Function

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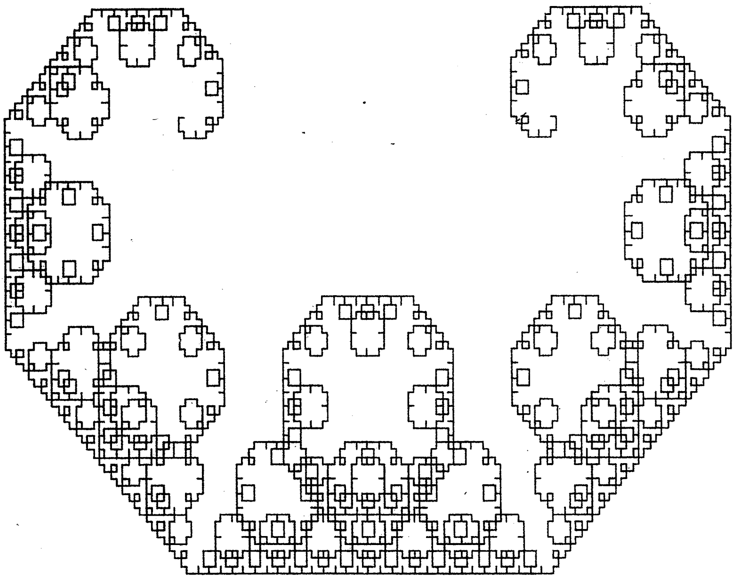

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FUNCTION is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof G B Preston. FUNCTION is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

FUNCTION deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of FUNCTION include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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EDITORIAL

Welcome again to our readers! Here is a summary of what you'll find in this issue of *Function*.

This time the front cover demonstrates that mathematics is also a form of art. The curve shown there is a product of the imagination of the French mathematician Paul Lévy, one of the first scientists to investigate the figures that we now know as fractals.

We have two feature articles in this issue. You will need to stand in front of a mirror as you read the first article: in it, Paul Grossman examines the question: "Why does a mirror interchange left with right, but not top with bottom?" In the second feature article, Ian Collings analyses an interesting number problem that was suggested to him by a non-mathematician.

In the History of Mathematics section, Michael Deakin relates the origin of the imaginary number i. In his usual lively style he tells us that although a widespread belief is that complex numbers were "invented" to solve certain quadratic equations, in fact they owe their origin to the process of solving cubic equations.

Another piece of evidence that mathematics is also an art form can be found in the Computers and Computing section. This time pretty graphs such as roses and butterflies are generated using a computer program based on equations expressed in polar coordinates.

We also include news of a recent advance in the problem of filling space with solid figures, and the usual problem section with solutions to previous ones and a few new problems to challenge your mind.

Happy reading!

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THE FRONT COVER
Lévy's Curve

Cristina Varsavsky, Monash University

The curve shown on the front cover is a product of the imagination of the French mathematician Paul Lévy (1886-1971), one of the first scientists to investigate the figures that were later called fractals. It belongs to a family of fractal curves which includes the better-known *Koch Snowflake*.

The curve is constructed recursively starting from a line segment (Figure 1(a)). This is replaced by two segments which form a right isosceles triangle with the old line segment as shown in Figure 1(b). This process is then repeated on each of the two new line segments, giving 1(c); then again on each of the four segments, 1(d); then on each of the eight segments, 1(e); and so on. The first loop appears after the fifth iteration, 1(f), and becomes more pronounced from then on.

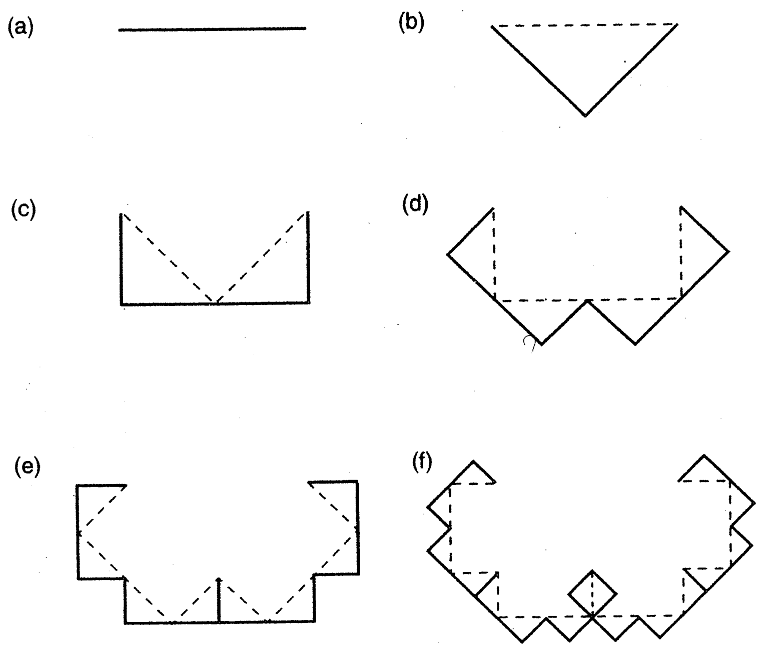

Figure 1

The curve on the front cover is the twelfth stage in the process described above. Lévy's curve is obtained as the limit of this process as the number of iterations tends to infinity. The limiting curve cannot be drawn, but the twelve iterations help us to picture it in our minds.

It is a simple exercise to work out the length of Lévy's curve. Let l be the length of the segment we start with. Using the familiar Pythagoras's Theorem we see that the length of each segment produced after the first iteration is $l \times \sqrt{\frac{1}{2}}$, giving a total length of $l \times 2 \times \sqrt{\frac{1}{2}} = l\sqrt{2}$. At the following stage, the length of each segment is $l \times \sqrt{\frac{1}{4}}$ and the length of the curve is $l \times 4 \times \sqrt{\frac{1}{4}} = l\sqrt{2^2}$. In general, after n iterations we will have 2^n line segments forming a curve of length $l\sqrt{2^n}$. It is obvious then that Lévy's curve is infinitely long.

Lévy's curve – or rather any stage in the process of obtaining it – can also be constructed starting from one of the ends. Let us see this with the curve obtained after four iterations, in which we number the sixteen segments starting from the left end as in Figure 2.

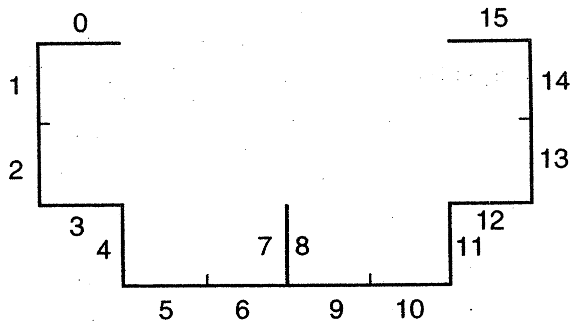

Figure 2

The wiggly curve is constructed with sixteen segments, all of the same length, drawn one after another. After each segment is drawn, there are only four possible directions – left, right, up and down – in which the following one may be drawn. The rule governing the choice of direction is related to the binary expression of the position of the segment (0 to 15) within the curve. We can express this in algorithmic form as follows:

1. Express in binary notation the position of the segment with respect to the starting point.
2. Count the numbers of 1's, p, in the binary expansion found in step 1.
3. Calculate the remainder, r, after division of p by 4.
4. If $r = 0$ move left. If $r = 1$ move down. If $r = 2$ move right. If $r = 3$ move up.

