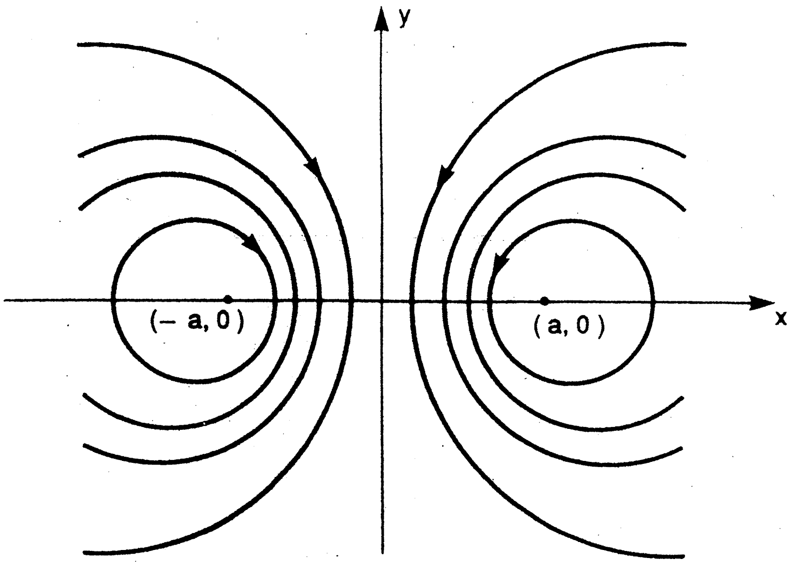
Function

A School Mathematics Magazine

Volume 18 Part 3 June 1994



Mathematics Department - Monash University

FUNCTION is a mathematics magazine produced by the Department of Mathematics at Monash University. The magazine was founded in 1977 by Prof. G.B. Preston. FUNCTION is addressed principally to students in the upper years of secondary schools, and more generally to anyone who is interested in mathematics.

FUNCTION deals with mathematics in all its aspects: pure mathematics, statistics, mathematics in computing, applications of mathematics to the natural and social sciences, history of mathematics, mathematical games, careers in mathematics, and mathematics in society. The items that appear in each issue of FUNCTION include articles on a broad range of mathematical topics, news items on recent mathematical advances, book reviews, problems, letters, anecdotes and cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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FUNCTION is published five times a year, appearing in February, April, June, August, and October. Price for five issues (including postage): \$17.00*; single issues \$4.00. Payments should be sent to the Business Manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the Business Manager.

*\$8.50 for *bona fide* secondary or tertiary students.

EDITORIAL

Welcome to this new issue of *Function* which we hope you'll find interesting!

This time we include two feature articles. In the first one, "Beyond Reasonable Doubt", Paul Lochert looks at how probabilistic arguments were used in the courtroom, first to convict the accused and later to overturn the decision. In the second feature article, "The Fifty and the Twenty", Michael Deakin presents different shapes of coins and analyses their geometry.

Our front cover depicts the streamlines for the flow of a fluid with two whirlpools. In the corresponding article, Ian Collings presents a mathematical discussion on how to produce these circles.

This issue also includes a book review: Robyn Arianrhod gives an interesting introduction to the recently published book *Newton for Beginners.*

In the History of Mathematics section, Michael Deakin introduces the quaternions, an interesting extension of the algebra of complex numbers, and discusses their relation to the dot and cross products of three-dimensional vectors.

In our regular Computers and Computing column, you will find algorithms for performing arithmetic with integers as large as you like, far beyond anything your calculator could handle, without any loss of accuracy.

As usual, there are solutions to some of the problems we have published, and of course a few new problems to challenge your mind.

Happy reading!

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THE FRONT COVER
Circles and Vortices

Ian Collings, Deakin University

We all know that a circle can be described as the path traced out by a point which moves in a plane so that it is a constant distance from a fixed point.

Suppose we have a fixed point $F(a,0)$ on the x-axis, and a point $P(x,y)$ moves so that it is always a distance d from F, as in Figure 1.

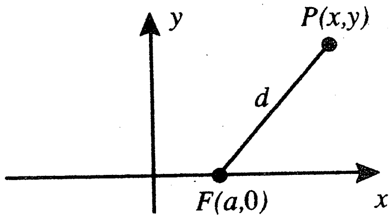

Figure 1

Using the formula for the distance between two points in the Cartesian plane, we obtain:

$$(x-a)^2 + y^2 = d^2.$$

What result do we obtain if a point P moves so that its distance from each of two fixed points is the same? Let the two points be $F(a,0)$ and $G(-a,0)$ as in Figure 2. Then $PF = PG$, so the distance formula gives:

$$(x-a)^2 + y^2 = (x+a)^2 + y^2.$$

This equation reduces to $4ax = 0$, and so $x = 0$, which is of course the equation of the y-axis. The point P moves along the y-axis as expected.

