let us say.[5] If we now use only the crude approximation $F(10) \approx 0.05 = \frac{1}{20}$ we estimate that the area of the region under the curve is: $\frac{2.7}{20} \times 10^{43} = 1.35 \times 10^{42}$.

This was the challenge that I set my colleague with the super-dooper new calculator (as it then was). I would like to say that I beat the machine, but honesty must prevail. Not only did it beat me, but it got a considerably more accurate answer. However, it then occurred to me that maybe if I made the problem *harder*, then I could beat the machine. So that is exactly what I did.

I suggested that we both integrate an even more runaway function: e^{x^3}. Now it is easy to see, by the same logic that applies to the Dawson Integral, that

$$\int_0^X e^{x^3}\,dx \approx \frac{e^{X^3}}{3X^2}$$

so that if we put $X = 10$, we get $\int_0^{10} e^{x^3}\,dx \approx \frac{1}{300} \times 10^{434.3}$. Now 0.3 is very close to $\log 2$, and since $10^{\log 2} = 2$ this time we have a value of $\frac{2}{300} \times 10^{434} \approx 6.7 \times 10^{431}$. This I did, although I wasn't especially quick. Nevertheless I did beat the machine, because it gave up and delivered an error message!

It occurred to me lately that perhaps modern machines can do better. I asked *Function's* editor, Dr Varsavsky, to try this problem. Next day, she emailed me to say that she had tried two different machines running different software:

> "I tested the integral with Maple 4 last night, running on a Pentium II (the fastest PC currently in the market) and I had the same results as with *Scientific Notebook*: .... the output is given immediately."

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[5] In fact 2.7 is a *very good* estimate, if we reflect that log 2.7=log 27 – log 10 = 3log 3 – 1 = 0.43.

The result that was "given immediately" was $6.571288988 \times 10^{431}$. Not only was this "given immediately", but it has more accuracy than I can achieve with a hand calculator. My best effort using improved approximations (and reached only after an embarrassing number of mistakes and false starts) was 6.57128×10^{431}. So, another win for the computer.

But I was undaunted. "All I need do", I thought, "is make the problem yet harder." So I did. This time I integrated $\int_0^{10} e^{x^4}\, dx$, and asked her to see how the machine went.

To do this one, we need more figures than the four I used for $\log e$ in the earlier crude estimates. My trusty *Casio fx*-570 tells me that $\log e = 0.434294481$, a value I used above, and we need to multiply this by 10^4, i.e. 10000. This gives 4342.94481, and $10^{0.94481} = 8.80682$. So $e^{10^4} = 8.80682 \times 10^{4342}$.

We now need to multiply this by a suitable factor. In the case of the Dawson Integral, this was $\frac{1}{2X} = \frac{1}{20}$ in the simplest (crudest) approximation. With the next case, we had $\frac{1}{3X^2} = \frac{1}{300}$ (again in the crudest approximation available). In our present case, this factor[6] becomes $\frac{1}{4X^3} = \frac{1}{4000}$ so we divide 8.80682 by 4 and adjust the powers of 10 to find

$$\int_0^{10} e^{x^4}\, dx = 2.2017 \times 10^{4339}.$$

A slightly more precise calculation gives 2.20203×10^{4339}. This is of course an enormous number. It has 4340 digits before its decimal point.

And no! I didn't do either of these calculations in my head (although there are some people who could).[7] I used a hand calculator.

[6] There is a relatively simple pattern behind the construction of these factors. Can you see it?

[7] A person who had memorised five digits of log *e* could easily come up with a quite respectable estimate in their head. 0.9 is about 3log 2 = log 8, and so we would expect to divide 8 by 4 and then adjust the powers of ten. With more digits memorised, we could do even better.

But nonetheless, I beat the machine. Here is Dr Varsavsky's report:

> "[In this case] the computer keeps calculating non-stop – I left it working while having dinner but *Maple* could not succeed in producing a result (neither an error/warning message)."

So, for the moment, we humans are ahead. Of course, if you wanted *really* accurate results, then you could tell the computer how to do the human thing, and once you did this, then it would always win after that! It would be both faster and more accurate.

I close by setting a couple of problems for the reader.

Problems

What is the value of $\int_0^{10} e^{x^5}\,dx$? What about $\int_0^{10} e^{x^6}\,dx$?