The following table shows the directions for the 16 segments in the fourth iteration:

n	Binary	p	r	Direction
0	0	0	0	left
1	1	1	1	down
2	10	1	1	down
3	11	2	2	right
4	100	1	1	down
5	101	2	2	right
6	110	2	2	right
7	111	3	3	up
8	1000	1	1	down
9	1001	2	2	right
10	1010	2	2	right
11	1011	3	3	up
12	1100	2	2	right
13	1101	3	3	up
14	1110	3	3	up
15	1111	4	0	left

The curve on the front cover, which consists of 2^{12} segments (some of them overlapping), was generated by a computer program based on this algorithm.

* * * * *

MIRROR IMAGES

Paul U A Grossman

Someone once asked me: "Why does a mirror interchange left with right, but not top with bottom? There is surely no reason why it should affect the image differently in the horizontal and vertical directions." We soon established that the questioner had a specific case in mind, namely a plane mirror mounted vertically in which he was looking at himself. Is there a paradox? Let us examine the effects produced by a plane mirror.

When a ray of light strikes a reflective surface, it continues its travel in the plane that contains its incoming path and is normal to the surface. The angle of reflection equals the angle of incidence. To the eye of an observer, these reflected rays appear to come from a source behind the mirror.

Let us now show that the back projections of all rays emanating from one point and reflected from a plane mirror intersect at one point behind the mirror, giving a "virtual" image. Let us also determine the position of this image. We shall introduce a cartesian coordinate system with its origin in the mirror surface, x and y axes in the plane of the mirror and the z-axis normal to it. Figure 1 shows the light source at point A with coordinates (x_A, y_A, z_A) and two rays emanating from it towards the mirror, one in the z-direction, the other remaining in the plane with equation $x = x_A$ and striking the mirror at angle φ to the normal.

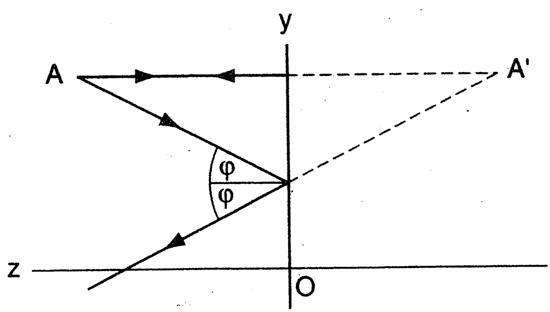

Figure 1

The projected extensions of the reflected rays (drawn by broken lines) intersect at A'. The x- and y-coordinates of A' must be the same as those of A, since both points lie on the same normal to the xy-plane. The two triangles formed by the points of impact and respectively A and A' are congruent, having a common side and two equal angles; hence $z_{A'} = -z_A$. The same arguments hold whatever the value of the angle φ, therefore all rays reaching the mirror from A in the plane with equation $x = x_A$ are reflected such that their back projections pass through A'.

What about rays leaving A outside the plane with equation $x = x_A$? Since we are free to choose the x and y directions arbitrarily, the above result must hold universally, thus

$$x_{A'} = x_A, \quad y_{A'} = y_A, \quad z_{A'} = -z_A$$

for all rays from point A that reach the mirror. Similar relations must hold for rays from any other point with positive z.

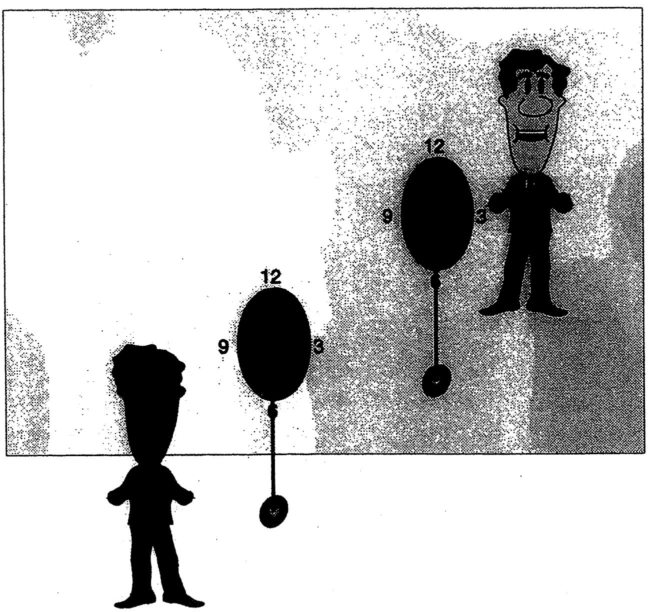

Figure 2

Thus my questioner looking into a vertical plane mirror facing, say, north would see top and bottom unchanged in the image, east and west unchanged, but his nose would seem to point north instead of south. Should you wish to experience the effect of the reversal in the z-direction, try to thread a needle while looking at it in a mirror.

In a horizontal mirror, whether it be the surface of a lake or a glass mirror on a ceiling, the z-axis is vertical and therefore up and down are interchanged.

So far we have checked only the position of the image in relation to that of the object. Let us now investigate the orientation of both object and image in relation to the observer. We shall limit the analysis to an object parallel to the mirror, i.e. $z = z_A$ for any point on the object. Let the object be a thin circular plate. The eye of the observer may be in one of three regions:

(a) further away from the mirror ($z_{obs} > z_A$),
(b) closer to the mirror ($z_{obs} < z_A$), or
(c) at the same distance as the object ($z_{obs} = z_A$).

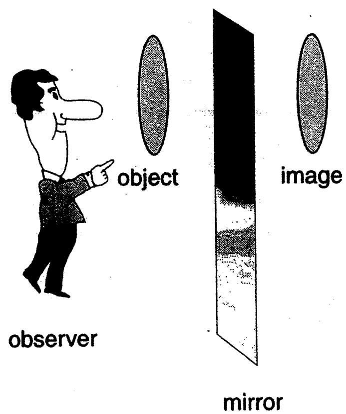

Figure 3

(a) The observer further from the mirror, as depicted in Figure 2 and shown diagrammatically in Figure 3, sees both the object and its image while looking (essentially) in the negative z-direction. If marks on the circumference of the plate are numbered clockwise, they will appear clockwise in the image, too. The only difference, apart from the size, is that the observer sees on the image the reverse side of the plate because of the reversal in the z-direction. If the plate is transparent, then any writing on it is equally readable on the image.

(b) An observer who is closer to the mirror than the object is (Figure 4) cannot see the object and its image without turning. He or she looks z-wards at the object and $(-z)$-wards at the image. Marks on the circumference, numbered clockwise on the plate, appear anticlockwise on the image; letters face the opposite direction.