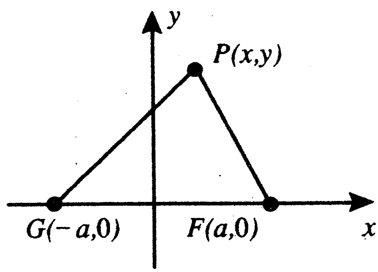

Figure 2

However, what result do we obtain if $PF = k \cdot PG$, where k is a non-negative constant? We already know about the case $k = 1$, but what if $0 \leq k < 1$ or if $k > 1$? If $PF = k \cdot PG$, then applying the distance formula, we obtain:

$$(x-a)^2 + y^2 = k^2[(x+a)^2 + y^2]. \tag{1}$$

If we expand both sides and collect terms, we obtain

$$(1-k^2)[x^2 - \frac{2a(1+k^2)}{1-k^2}x + a^2] + (1-k^2)y^2 = 0, \ k \neq 1.$$

Completing the square on the terms in the square brackets we obtain, after a little algebra,

$$\left[x - \frac{a(1+k^2)}{1-k^2}\right]^2 + y^2 = \frac{4k^2a^2}{(1-k^2)^2}, \quad k \neq 1 \tag{2}$$

which represents a circle with centre $\left(\frac{a(1+k^2)}{1-k^2}, \ 0\right)$ and with radius $r = \frac{2ka}{|1-k^2|}$. As k varies, we obtain a set of circles; the centres all lie on the x-axis but take up different positions together with different radii for different values of k. If $k = 0$, then we obtain $(x-a)^2 + y^2 = 0$; i.e. a "circle" centred at $(a, 0)$ with zero radius.

As k increases from 0 to 1, the centre moves along the x-axis to the right of $x = a$ and the radius increases in magnitude. When $k = 1$ we obtain, from Equation (1), the y-axis. This line may be thought of as a "circle" with its centre at plus or minus infinity on the x-axis and with a radius of infinite magnitude.

As k increases from 1 to infinity, the centre moves from left to right along the negative x-axis and the radius decreases to zero when the centre is $(-a, 0)$.

The situation is depicted on the front cover. The arrows on the diagram refer to a physical interpretation of the problem in fluid dynamics. In this interpretation, the circles represent *streamlines* (lines following the flow), with the arrows indicating the direction of flow. There are two vortices (whirlpools): one at $(a, 0)$ and one at $(-a, 0)$.

If the streamline along the y-axis is replaced with a solid wall, then the pattern of flow is unchanged if we assume that the fluid slides smoothly along the wall without being slowed down. In this case, the circles in the region $x > 0$ represent the streamlines in a fluid with a vortex located at $(a, 0)$ and a wall at $x = 0$. This situation is depicted in Figure 3.

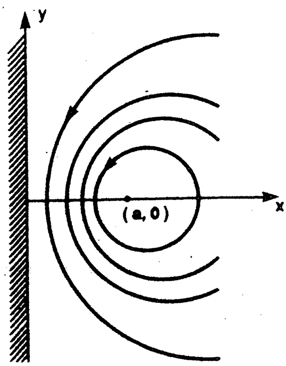

Figure 3

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Dr Ian Collings is a Senior Lecturer in Mathematics at Deakin University with a special interest in applied mathematics. He holds a Ph.D. from Melbourne University and is currently undertaking research in continental shelf waves and mathematics in sport. His golf handicap is 11.

BEYOND REASONABLE DOUBT

Paul Lochert, Monash University

In the practice of criminal law in Australia and many other societies, the accused is presumed innocent until proven guilty beyond reasonable doubt. Increasingly, expert witnesses are being utilised to provide information based on the findings of their research, which will often involve technical information that is likely to be foreign to most others in the courtroom. In this search for greater precision or exactitude, is there a place for the statistician? Can "beyond reasonable doubt" be defined as some level of certainty or some other quantifiable measure?