Can you generalise these results to $\int_0^{X} e^{x^n}\,dx$?

* * * * *

> Why do we do mathematics? We mainly do mathematics because we *enjoy* doing mathematics. But in a deeper sense, why should we be paid to do mathematics? If one asks for the justification of that, then I think one has to take the view that mathematics is part of the general scientific culture. We are contributing to a whole, organic collection of ideas, even if the part of mathematics which I am doing now is not of direct relevance and usefulness to other people. If mathematics is an integrated body of thought, and every part is potentially useful to every other part, then we are all contributing to a common objective.
>
> – Michael Atiyah

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 22.1.1 (A Begay, Lupton, Arizona, USA)

Let $S(n)$ be the smallest positive integer such that $S(n)!$ is divisible by n, where $m!$ denotes $1 \times 2 \times 3 \times \ldots \times m$ (the factorial function).

(a) Prove that if p is prime then $S(p) = p$.

(b) Calculate $S(42)$.

(The function $S(n)$ is called the *Smarandache function.*)

SOLUTION

(a) Let p be prime. Then $p!$ is divisible by p, but none of $1!, 2!, \ldots, (p-1)!$ are, because p does not divide $1, 2, \ldots, p-1$. Hence $S(p) = p$.

(b) $S(42)!$ is divisible by 42, so it must be divisible by the prime divisors of 42, namely 2, 3 and 7. The smallest factorial divisible by 2, 3 and 7 is clearly 7!, and, since 2, 3 and 7 are distinct primes, any number divisible by all of them must be divisible by their product. Hence $S(42) = 7$.

Solutions were received from Carlos Alberto da Silva Victor (Nilópolis, Brazil), Keith Anker (Glen Waverley, Vic), and the proposer.

PROBLEM 22.1.2 (Julius Guest, East Bentleigh, Vic)

An equilateral triangle ABC is inscribed in a circle. Let D be any point on the minor arc subtended by $\overline{AB}$. Prove that $DC = DA + DB$.

SOLUTION

The solution to this problem actually appeared in the previous issue of *Function*! It is one of the special cases of Ptolemy's theorem discussed by Ken Evans in his article "Ptolemy and his Theorem" (*Function Vol* 22 *Part* 2). Here is another proof, which doesn't use Ptolemy's theorem.

First, note that the angles ABC and ADC are equal, since they subtend the same arc, AC. Hence $ADC = 60°$. Let E be the point on $\overline{CD}$ such that $DA = DE$; then triangle ADE is equilateral (Figure 1). Now, $AB = AC$ (since ABC is equilateral), and $AD = AE$ (since ADE is equilateral). Let angle BAD have measure α. Then $BAE = 60° - \alpha$ (since ADE is equilateral), so $CAE = \alpha$ (since ABC is equilateral). Therefore the angles BAD and CAE are equal, so the triangles ABD and ACE are congruent (by the "two sides and the included angle" rule). Thus $EC = DB$, so $DC = DE + EC = DA + DB$.

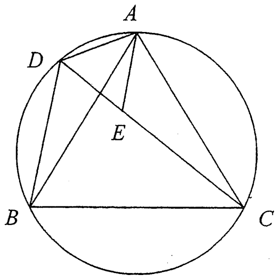

Figure 1

Solutions were received from Carlos Victor, Claudio Arconcher (São Paulo, Brazil), and Keith Anker.

PROBLEM 22.1.3

Let ABC be an equilateral triangle, and let P be a point inside ABC such that $AP = 3$, $BP = 4$, and $CP = 5$. Prove that the angle APB has radian measure $5\pi/6$.

SOLUTION

Construct an equilateral triangle APD on $\overline{AP}$, as shown in Figure 2. Construct the interval $\overline{BD}$. Then, since APD is equilateral, $\angle DAP = \pi/3$, so $\angle BAD = \pi/3 - \angle BAP$. Since the triangle ABC is equilateral, $\angle BAC = \pi/3$, so $\angle CAP = \pi/3 - \angle BAP$. Therefore $\angle BAD = \angle CAP$. We also know that $AD = AP$ and $AB = AC$, so the triangles ABD and ACP are congruent ("two sides and the included angle"). Hence $BD = CP = 5$. Since $PD = 3$ (because APD is equilateral), and $BP = 4$, it follows from the converse of Pythagoras's theorem that $\angle BPD = \pi/2$. But $\angle APD = \pi/3$ because APD is equilateral, so $\angle APB = 5\pi/6$.