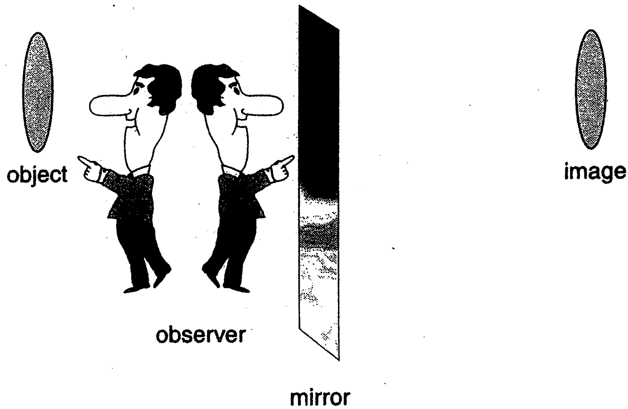

Figure 4

The observer turning to look from object to image may rotate about any axis but we are in the habit of turning while standing upright, i.e. turning about a vertical axis. In the process we carry with us what we know as the right and the left sides of our bodies. Unlike east, west, etc., left and right do not denote an independent direction in space but are subjective. The change from clockwise to anticlockwise direction is readily perceived as an interchange between left and right. If, in order to see the image, we stood on our heads (which is a legitimate way of rotating our bodies) and we kept calling the direction of our feet "down" and that of our heads "up", then we would argue that a mirror turns top to bottom but retains left and right.

(c) An observer at the same distance as the object cannot see the object, only its image. This happens when we look at our eyes in the mirror and this is how my questioner first thought there was a paradox. While we cannot see our eyes, only their image, we know which is left and which is right and we imagine what they would look like to someone facing us. Thus we place ourselves into the shoes of an observer in position (b) listed above who looks in the positive z-direction to see us but would have to rotate to see the image.

In summary, we have seen that in the directions parallel to the face of the mirror the image is unchanged from the object. This confirms what my questioner tried to say in his second statement, although he did not express it precisely. The fallacy in his first question arises from the fact that left and right, unlike top and bottom, do not define a direction in space, but depend on the orientation of the observer.

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Paul U A Grossman is a physicist who has retired from the position of Principal Research Scientist with the CSIRO. His main research interest was rheology (the study of deformation and flow of materials). His hobbies include music and he is also active in Amnesty International.

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But was it Really Cheating?

The Cambridge mathematician John Littlewood (1885-1977) recalled how he had once "cheated" in an exam. After unsuccessfully attempting to answer one of the questions, he left his seat to get some more paper. On the way, he noticed that another student had placed a mark beside that question to indicate that he had done it. Knowing that this particular student was one of the less capable ones, Littlewood inferred that the question must be amenable to a simple solution. On trying it again, this time specifically looking for a simple way of doing it, he was able to solve it. "The perfectly highminded man would no doubt have abstained from further attack", he wrote later. "I wish I had done so, but the offence does not lie very heavily on my conscience".

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DIFFERENCES OF SQUARES

Ian Collings, Deakin University

Some time ago I received a letter from an "ordinary Australian" who obviously enjoyed playing with numbers. He had noted that every natural number could be expressed as the difference between two squared rational numbers, where the smaller number was between 0 and 1.

For example, the number 5 could be written as

$$5 = \left(2\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2$$

Other examples he presented were $63 = 8^2 - 1^2$ and

$$100,895,598,169 = \left(317,640\frac{428,569}{635,280}\right)^2 - \left(\frac{428,569}{635,280}\right)^2$$

I thought his results were interesting, so I decided to explore whether this holds for any natural number. The question is: given a natural number N, is it possible to find x and y such that

$$N = (x+y)^2 - y^2$$

with x a natural number, and y a fraction, $0 \leq y \leq 1$?

Perhaps it is simpler to have N expressed as

$$N = x(x+2y) \tag{1}$$

which of course follows from the previous expression after expanding the square and simplifying.

Three simple cases follow after a quick inspection of the equation (1):

1. $N = x^2$ for some natural number x. In this case $y = 0$.
2. $N = x(x+1)$ for x a natural number, for example $56 = 7 \times 8$. In this case $y = \frac{1}{2}$ and therefore
$$N = \left(x+\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$
3. $N = x(x+2)$ for x a natural number, for example $63 = 7 \times 9$. Here $y = 1$ and then
$$N = (x+1)^2 - 1^2$$

3. $N = x(x+2)$ for x a natural number, for example $63 = 7 \times 9$. Here $y = 1$ and then

$$N = (x+1)^2 - 1^2$$

For a general value of N, we could proceed as follows.

Let x be the largest natural number such that $x^2 \leq N$. We write

$$\begin{aligned} N &= x^2 + (N - x^2) \\ &= x\left(x + 2\frac{N - x^2}{2x}\right) \end{aligned} \qquad (2)$$

which is in the same form as (1) with $y = \frac{N-x^2}{2x}$. The number y is evidently a non-negative rational number. But is it less than or equal to 1?

Let's see. Because of the way x has been chosen, we have

$$N < (x+1)^2$$

therefore

$$N < x^2 + 2x + 1$$

and because N and x are natural numbers, we have

$$N \leq x^2 + 2x$$

Then it follows that

$$\frac{N - x^2}{2x} \leq 1.$$

If we take, for example, $N = 513$, we have $22^2 = 484 \leq 513$ and $23^2 = 529 > 513$. Then $x = 22$ and $y = \frac{513-22^2}{2\times 22} = \frac{29}{44}$, which gives us

$$513 = \left(22\frac{29}{44}\right)^2 - \left(\frac{29}{44}\right)^2.$$

A related question you might consider is whether it is possible to write any natural number as the difference of the *cubes* of two rational numbers where the smaller number lies between 0 and 1.

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HISTORY OF MATHEMATICS

Laputa or Tlön?[1]

Michael A B Deakin

My title gives two possible answers to the question: "What kind of dream-world do you mathematicians inhabit?" We sometimes find ourselves relegated to "Cloud Nine", "off the planet", etc. Some of the public – even the influential public – see Pure Mathematics as consisting of airy-fairy flights of imagination indulged in by a few rare nuts of a freakish turn of mind. Such a view places us in *Laputa*. Some of you may know of it. After Gulliver had left the mini-micro world of Lilliput, and had done with the super-dooper extra Texans of Brobdignag, he visited several places, among them a land of airborne floating islands, peopled by impossibly impractical researchers. This was *Laputa.*

This land does not exist, being a product of Jonathon Swift's embittered and satirical mind. Nonetheless, it is very real, for it stands as a symbol of impractical and dilettantish research. *Laputa* lives on as an image of the most convolutedly abstract, narrowly academic, deliberately useless thinking that mankind can produce.

Mathematicians do not, outside the realms of fiction, inhabit *Laputa.*

We live, in fact, in *Tlön* – a world at once much more dreamy and abstract, much more here and now, and much less known to the average reader of this article. *Tlön* is a fictitious land, the invention of the *heresiarchs of Uqbar*, a country which also does not exist. *Uqbar* was the result of a conspiracy by a secret society known as *Orbis Tertius* – itself a fiction invented by the Argentine writer Jorge Luis Borges who, you will by now be pleased to hear, was a real person.

Tlön is thus a fiction cubed, and yet it is our reality, for it is Borges's symbol of the way in which intellectual frameworks affect our perception of the world. (I shall not, in an essentially mathematical article, enter the controversy over the metaphysics or the theology of *Tlön.*) Mathematics in its development affects and is affected by the intellectual framework we inherit.

[1]Dr Deakin is currently overseas. We reprint here an edited text of a schools' lecture he gave in 1979, and which also appeared in *Function*, *Vol.* 3, *Part* 5.

Had Swift been told in 1726 that mathematicians were investigating the square root of -1, he would undoubtedly have relegated them to *Laputa*.