The legal profession likes to rely on precedents. Thus they often search for a prior case that can be identified with the current case and assert that the previous finding will again hold. I will consider a case that could form a precedent, where the conviction was based purely on the likelihood or chance of an event occurring, thus appearing to quantify "beyond reasonable doubt". The case was subsequently appealed against and the finding was overturned, again on probabilistic arguments.

The case in question is that of Malcolm and Janet Collins *v.* California (1965).

The crime in question involved the aggravated robbery of an elderly woman. No witnesses to the crime other than the woman involved were located by the police. A number of witnesses saw a couple leave the scene of the crime. From these witnesses the police built up a profile of the suspects as follows. A couple was seen driving a yellow car away from the scene of the attack, the female was caucasian with blonde hair tied in a pony tail, and the male was negroid with a beard and moustache.

The police, by checking owners of yellow (or at least partially yellow) cars, located a couple, Malcolm and Janet Collins, who fitted the profile developed. No witness, including the elderly woman who was attacked, could positively identify either of the accused.

The case for the prosecution took the following format.

A mathematician (expert witness) was called to explain the multiplication law of probability. That is, if two events A and B are independent then the probability of both occurring is given by the product of the individual probabilities, namely:

$$P(A \cap B) = P(A) \cdot P(B).$$

Next the attorney presented the following estimates for the probability of each characteristic and claimed (without documentation) that they were conservative estimates.

$$\begin{aligned} P \text{ (Negro male with beard)} &= 1/10 \\ P \text{ (Male with moustache)} &= 1/4 \\ P \text{ (Interracial couple in car)} &= 1/1000 \\ P \text{ (Female with blonde hair)} &= 1/3 \\ P \text{ (Female with pony tail)} &= 1/10 \\ P \text{ (Yellow car)} &= 1/10 \end{aligned}$$

The attorney then applied the multiplication law provided by the expert witness and found that the probability of a couple meeting the profile, even using conservative estimates, was 1 in 12 million. The police had clearly carried out their duties well to succeed in locating such a rare event and clearly it was most unlikely that the accused couple were not in fact the culprits.

The jury was impressed and convicted the couple.

From this precedent it appears that "beyond reasonable doubt" has been quantified to be any event less likely than 1 in 12 million.

But wait! There was an appeal in the Supreme Court, Los Angeles County (1968). It was successful. It was again based solely on probabilistic arguments.

At the appeal the proceedings were questioned on the following grounds.

First, the prosecution did not supply evidence to support any of the individual probabilities or that they were approximately accurate. Any change in even one individual probability would change the final chance in a way that may not make the event appear so unlikely that a jury would accept that no other such couples exist.

Second, only the special case for computing joint probabilities was given and no evidence was provided to show that the events were independent. The multiplication law says in general that the probability of event A and event B occurring is given by the probability of event A occurring given that event B has occurred, multiplied by the probability of event B occurring, i.e.

$$P(A \cap B) = P(A|B) \cdot P(B).$$

There are various sociological studies that would imply that at least some of these events are dependent. For example: in studies of interracial marriages between a negro male and a caucasian female there is a disproportionate number of blonde-haired blue-eyed females. Thus although it may be the case that

$$P\text{ (Female with blonde hair)} = 1/3,$$

it could nevertheless be true that

$$P\text{ (Female with blonde hair} \mid \text{interracial marriage)} = 2/3.$$

The introduction of these dependencies will change the probability of such a profile.

Although these two arguments put in question the 1 in 12 million, they were not the ones that reversed the decision.

The appeal judge reversed the decision by seeking the answer to the question:

> "What is the probability that there is more than one such couple, given that one exists?"

Clearly at least one such couple exists, since they were found and prosecuted. Accepting the prosecution's estimate of 1 in 12 million, the judge concluded that taking a population size of 12 million there is a 41% chance of there being more than one such couple. This chance was so high that the judge could not accept "beyond reasonable doubt" that Malcolm and Janet Collins were the guilty party.

The judge in his findings stated:

> No mathematical equation can prove beyond reasonable doubt
>
> (1) that the guilty couple *in fact* possessed the characteristics described by the witnesses, or even
>
> (2) that only *one* couple possessing those distinctive characteristics could be found in the entire Los Angeles area.

The following argument shows one approach which gives the result obtained by the judge.