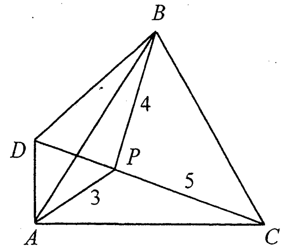

Figure 2

Solutions were received from Julius Guest, Carlos Victor, and Keith Anker.

PROBLEM 22.1.4 (based on a problem in *Mathematics and Informatics Quarterly*)

Let *COFFEE* and *BREAK* be two numbers written in decimal notation, where each letter stands for a digit. Find the numbers, given that $\sqrt{COFFEE}$ and $\sqrt[4]{BREAK}$ are integers.

SOLUTION (based on the solutions by Carlos Victor and Keith Anker)

It is usual in problems of this type to assume that different letters never stand for the same digit, and we will do so here. Let $\sqrt{COFFEE} = n$ and $\sqrt[4]{BREAK} = p$. Then $10^5 \le n^2 < 10^6$ and $10^4 \le p^4 < 10^5$, so $100\sqrt{10} \le n < 10^3$, which yields $317 \le n \le 999$, and $10 \le p < 10\sqrt[4]{10}$, yielding $10 \le p \le 17$. The possible values of *BREAK* are the fourth powers of the numbers from 10 to 17: 10000, 14641, 20736, 28561, 38416, 50625, 65536, and 83521. Ignoring numbers containing repeated digits, we are left with just four numbers to check. We deduce that the possible values for E are 4, 5 and 7; however, since *COFFEE* is a perfect square, it cannot end in 7. Moreover, no perfect square can end in 55, since perfect squares ending in 5 always end in 25, as may be readily shown. Therefore $E = 4$, and $BREAK = 38416$. The trailing 44 in *COFFEE* requires that n ends in 12, 38, 62 or 88, which means we now only need to consider the values of n between 317 and 999 that satisfy this extra condition. On checking these values, we find that only 838 works, giving $COFFEE = 702244$.

By not assuming that different letters necessarily represent different digits,Carlos Victor produced several other answers: *COFFEE* = 473344 with *BREAK* = 38416, and *BREAK* = 10000 with a number of possibilities for *COFFEE*. Julius Guest also submitted a solution.

PROBLEM 22.1.5 (from *Parabola*, University of New South Wales)

Let x be a real number with $x \neq \pm 1$. Simplify

$$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \dots + \frac{2^n}{x^{2^n}+1}.$$

SOLUTION (from *Parabola*)

Observe that $-\frac{1}{y-1} + \frac{1}{y+1} = \frac{-2}{y^2-1}$ for any $y \neq \pm 1$. Adding and subtracting $\frac{1}{x-1}$ in the given expression, we have

$$\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \dots + \frac{2^n}{x^{2^n}+1}$$

$$= \frac{1}{x-1} - \frac{1}{x-1} + \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \dots + \frac{2^n}{x^{2^n}+1}$$

$$= \frac{1}{x-1} - \frac{2}{x^2-1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \dots + \frac{2^n}{x^{2^n}+1}$$

$$= \frac{1}{x-1} - \frac{4}{x^4-1} + \frac{4}{x^4+1} + \frac{8}{x^8+1} + \dots + \frac{2^n}{x^{2^n}+1}$$

$$= \frac{1}{x-1} - \frac{8}{x^8-1} + \frac{8}{x^8+1} + \dots + \frac{2^n}{x^{2^n}+1}$$

$$= \dots$$

$$= \frac{1}{x-1} - \frac{2^{n+1}}{x^{2^{n+1}}-1}$$

Solutions were received from Carlos Victor, Julius Guest, and Keith Anker.

PROBLEM 22.1.6

Prove that $\int_0^{\pi/2} \frac{1}{1+(\tan x)^k}\, dx = \frac{\pi}{4}$ for any real value of k.