In point of mathematical nicety, as subtraction is a simpler operation than division, negative numbers, arising from the former, logically precede fractions, which owe their genesis to the latter.

None of which, of course, reflects the actual sequence of historical acceptance. The Greeks of Pythagoras's time had used not only fractions, but irrational numbers. Yet 23 centuries later, Euler regarded negative numbers as "imaginary quantities". The point is that I can visualise (for example) $\frac{3}{4}$ of an apple, without undue mental strain. To imagine -1 apple, however, involves my envisioning some sort of "hole" in the fabric of the universe.

In discussing the square roots of such quantities, we enter a world of unreality that daunted our mathematical forebears. To this day, we speak of "real numbers" and "imaginary numbers" – the latter being those that, when squared, give rise to negative (nowadays respectably real) numbers. A "complex number" is the sum of a real and an imaginary number.

Now you may have been told, or you may have imagined for yourself, that complex numbers were invented because of some aesthetic need for completeness. The equation $x^2 - 1 = 0$ has two solutions, while $x^2 + 1 = 0$ has none. We are being unfair to the second equation.

Life was never so simple. The point is rather different, insofar as historians have been able to piece it together – and this is a difficult matter.

Consider the quadratic equation

$$ax^2 + bx + c = 0.$$

You and I know that we can solve this, if all else fails, by using the formula

$$x = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac}),$$

if $b^2 \geq 4ac$.

This much was known to the Greeks of antiquity. It was the genius of Renaissance Italy to solve the next problem in line – to wit

$$ax^3 + bx^2 + cx + d = 0.$$

It is possible simply to write down a formula (a very messy one) for the roots of this equation. But no understanding lies that way. Let us first simplify the problem. Observe, to begin with, that we can divide through by a (unless, of course, $a = 0$, a trivial case).

This gives, with change of notation,

$$x^3 + Ax^2 + Bx + C = 0.$$

This form of the equation may be simplified yet further.[2]

I will deal with two specific examples:

$$x^3 - 3x^2 + 6x - 8 = 0 \tag{1}$$

$$x^3 + 6x^2 - 9x - 14 = 0. \tag{2}$$

To simplify equation (1), put $x = y + h$. This yields

$$y^3 + (3h - 3)y^2 + (3h^2 - 6h + 6)y + (h^3 - 3h^2 + 6h - 8) = 0.$$

Now choose $h = 1$, to make the coefficient of y^2 equal to zero:

$$y^3 + 3y - 4 = 0. \tag{3}$$

A similar process applied to equation (2) gives (with $h = -2$)

$$y^3 - 21y + 20 = 0. \tag{4}$$

You may check that equation (3) has the single root $y = 1$ (i.e. $x = 2$), and equation (4) has three roots $y = -5, 1, 4$ (i.e. $x = -7, -1, 2$).

We may similarly reduce all cubic equations to the standard form

$$y^3 + 3Hy + G = 0. \tag{5}$$

The solution of equation (5) is nowadays widely attributed to the Italian mathematician Tartaglia (1500?-1577), although there is some dispute about this among historians of Mathematics.

The key insight now is the observation that

$$y^3 - 3pqy + (p^3 + q^3) = (y + p + q)(y^2 - [p + q]y + [p^2 + q^2 - pq]). \tag{6}$$

(You can check this factorisation by expanding the brackets.) From (6) it follows that the equation

$$y^3 - 3pqy + (p^3 + q^3) = 0 \tag{7}$$

[2]These processes were discussed in *Function*, *Vol.* 17, *Part* 2.

may be solved. One root is $y = -(p+q)$. The others (if present) may be found by setting the quadratic factor equal to zero.

Now compare equations (5) and (7). Equation (5) may be solved if we can determine p and q to satisfy

$$pq = -H, \quad p^3 + q^3 = G,$$

or

$$p^3q^3 = -H^3, \quad p^3 + q^3 = G.$$

Thus p^3 and q^3 are the roots of a quadratic equation

$$t^2 - Gt - H^3 = 0.$$

(This holds because $(t-p^3)(t-q^3) = t^2 - (p^3+q^3)t + p^3q^3$.)

It follows that

$$p = \left[\frac{1}{2}(G + \sqrt{G^2 + 4H^3})\right]^{1/3}$$

$$q = \left[\frac{1}{2}(G - \sqrt{G^2 + 4H^3})\right]^{1/3}$$

Applying this procedure to equation (3), for which $H = 1$, $G = -4$, we find, after some work,

$$p = (\sqrt{5} - 2)^{1/3}, \quad q = -(\sqrt{5} + 2)^{1/3}.$$

We may now check that $p = \frac{1}{2}(\sqrt{5} - 1)$, $q = -\frac{1}{2}(\sqrt{5}+1)$. As the root y is $-(p+q)$, we find

$$y = -\frac{1}{2}(\sqrt{5} - 1) + \frac{1}{2}(\sqrt{5} + 1) = 1,$$

as expected.

Turn now to equation (4). In this instance $H = -7$, $G = 20$, so that we reach

$$p = (10 + 9\sqrt{-3})^{1/3}$$

$$q = (10 - 9\sqrt{-3})^{1/3}.$$

We can proceed no further, as our formulae involve the dreaded square roots of negative numbers. (This is all the more maddening, as we know that three perfectly respectable real roots are there waiting for us in the wings.)

The decisive step seems to have been taken by another Italian mathematician, Bombelli (1526-1572). Writing i for $\sqrt{-1}$, without thought as to whether or not i exists, we may discover

$$p = -\frac{1}{2} + \frac{3\sqrt{3}}{2}i$$

$$q = -\frac{1}{2} - \frac{3\sqrt{3}}{2}i,$$

results which may be checked by cubing and writing -1 in place of i^2, wherever it occurs.

Now we put $y = -(p+q) = \frac{1}{2} - \frac{3\sqrt{3}}{2}i + \frac{1}{2} + \frac{3\sqrt{3}}{2}i = 1$, which is one of the three roots. Note, however, that to find this *real* root, we *had* to have recourse to complex numbers. (It is a theorem, and you can prove it from the formulae given in this article, that this is always the case when a cubic has three real roots.) Bombelli passed through the valley of the shadow and emerged unscathed. To re-enter reality, he had to travel through *Tlön*.

But once *Tlön* has been sighted, be it ever so briefly, there is no turning back. Like all early voyagers, he left confused maps and log books. "I have found", he wrote, "a new sort of cube root, easily distinguished from the others". What he had found (haven't we just been saying it?) was a new sort of square root.

The matter depends on how you see it. We look back on Bombelli's achievement with four centuries of cheaply inherited wisdom. Naturally he had discovered cube roots. After all, they cropped up in connection with cubic equations. Any old fool can write $i^2 = -1$, and even invent a play algebra around it. He will inhabit *Laputa* and never sight *Tlön*. To qualify for residence in *Tlön*, one needs not only to entertain zany ideas, but to know what to do with them, and how to take them to human purposes.

Let us take a simple cubic – a very simple one – namely $x^3 - 1 = 0$. This is readily factorised to give

$$(x-1)(x^2+x+1) = 0$$

and we recover the solitary root $x = 1$, as we should expect. But now we have the possibility that

$$x^2 + x + 1 = 0. \tag{8}$$

We cannot shirk it, as our earlier excuse, that square roots of negative numbers do not exist, no longer holds water. We have ourselves gainsaid it.