Define:

X to be the number of couples fitting the profile in n couples

$$\begin{aligned} p &= P\text{ (couple fitting the profile)} \\ &= 1/12{,}000{,}000 \end{aligned}$$

Then it can be assumed that X follows a binomial distribution:

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, 2, 3, \ldots, n.$$

If n is large and p is small, it can be shown that the binomial distribution can be approximated by the formula

$$P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!} \qquad x = 0, 1, 2, 3, \ldots$$

where $\lambda = np$ and e is the base of natural logarithms. (This result is known as the Poisson approximation to the binomial distribution.) This gives

$$P(X=0) = e^{-\lambda}$$

$$P(X=1) = \lambda e^{-\lambda}$$

$$\begin{aligned} P(X>0) &= 1 - P(X=0) \\ &= 1 - e^{-\lambda} \end{aligned}$$

$$\begin{aligned} P(X>1) &= 1 - [P(X=0) + P(X=1)] \\ &= 1 - (e^{-\lambda} + \lambda e^{-\lambda}) \\ &= 1 - e^{-\lambda}(1+\lambda) \end{aligned}$$

We require the conditional probability of more than one couple given at least one exists, i.e.

$$\begin{aligned} P\text{ (more than 1 couple } | \text{ at least 1 couple)} &= P(X>1|X>0) \\ &= P[(X>1)\cap(X>0)]/P(X>0) \end{aligned}$$

Since $(X > 1)$ is a subset of $(X > 0)$, $P[(X > 1) \cap (X > 0)] = P(X > 1)$. So

$$\begin{aligned} P(X>1|X>0) &= P(X>1)/P(X>0) \\ &= \frac{1-e^{-\lambda}(1+\lambda)}{1-e^{-\lambda}} \\ &= \frac{e^{\lambda}-(1+\lambda)}{e^{\lambda}-1} \end{aligned}$$

Now, taking $n = 12,000,000$ (not an unreasonable number for California in 1968) gives $\lambda = 1$.

Hence

$$P(X > 1 | X > 0) = \frac{e-2}{e-1}$$
$$= 0.418$$

This result is consistent with the result obtained by the judge, but it is not clear that it is the answer to the question that should have been posed, and it has moved the argument away from the issue of quantifying "beyond reasonable doubt". This solution answers the question before the police commence searching, with the existence of at least one couple being a consequence of the witness reports. Has the apprehension of one couple changed the required conditioning? The real question to be answered is "What is the probability that the first couple apprehended is in fact the guilty couple?" These issues have been discussed in articles by Watterson (1982), Clark and MacNeil (1977), and Eggleston (1980).

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THE FIFTY AND THE TWENTY

Michael A.B. Deakin, Monash University

In this country most coins are circular and this is the case in other countries also. But other shapes have been used or are still in use. Very early in the history of New South Wales, the *Holey Dollar* was current. This was an annular coin, i.e. the coin was circular and had a circular hole at its centre. In Papua New Guinea, early this century, the standard coin was circular, but had a square hole at its centre. Other countries go in for square coins (but with rounded corners) or for scalloped edges on otherwise circular coins.

But even more unusual shapes have been used. In my boyhood I collected coins and one of the treasures of my collection was from 18th-Century France. I had had this coin for quite some time before I realised that it was not in fact quite circular but was a 22-gon (though rather worn). More recently, Britain has had a dodecagonal (12-sided) threepence, and our 50c piece is also so shaped.

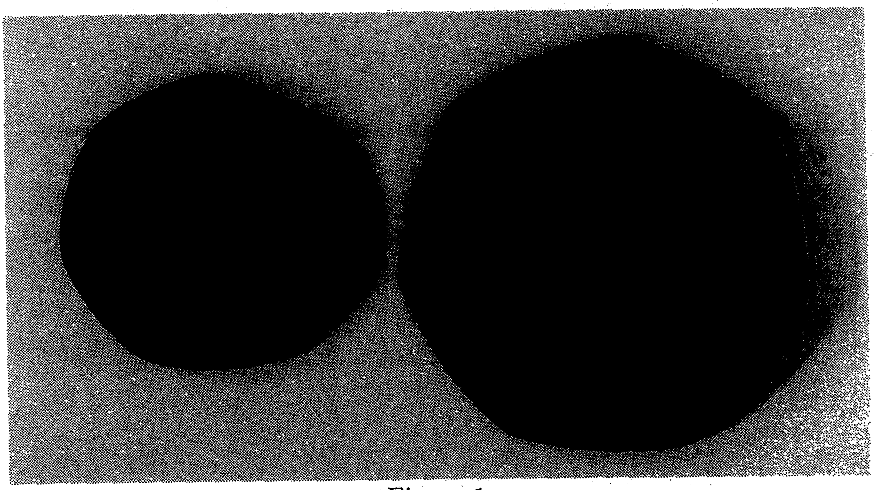

Figure 1

Britain today has two coins (Figure 1) that surely qualify as having one of the most unusual shapes ever used. These are the 20-pence and 50-pence pieces. Both have the same shape and clearly that shape is not circular. Close inspection shows that it is made up of seven curved sides.