SOLUTION

$$\begin{aligned}
I &= \int_0^{\pi/2} \frac{1}{1+(\tan x)^k}\, dx \\
&= \int_0^{\pi/2} \frac{1}{1+(1/\cot x)^k}\, dx \\
&= \int_0^{\pi/2} \frac{(\cot x)^k}{(\cot x)^k+1}\, dx \\
&= \int_0^{\pi/2} \frac{[\tan(\pi/2-x)]^k}{[\tan(\pi/2-x)]^k+1}\, dx \\
&= \int_{\pi/2}^{0} \frac{(\tan u)^k}{(\tan u)^k+1}(-du) \quad (u = \pi/2-x) \\
&= \int_0^{\pi/2} \frac{(\tan u)^k}{(\tan u)^k+1}\, du \\
&= \int_0^{\pi/2} \left(1-\frac{1}{(\tan u)^k+1}\right) du \\
&= \frac{\pi}{2} - I
\end{aligned}$$

Therefore $2I = \frac{\pi}{2}$, so $I = \frac{\pi}{4}$.

Note that I is an improper integral, because $\tan x$ is not defined when $x = \pi/2$, and $(\tan x)^k$ is not defined for negative values of k when $x = 0$. A rigorous proof would proceed along the lines we have shown, but some of the steps would need to be justified carefully.

Solutions to this problem were received from Carlos Victor, Julius Guest, Keith Anker, Claudio Arconcher, and John Barton (Carlton North, Vic).

More on an earlier problem

K R S Sastry (Bangalore, India) has written to us about Problem 21.4.2, a solution of which appeared in the February 1998 issue. The problem was to find all three quadratic polynomials $p(x) = x^2 + ax + b$ such that a and b are roots of the equation $p(x) = 0$. Mr Sastry has considered a generalisation of a slightly different problem: determine all polynomials $P_n(x) = x^n + a_1x^{n-1} + \dots + a_n$ where the a_i are integers and where $P_n(x)$ has as its n zeros precisely the numbers $a_1, a_2, \dots, a_n$ (counted in their multiplicities). (Our problem was not quite the same as this problem with $n = 2$, because we allowed a and b to be the same root.) Mr Sastry's solution is:

(i) $P_n(x) = x^n, n = 1, 2, 3, \dots$

(ii) $P_2(x) = x^2 + x - 2 = (x-1)(x+2)$

(iii) $P_3(x) = x^3 + x^2 - x - 1 = (x-1)(x+1)^2$

(iv) $P_n(x) = x^{n-2}P_2(x),\ n \geq 2$

(v) $P_n(x) = x^{n-3}P_3(x),\ n \geq 3$

This solution was published in the September 1994 issue of *Crux Mathematicorum*. The problem had appeared in an earlier issue of that journal, and had also been used in the 18th Austrian Mathematical Olympiad.

PROBLEMS

Readers are invited to send in solutions (complete or partial) to any or all of these problems. All solutions received by 10 August 1998 will be acknowledged in the October issue, and the best solutions will be published.

PROBLEM 22.3.1 (Julius Guest, East Bentleigh, Vic)

Let $ABCD$ be a square, and let P be a point inside it such that $AP = 40$, $BP = 30$, and $CP = 50$. Find the side length of the square.

PROBLEM 22.3.2 (K R S Sastry, Bangalore, India)

Prove that, in any triangle, the inradius equals one third of an altitude if and only if the side lengths of the triangle are in arithmetic progression.

PROBLEM 22.3.3 (from *Mathematical Spectrum*)

Solve the simultaneous equations

$$ax + by = 2 \quad (1)$$

$$ax^2 + by^2 = 20 \quad (2)$$

$$ax^3 + by^3 = 56 \quad (3)$$

$$ax^4 + by^4 = 272 \quad (4)$$

PROBLEM 22.3.4 (from *Alpha*)

Let A, B, C, D be four points in three-dimensional space that do not all lie in the same plane. Let the midpoint of $\overline{AB}$ be M, and let the midpoint of $\overline{CD}$ be N. Prove that $\frac{1}{2}(AD + BC) > MN$.

PROBLEM 22.3.5 (from *Alpha*)

Prove, for any natural number n, that the inequalities

$$2\sqrt{n+1} - 2\sqrt{n} < \frac{1}{\sqrt{n}} < 2\sqrt{n} - 2\sqrt{n-1}$$

are satisfied.

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