We apply the formula to equation (8), to find

$$x = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

We may write $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$ and check that $\omega^2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$ and $\omega^3 = 1$. Also, $(\omega^2)^3 = 1$, as we may verify the matter by calculation.

A similar analysis may be applied to the quadratic factor in equation (6). This now factorises into

$$(y + \omega p + \omega^2 q)(y + \omega^2 p + \omega q).$$

Equation (7) has two other roots besides $y = -(p+q)$. They are $y = -(\omega p + \omega^2 q)$ and $y = -(\omega^2 p + \omega q)$. I leave it as an exercise to you, the reader, to check that in equation (4), these produce for us $y = -5$ and $y = 4$, the two other roots whose whereabouts may have troubled you.

Our picture is still unsatisfactory, however. It is still possible to object that all this is very fine, but that these calculations involving i are disreputable and suspect, because i does not exist. The objection is, in essence, that, thinking to find *Tlön*, we have drifted off to *Laputa*. What is required is a proof that the imaginary numbers are every bit as real as the real numbers.

We need to show that it is possible to represent complex numbers and their properties entirely in terms of the properties of the more familiar real numbers. The first successful proof along these lines was due to Carl Friedrich Gauss (1777-1855), who is often regarded with Archimedes and Newton as one of the very greatest mathematicians of all time.

Modern texts, however, tend to follow a later and simpler treatment, due to Hamilton (1805-1865). On this account, complex numbers are pairs of real numbers $[a, b]$ that add and multiply according to the laws

$$[a,b] + [c,d] = [a+c,\ b+d]$$

$$[a,b] \cdot [c,d] = [ac - bd,\ ad + bc].$$

The imaginary numbers are those pairs $[0, b]$, for which the first number is zero, and i is an abbreviation for $[0, 1]$. We may now calculate i^2 as $[0,1] \cdot [0,1]$, whose value is found quite readily to be $[-1, 0]$.

This is not exactly the real number -1, but is so close to it in behaviour that we abbreviate it to -1. A similar convention applies in the case of any other complex number whose second member is zero. These numbers are referred to as "real", although there is a slight misuse of language involved here.

The point of these manoeuvres is that they demonstrate conclusively that there is no mystery to the complex numbers, after all. We can happily use them, as Bombelli did, and know we're not talking nonsense. We will be safe living in *Tlön*.

That we have come to live there is perhaps best indicated by the fact that those eminent realists, the electrical engineers, treat alternating currents and voltages as complex quantities, and combine the resistance, inductance and capacitance of a circuit into one complex quantity – the impedance. $\sqrt{-1}$ is here to stay.

Further Reading

Gulliver's voyage to *Laputa* is described in Book Three of *Gulliver's Travels*, which should be readily accessible to the reader. *Tlön* is described in the short story *Tlön, Uqbar, Orbis Tertius*, which is less widely available. The best English translation is to be found in the collection *Labyrinths*, edited by D A Yates and J E Kirby and published in the Penguin Modern Classics series.

My account of complex numbers and the solution of cubic equations is based on that given by C V Durell and A Robson in *Modern Algebra, Vol.* II, published in 1937. Modern treatments of complex numbers are easily accessible. There are not so many good treatments of cubic equations. However, *College Algebra*, by J R Rosenbach, E A Whitman, B E Meserve and P M Whitman (published by Ginn) has a good account.

My treatment of the history of these matters is based on J N Crossley's *The Emergence of Number* and on P L Rose's *The Italian Renaissance of Mathematics*.

The uses of $\sqrt{-1}$ in electrical engineering are to be found in almost any standard text on AC circuit theory. In this context, i represents current and $\sqrt{-1}$ is denoted by j. Note that ω is angular frequency, and hence is not used to abbreviate $(-1+\sqrt{-3})/2$.

* * * * *

COMPUTERS AND COMPUTING

About Polar Coordinates and Pretty Graphs

Cristina Varsavsky

We usually locate a point P in the plane by quoting its coordinates (x, y) with respect to two perpendicular axes. The number x is the distance of the point from the vertical axis, while y gives the distance from the horizontal axis. The intersection of the two axes is the origin O; distances are measured horizontally from left to right and vertically from bottom to top. We call (x, y) the *cartesian coordinates* of the point P.

Another way of representing points in the plane is by *polar coordinates.* The polar coordinates of P are r, the distance of the point P from the origin, and φ, the angle between OP and the positive horizontal axis, measured in the anticlockwise direction, as shown in Figure 1. We call r the *radial distance* and φ the *polar angle*, which we'll measure in radians. For example, the point with polar coordinates $(2, \pi/4)$ is at a distance 2 from the origin and 45 degrees anticlockwise from the horizontal axis. This is represented in Figure 2.

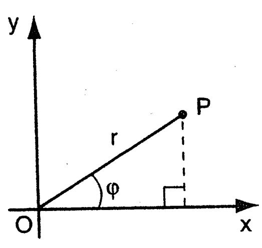


Figure 1

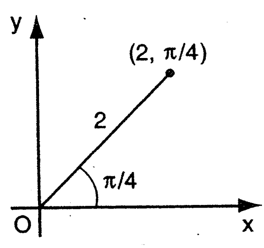


Figure 2

It is convenient to allow r and φ to take negative values: negative polar angles are measured clockwise; negative radial distances are measured in the opposite direction. For example, $(3, \pi/2)$ represents the same point as $(-3, -\pi/2)$ or $(3, -3\pi/2)$.

From Figure 1, we can see that the relationship between the cartesian and polar coordinates of a point is given by

$$\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned} \tag{1}$$

You have certainly learnt how to represent equations graphically, using cartesian coordinates. Take, for example, the equation

$$y = 3x^2 - 2x + 24 \tag{2}$$

The graph is the collection of points (x, y) in the cartesian plane satisfying the mathematical equation (2).

Similarly, equations could be expressed in polar coordinates; such equations are called *polar equations*. Examples of these are: $r = 4$, $r = 2\sin(4\varphi)$, $r = 2\varphi + 1$ and $\varphi = r^2$.

What do the graphs of polar equations look like? Let us start with the first, namely $r = 4$. Any point on the graph of this equation is at a distance 4 from the origin, and since φ is not present in the expression, all angles from 0 to 2π are possible. Hence this expression represents the circle with its centre at the origin and radius 4.

Sketching polar equations is not always as easy as it was in the previous example. A simple computer program could be a very helpful tool to obtain the graphs. We will design one such program in QuickBasic to plot the polar equation $r = 2\sin(4\varphi)$.

What do we need to do? For a sufficiently large number of angles in the range from 0 to 2π, we need to calculate the corresponding radial distances and plot those points on the screen. To describe location on the screen in the same way we do with cartesian coordinates, we will include the statement

```
window (-a, -b) - (a, b)
```

This allows us to locate any point on the screen by giving its x- and y-coordinates with $-a < x < a$ and $-b < y < b$. The point $(0, 0)$ is in the middle of the screen, $(-a, -b)$ is at the bottom left corner, and (a, b) is at the top right corner. The numbers a and b have to be chosen in such a way that the graph fits on the screen, and also in a ratio that would make a square appear as a square rather than a rectangle. In our case we choose $a = 4$ and $b = 3$. (This works fine in my IBM compatible computer with a VGA monitor, but you may need to change this ratio for your screen.) The whole graph will fit on this screen: this is because the sine function takes values only between -1 and 1, and therefore the radial-distance could not be any larger than 2.