Figure 2 shows the outline of the 50-pence piece, with one vertex (A) singled out, as therefore is also its opposite side BC. Draw a circle with centre A and radius AB. Note that BC is an arc of this circle.

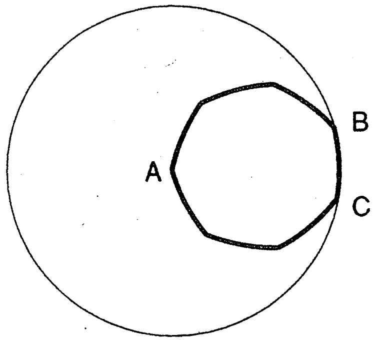

Figure 2

Any regular n-gon can be modified in this way, so that straight sides are replaced by circular arcs, as long as n is odd. The very simplest case is that for which $n = 3$. See Figure 3.

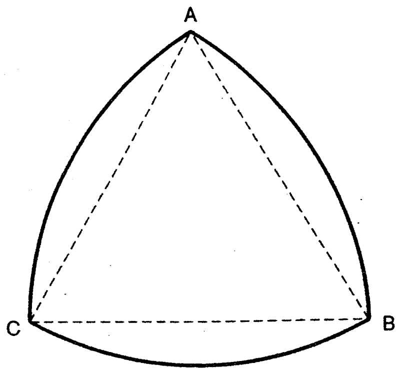

Figure 3

Here ABC is an equilateral triangle but side BC has been replaced with a circular arc drawn with centre A and radius AB ($= AC$). Similarly for sides AB and CA.

Figure 4 illustrates, and a little thought makes it clear, that this modified triangle can be made to turn between a pair of parallel lines. One line will touch a vertex about which the curve will pivot, while the other line will be tangent to a side which rolls upon it. So we could make a roller in this shape and allow the top line to move relative to the bottom by having rollers so shaped rotate between the two.

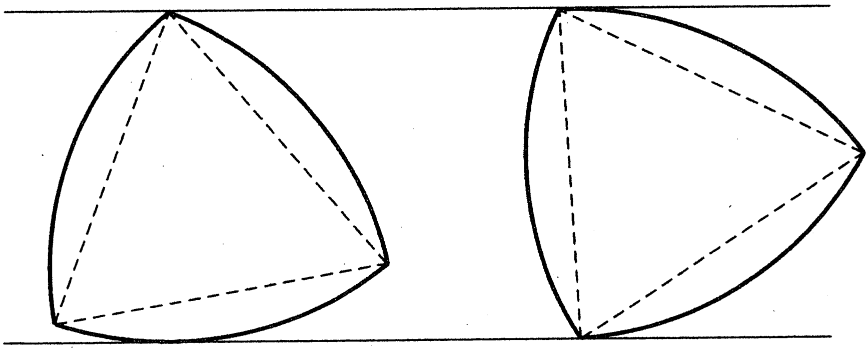

Figure 4

The *breadth* of a curve in a given direction is defined to be the distance between two parallel lines which are perpendicular to the given direction and which touch the curve on opposite sides. In the present case, the breadth of the curve ABC does not depend on the direction and for this reason ABC is referred to as a *curve of constant breadth.*

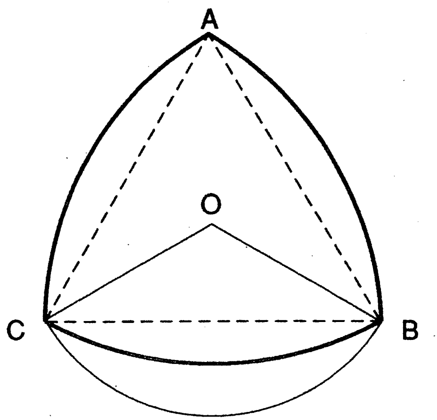

Figure 5

We cannot, however, make a practical *wheel* from the curve ABC. See Figure 5. O is the centre of ABC and the outer of the curved arcs BC is an arc of a circle with centre O and radius OB $(= OC = OA)$. The inner arc BC thus does not preserve a constant distance from O. If we pivoted the curve at O and rolled such a wheel along a flat road, O would ride up and down.