To obtain the graph of the equation, we need to plot some points on the graph and join them with lines. These points must be taken at sufficiently small intervals to make the graph look reasonably smooth. So we take angles from 0 to 2π at intervals of 0.01. Determining the cartesian coordinates of these points is straightforward if we recall equations (1):

$$x = r\cos\varphi = 2\sin(4\varphi)\cos\varphi$$

and

$$y = r\sin\varphi = 2\sin(4\varphi)\sin\varphi$$

The program follows:

```
REM Polar Plots
SCREEN 9
pi = 3.1416

REM This sets the window size
a = 4: b = 3
WINDOW (-a, -b) - (a, b)

REM This draws the cartesian axes and the ticks on them
LINE (0, -b) - (0, b)
LINE (-a, 0) - (a, 0)
For i = -a TO a: LINE (i, -.05) - (i, .05) : NEXT i
FOR i = -b TO b: LINE (-.05, i) - (.05, i) : NEXT i

PSET (0, 0)
FOR phi = 0 TO 2 * pi STEP .01
        x = 2 * SIN(4 * phi) * COS(phi)
        y = 2 * SIN(4 * phi) * SIN(phi)
        LINE - (x, y)
NEXT phi
END
```

Now everything is ready for us to obtain the graph of the polar equation $r = 2\sin(4\varphi)$. As we run the program, we see how an eight-petal rose forms on the screen (see Figure 3).

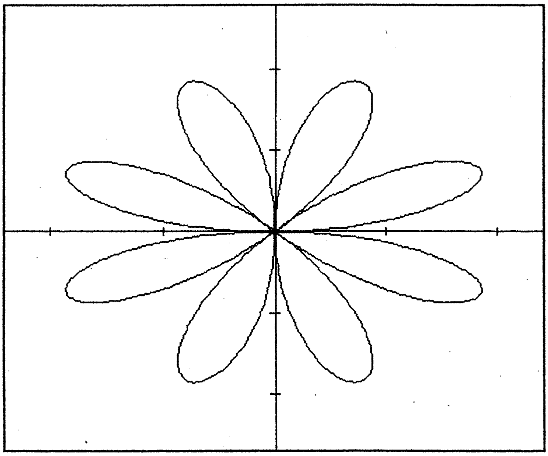


Figure 3

To plot any other polar equation you need to change the expressions for x and y in the program. We could explore what happens if a small change is made to the previous polar expression. Say we have $r = 2\sin(2\varphi)$. How does the graph change? The computer output, a four-petal rose, is shown in Figure 4. So a 2 in the argument of sine produces four petals, a 4 produces eight petals. How many petals will we get with a 6, or with a 3? Can you find a pattern?

Similarly, let us explore the number multiplying the sine. What if we had $r = \sin(4\varphi)$, or $r = 3\sin(4\varphi)$, or $r = (1/2)\sin(4\varphi)$? How does the graph change?

Another possibility is to use different numbers in the definitions of x and y. Figure 5 shows the graph for the following equations:

$$x = 2\sin(5\varphi)\cos\varphi$$

$$y = 2\sin(6\varphi)\sin\varphi$$

while Figure 6 shows the graph for the equations

$$x = 2\sin(8\varphi)\cos\varphi$$

$$y = 2\sin(8\varphi)\sin(3\varphi)$$

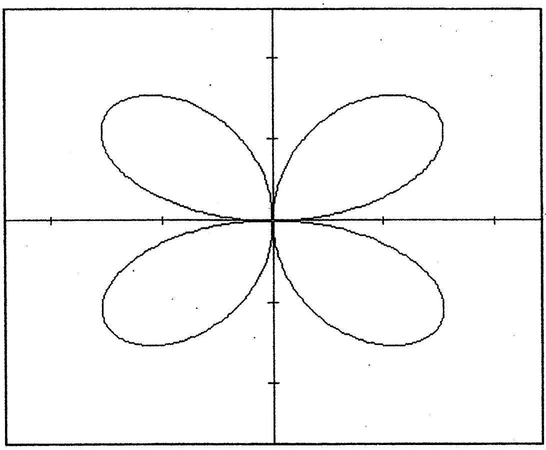


Figure 4

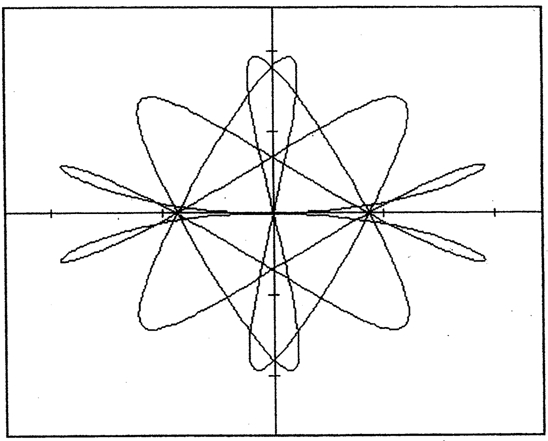


Figure 5

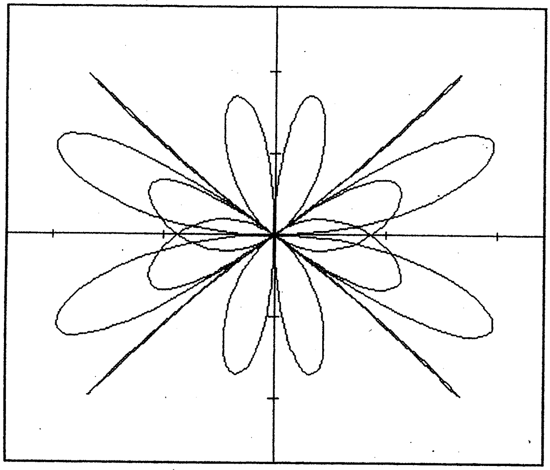


Figure 6

Many other striking curves can be obtained through simple polar expressions. Try the following with all the variations you can think of:

$$\begin{aligned} r &= 1+\cos\varphi \\ r &= 1+\sin(2\varphi) \\ r &= 2+\sin(3\varphi) \end{aligned}$$

You may need to change the window settings for some of them. Also, observe that in the above program, we start from the origin (PSET (0,0)). Since the point (0, 0) does not belong to the graphs of the equations listed above, you will need to change the starting points.

Although in the previous examples expressions were plotted for φ between 0 and 2π, there is no real reason to restrict φ to that range. Take for example the graph of the polar equation

$$r = e^{\cos\varphi} - 2\cos(4\varphi) + \sin^3(\varphi/12)$$

It's depicted in Figure 7. The complete butterfly appears if we take φ between 0 and 13π.

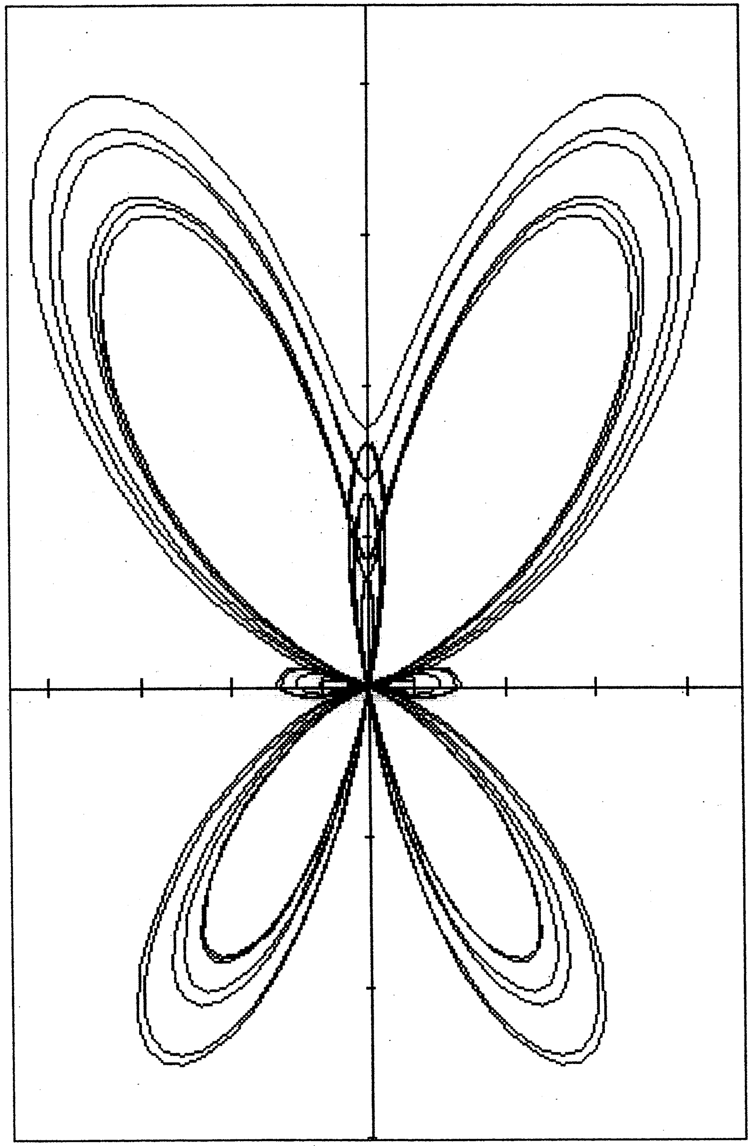


Figure 7

NEWS

Of Bees and Peas: A Recent Advance in a Close-Packing Problem

A recent issue of the journal *Nature* (*Vol.* 367, 17 February 1994) carries a report of an advance in an old problem. The problem is: how can 3-dimensional space be filled with solid figures without leaving gaps, in such a way that all of the figures have a given equal volume, and their surface areas are as small as possible?

The analogous problem in two dimensions is rather easier, and the solution is known. In this case, the problem is to fill the plane with figures of an equal fixed area, so that their perimeters are as small as possible. The figure with the smallest perimeter for a given area is a circle, but of course you can't fill the plane with circles without leaving gaps. Of the regular polygons, only the equilateral triangle, the square and the regular hexagon can fill the plane, and of these the hexagon has the smallest perimeter for its area. The problem does not require the figures to be regular polygons – in fact, it doesn't even say that all of the figures must be congruent – but it turns out that the hexagon is indeed the solution. Bees exploit this fact by building their honeycomb cells in a hexagonal pattern so as to use the smallest amount of wax for a given cross-sectional area of the cells.

In three dimensions, matters become a lot more complicated. The sphere is the solid figure with the smallest surface area for a given volume. It is not possible to fill space with spheres, but you might imagine that an initial arrangement of spheres could be "squashed" to obtain the answer. This idea must have occurred to Stephen Hales, one of Isaac Newton's contemporaries. In an ingenious if somewhat unorthodox attempt to solve a pure mathematical problem by carrying out a physical experiment, Hales filled a barrel with peas and squashed them down. He found that the peas were deformed into various irregular shapes with 13 or 14 faces, a result which didn't shed a great deal of light on the problem.

The only one of the five regular polyhedra that fills space is the cube, and it is certainly not the solution. A rather better candidate is a solid known as a *truncated octahedron.* A truncated octahedron is a solid figure with six square faces and eight hexagonal faces; it can be produced by starting with a regular octahedron and slicing off each vertex at the appropriate place (see Figure 1). Truncated octahedra *do* fill space, and in fact crystals with a truncated octahedral structure are known in nature. A little over one hundred years ago, Lord Kelvin showed that by subtly curving the faces of a truncated octahedron in a certain way, the surface area could be slightly reduced while the volume stayed constant, and the resulting solid would still fill space.

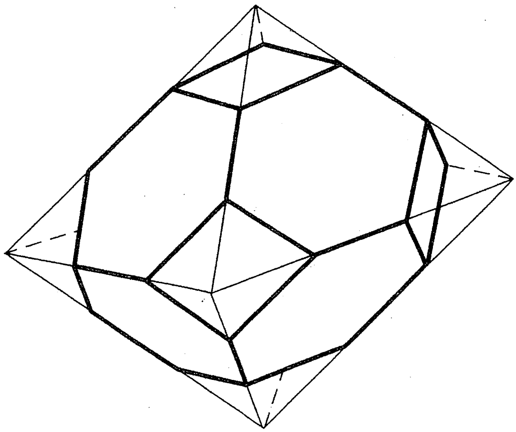


Figure 1

The recent advance, by D Weaire and R Phelan, is a modest improvement on Kelvin's long-standing result. Weaire and Phelan, working with the aid of a computer graphics package, found an arrangement of figures which reduces the surface area by 0.3 per cent compared with Kelvin's arrangement. As in the earlier arrangement, the figures are slightly modified polyhedra, but unlike Kelvin's arrangement, the new arrangement uses figures of two different shapes: one with 12 faces and one with 14. Whether this really is the solution to the problem remains to be seen.

* * * * *

LETTER TO THE EDITOR

Telecom 1994 Australian Mathematical Olympiad

Following the publication of the Telecom 1994 Australian Mathematical Olympiad problems in *Function*, *Vol.* 18, *Part* 2 (April 1994), we received the following letter from J C Barton of North Carlton.

> Dear Editors,
>
> To keep my hand in, so to speak, I tried some of the problems of the Telecom 1994 AMO given to the schools on 8/9 February 1994.
>
> There are answers to Nos. 1, 2, 3, 6, 8 and a few comments thrown in. I found them quite challenging.
>
> I find it hard to rake up any interest in questions like Nos. 5 and 7.
>
> For No. 4 I would hope that those candidates who gave as their sole answer that, by inspection, $f(t)$ equals t squared is a solution would get the majority of the marks.
>
> Yours faithfully,
> J C Barton

Enclosed with the letter were detailed solutions to Problems 1, 2, 3, 6 and 8, which are too long for us to reproduce here. Any reader who is interested in obtaining a copy is invited to contact one of the Editors.

If you have solved some of the problems in *Function* and would like to share your solutions with other readers, or if you have some new problems or anything else of interest, let us know by writing to the Editors at the address inside the front cover. If you prefer to communicate electronically, note that our new e-mail address, "function@maths.monash.edu.au", is now available.

* * * * *

PROBLEM CORNER

SOLUTIONS

In Problems 18.2.1-3, ABC is a triangle in which the sides opposite the vertices A, B and C have lengths a, b and c respectively. The lengths of the medians from A and B are denoted respectively by m_a and m_b. The length of a median is related to the side lengths by the equation $4m_a^2 = 2b^2+2c^2-a^2$ (and similarly for m_b); see the article "Means and triangle medians" by K R S Sastry in *Function, Vol.* 18, *Part* 2, where this result is derived.