Thus a curve of constant breadth may make a perfectly good roller, but is no good at all as a wheel.

The British coins are thus curves of constant breadth. It is desirable that coins have this property: it aids their use in slot machines, for example.

However, the question arises as to why this rather peculiar shape was chosen, and not the circle, which of course is also a curve of constant breadth.

The usual reason for choosing an unusual shape is to distinguish two otherwise similar coins. Australian 50c and 20c coins differ in shape to avoid the confusion that might otherwise occur – as it does with the 10c and $1 coins. This reason, however, does not apply to the British coins. So presumably they are shaped as they are for purely aesthetic reasons.

There is a slight saving in metal in that if we inscribe the shape in a circle, the circle has a slightly larger area; however, the difference is very small. (As an exercise you could work out what it is.)

And why *seven* sides? All I can think of here is that the corners on a three- or five-sided coin were deemed to be too sharp (as another exercise, can you work out the angle at which the circular arcs intersect?) and that nine-, eleven-, ...-sided coins were too nearly circular.

In fact, our 50c piece gives no trouble in slot machines (as a third exercise, determine the maximum extent of deviation from circularity in the case of a dodecagon) and I failed for a time to realise that my 22-gonal coin was not strictly circular.

There are other curves of constant breadth beyond those described here and there is much more to be said of them. There is a short passage (pp. 62-63) on the topic in E.P. Northrop's *Riddles in Mathematics* and a very nice article by Martin Gardner in *Scientific American* (Feb. 1963). The fullest account of which I know is that of H. Rademacher and O. Toeplitz in Chapter 25 of their book *The Enjoyment of Mathematics*.

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BOOK REVIEW

Newton for Beginners

by William Rankin
(Allen and Unwin, $16.95, pb, 176 pp.)

Reviewed by Robyn Arianrhod, Monash University

The story of the birth of modern (mathematical) science is absolutely fascinating; it culminates in the work of the amazing Sir Isaac Newton, and I can think of no more lively and accessible introduction to the story than the recently published book *Newton for Beginners.* This is the latest in the popular series of *Beginners* books; its format consists of a narrative containing historical sketches and outlines of relevant ideas, and an abundance of illustrations: drawings, cartoons and occasional photos. (A sample page appears in Figure 1.)

Newton for Beginners is a great read, with much colour and drama as well as a surprising amount of information on the history of mathematics. Even most of the jokes contain bits of useful information, often as a tiny label under a drawing, or in the reappearance of a character from an earlier period in history in order to show how ideas have developed.

Newton was a genius, who discovered the nature of the rainbow and developed the first comprehensive theory of light itself, who invented the basic mathematical techniques we refer to as *calculus*, and who discovered the law of universal gravitation which we use today to launch satellites and rockets into space, and to explain the ocean's tides. Even geniuses draw on the ideas of other people, however. Indeed, two of the most fascinating things about Newton are his synthesis of the best ideas of his predecessors, as described in the sample page in Figure 1, and the consequent fights he got into with people who accused him of plagiarism. The most bitter feuds were with Robert Hooke and the famous Gottfried Leibniz, who also invented the calculus. In fact, *Newton for Beginners* gives some wonderful insights into the "humanness" of this great intellectual. For example, in order to verify his ideas experimentally, Newton went to extraordinary lengths which were almost childlike in their directness and enthusiasm, lengths which included sticking a blunt needle in his eye to explore the true nature of colour (as seen when the optic nerves are stimulated). Furthermore, whilst employed by the Royal Mint, Newton

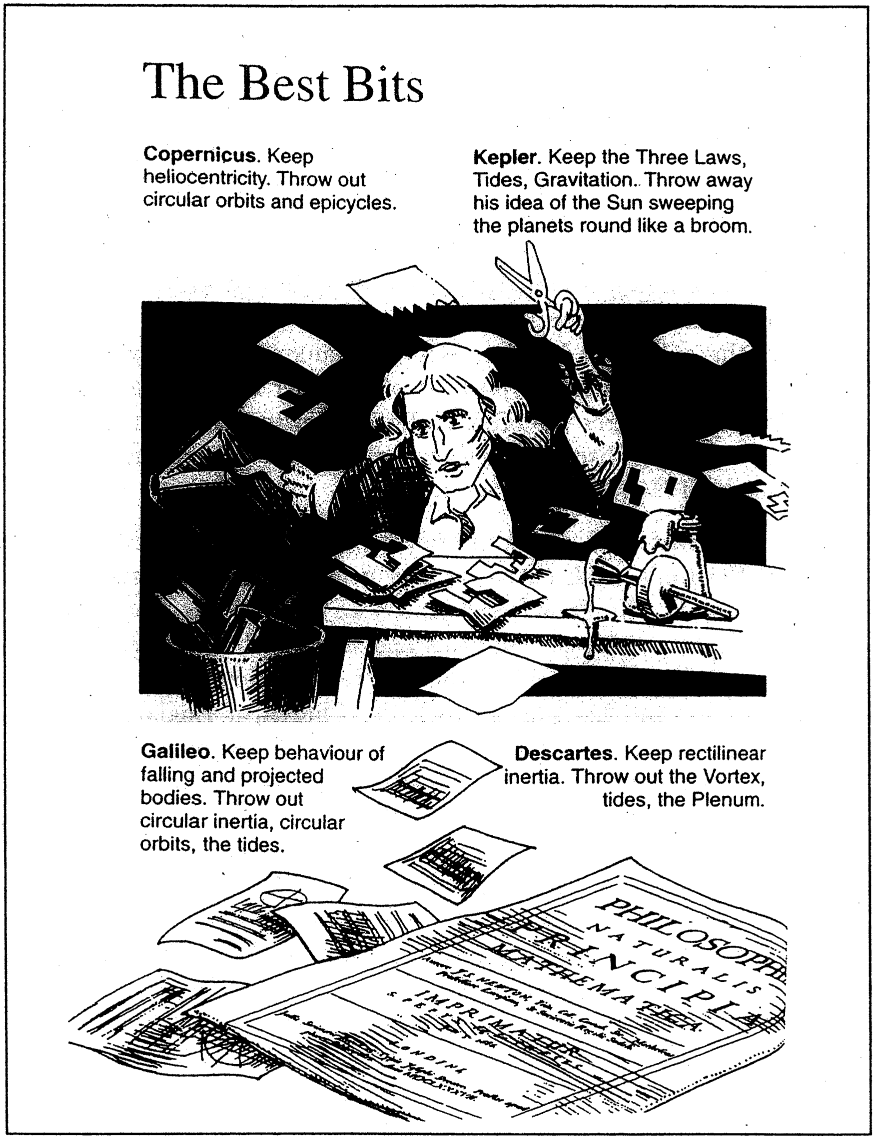
The Best Bits
Copernicus. Keep heliocentricity. Throw out circular orbits and epicycles.
Kepler. Keep the Three Laws, Tides, Gravitation. Throw away his idea of the Sun sweeping the planets round like a broom.
Galileo. Keep behaviour of falling and projected bodies. Throw out circular inertia, circular orbits, the tides.
Descartes. Keep rectilinear inertia. Throw out the Vortex, tides, the Plenum.

Figure 1

threw himself with customary fervour into the business of catching counterfeiters in a manner that would rival that of Sherlock Holmes.

As I've said, the context in which a genius flourishes is vital, and at least half the book is devoted to the history of mathematics before Newton. It actually took me some time to get into *Newton for Beginners* – the format takes a while to decipher: which bits are real history, which bits are just comics? (I decided that most of the "bits" *are* about real history.) The narrative contains some grammatical and typographical mistakes, and it took a while for these to be overshadowed by the drama of the story, partly because the ancient history presented is sketchy and not particularly well written, and doesn't immediately appear to have a lot to do with the main subject. By the time I got to the part about Galileo, however, I found that the author and artist, William Rankin, had really found his stride. He paints a humorous portrait of a rather cantankerous Galileo, and he obviously admires the often-maligned Newton. Indeed, in the last page of the book, he discusses reasons for what he believes are common misjudgements of Newton's character.

There is a great deal of information in this little book; even the influences of Newton's work on succeeding generations, including those of the Enlightenment, the French and American Revolutions, and above all on Einstein, are considered. Galileo's and Kepler's brushes with the Inquisition are discussed, as are Newton's secret rejection of the idea of the Christian Trinity and his run-ins with the Church of England. In this story there are magic and jealousy, revolutions and executions, but above all, there is the incredible power of mathematics as the language for describing nature. There is not much actual mathematics in the book (it *is* aimed at a popular market), and it is interesting to note topics where a mathematical equation (such as Newton's Law of universal gravitation) would have provided a much simpler description than the words used – which, of course, is why we have mathematics! Nevertheless, it is vital, I believe, that students of mathematics know some of its history, and therefore I recommend this book highly to readers of *Function* as well as to the general public.