PROBLEM 18.2.1

Is there an isosceles triangle ABC with $a = c$ in which $m_a = a$?

SOLUTION

Yes. Substituting $m_a = a$ and $c = a$ in the equation above yields:

$$4a^2 = 2b^2 + 2a^2 - a^2$$

Solving this equation for b gives $b = \sqrt{\frac{3}{2}}a$. The answer can be written in the form $(a, b, c) = (\sqrt{2}k, \sqrt{3}k, \sqrt{2}k)$, where k is any positive real number. Note that $2\sqrt{2}k > \sqrt{3}k$, so the sides do form a triangle.

PROBLEM 18.2.2

Is there an isosceles triangle ABC for which $m_a = \frac{1}{3}(a + b + c)$, the arithmetic mean of the lengths of all three sides?

SOLUTION

Yes. We begin by looking for solutions in which the two equal sides are b and c. In order to find all such triangles, we need to solve the equation:

$$4\left[\frac{1}{3}(a + b + b)\right]^2 = 2b^2 + 2b^2 - a^2$$

Upon expanding the brackets and collecting terms, we obtain:

$$13a^2 + 16ab - 20b^2 = 0$$

The equation can be factorised as follows:

$$(13a - 10b)(a + 2b) = 0$$

Only the first factor gives positive solutions for both a and b. Therefore the answer is $(a, b, c) = (10k,\ 13k,\ 13k)$ for any positive real number k. These side lengths clearly form a triangle.

A second set of answers to the problem can be found, corresponding to the case where the side with length a is one of the two equal sides. We can assume without loss of generality that $a = b$. Proceeding in a similar manner as before, we obtain $(a, b, c) = ((-8 + 9\sqrt{2})k,\ (-8 + 9\sqrt{2})k,\ 7k)$. Checking that these side lengths form a triangle amounts to showing that $2(-8+9\sqrt{2}) > 7$. This inequality can be checked numerically on a calculator; alternatively, it can be established more rigorously by noting that it is equivalent to $18\sqrt{2} > 23$, which is true because $(18\sqrt{2})^2 = 648 > 529 = 23^2$.

PROBLEM 18.2.3

Under what circumstances (if any) is it possible to have a triangle ABC in which $m_a = \sqrt{bc}$ and $m_b = \sqrt{ca}$?

SOLUTION

In his article, Sastry proved that $m_a = \sqrt{bc}$ if and only if $a = \sqrt{2}|b - c|$. Therefore, if $m_a = \sqrt{bc}$ and $m_b = \sqrt{ca}$ then $a = \sqrt{2}|b-c|$ and $b = \sqrt{2}|c-a|$. We may assume without loss of generality that $a \leq b$. Three cases now arise.

Case 1: $a \leq c \leq b$. Then $a = \sqrt{2}(b - c)$ and $b = \sqrt{2}(c - a)$. Eliminate b from these equations to obtain $a = \frac{2-\sqrt{2}}{3}c$. Now substitute this result into the expression for b to obtain $b = \frac{2+\sqrt{2}}{3}c$. The answer is therefore $(a, b, c) = ((2 - \sqrt{2})k,\ (2 + \sqrt{2})k,\ 3k)$ for any positive real number k. It is easily checked that $a + c > b$, so the sides form a triangle.

Case 2: $a \leq b \leq c$. Then $a = \sqrt{2}(c - b)$. Proceeding as before, we obtain the answer $(a, b, c) = ((2 - \sqrt{2})k,\ (2 - \sqrt{2})k,\ k)$.

Case 3: $c \leq a \leq b$. Then $a = \sqrt{2}(b - c)$ and $b = \sqrt{2}(a - c)$. The answer is $((2 + \sqrt{2})k,\ (2 + \sqrt{2})k,\ k)$.

PROBLEM 18.2.4

Express each integer from 1 to 100 in terms of an equation involving all four digits 1, 9, 9, 4 (in *that* order) and any other mathematical symbols.

SOLUTION

This problem and a solution to it were submitted by R D Coote and the Year Eleven Three Unit class at Katoomba High School, NSW. The class came up with a complete list from 1 to 100 in less than a week.

The expression "any other mathematical symbols" could mean different things to different people, depending on how many symbols they happen to be acquainted with! The class at Katoomba used the operations of addition, subtraction, multiplication, division, negation (i.e. forming the negative of a number), raising to a power, square root, and factorial, together with juxtaposition of digits (i.e. writing one digit after another, as in "19"), and the use of brackets.

There is probably no practical way of solving this problem systematically; it is a matter of using trial and error, and applying some ingenuity! The complete list of answers is too long to reproduce here, so we just list some of the more difficult ones to find.

$$29 = 1 \times \sqrt{9} \times 9 + \sqrt{4} \qquad 42 = 1 \times 9 + 9 + 4!$$
$$51 = 1 + (\sqrt{9})! \times 9 - 4 \qquad 61 = 19 \times \sqrt{9} + 4$$
$$65 = -1 + (\sqrt{9})! \times (9 + \sqrt{4}) \qquad 69 = [-1 + (\sqrt{9})!] \times 9 + 4!$$
$$70 = (-1 + 9) \times 9 - \sqrt{4} \qquad 89 = 1 - (\sqrt{9})! + 94$$
$$93 = -1^9 + 94 \qquad 100 = 1 + 9 \times (9 + \sqrt{4})$$

R D Coote points out that the year 1994 lends itself very well to this problem because the digits are all perfect squares, allowing the square root to be used in many of the expressions.

PROBLEMS

PROBLEM 18.4.1 (Ian Collings, Deakin University)

What scores are possible in an Australian Rules football match in which the number of goals multiplied by the number of behinds equals the number of points (e.g. 7, 7, 49)? (1 goal = 6 points, 1 behind = 1 point.)

More generally, what scores satisfying this condition are possible in a game in which each goal is worth p points?

PROBLEM 18.4.2

The fraction $\frac{16}{64}$ is unusual among the fractions with two-digit numerators and denominators, because it can be simplified to the correct answer, $\frac{1}{4}$, by "cancelling" the 6's, even though the cancellation is not a mathematically valid operation. Find all fractions with this property.

If you want a more challenging and open-ended problem, you might like to explore the possibilities that arise if the numbers are permitted to have more than two digits.

PROBLEM 18.4.3

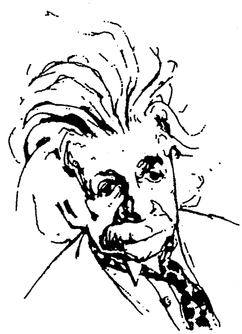
The guests at a party are each asked how many of the other guests they know. Every guest gives a different answer. Prove that at least one of the guests must be lying. (Assume that "knowing someone" is symmetric, i.e. if A knows B then B also knows A.)

* * * * *

MATHEMATICAL NURSERY RHYME

Over the water and over the lea,
All the world over, technicians agree
One formula transcends all formulae :–
E equals M times the square of C,
Found by that wondrous Albert E.,
Who showed how atomic energy
Derives from mass and light's velocity –
Strange truth that may shape man's destiny.

From: *The Surprise Attack in Mathematical Problems*
by L A Graham (Dover, 1968)



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