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HISTORY OF MATHEMATICS

The Fourfold Way[1]

Michael A.B. Deakin

Student:	I seem to have got stuck on this problem, sir.
Teacher:	You have forgotten it again! How many times do I have to say it? *You can't add a scalar to a vector!*

A common enough conversation, one would have thought. Let's see what lies behind it. Matters are not as simple as the teacher would have us believe.

Each point in two-dimensional space may be represented by means of a number pair (a, b). We learn later that such number pairs may be thought of as complex numbers $a + bi$, where a is called the *real part* and b is called the *imaginary part* of the number.

To add or subtract two complex numbers we simply perform the operations on their corresponding real and imaginary parts, namely

$$(a + bi) \pm (c + di) = (a \pm c) + (b \pm d)i$$

The multiplication of complex numbers uses the fact that $i^2 = -1$ when we expand in the usual way using the distributive law:

$$\begin{aligned}(a + bi) \cdot (c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

These two operations, together with their properties, provide the complex numbers with an algebraic structure. We will refer to the complex numbers with the algebraic structure described above as *complex algebra.*

Now, our world of experience is three-dimensional, so that an extension of the complex numbers is needed if we are to have an algebra applicable to it. The obvious thing to do is to try to form an algebra of number triples $[a, b, c]$ or $a + bi + cj$.

We would like the triples $[a, b, 0]$ to obey complex algebra, so we choose $i^2 = -1$. Now investigate ij – set it equal to $a + bi + cj$. Then

$$i(ij) = i(a + bi + cj)$$

[1]Parts of this article appeared in *Function, Vol.* 5, *Part* 3 under a different title. This material is here reprinted during Dr Deakin's current absence overseas.

and assuming that $i(ij) = (ii)j$ and that $ia = ai$, etc., we find

$$-j = ai - b + cij.$$

Replacing ij in the right-hand side by its corresponding expression, we have

$$\begin{aligned} -j &= ai - b + cij \\ &= ai - b + c(a + bi + cj) \\ &= (-b + ca) + (a + cb)i + c^2 j \end{aligned}$$

so that, equating coefficients of j, we find $c^2 = -1$, which is no help, as we are looking for real a, b, and c.

Thus we reach an impasse. In fact, it may be shown that if x, y, z represent triples of numbers such as $[a, b, c]$, no algebra of such triples can exist if we make the two natural requirements (where 0 stands for $[0, 0, 0]$):

(i) $xy = 0$ implies $x = 0$ or $y = 0$;
(ii) $x(yz) = (xy)z$.

Indeed, we might expect that any usable algebra would obey these requirements.

This was the situation that confronted the mathematician Sir William Rowan Hamilton (1805-1865) when he crossed Brougham Bridge, Dublin, one evening in 1843, while walking with his wife.

Various accounts by Hamilton survive, describing what happened. This one is quoted from Crewe's *A History of Vector Analysis*:

> "[Quaternions] started into life, or light, full grown ... . That is to say, I then and there felt the galvanic circuit of thought close, and the sparks that fell from it were the fundamental equations between i, j, k; *exactly such* as I have used them ever since. I pulled out, on the spot, a pocketbook, which still exists, and made an entry, on which, *at that very moment*, I felt that it might be worth my while to expend the labour of at least ten (or it might be fifteen) years to come. But then it is fair to say that this was because I felt a *problem* to have been at that moment *solved*, an intellectual *want relieved*, which had *haunted* me for at least *fifteen* years before."

A different version was given by Bamford Gordon [Prof. G.B. Preston] in *Function, Vol.* 8, *Part* 3. This quotes from a letter from Hamilton to his son.

> "In October 1843, having recently returned from a meeting of the British Association in Cork, the desire to discover the laws of the multiplication of triplets regained with me a certain strength and earnestness, which had for years been dormant, but was then on the point of being gratified, and was occasionally talked of with you. Every morning in the early part of the above cited month, on my coming down to breakfast, your brother William Edwin and yourself used to ask me, 'Well, Papa, can you multiply triplets?' Whereto I was always obliged to reply, with a sad shake of the head, 'No, I can only add and subtract them.' But on the 16th day of the same month – which happened to be a Monday and a Council day of the Royal Irish Academy – I was walking in to attend and preside, and your mother was walking with me, along the Royal Canal, to which she had perhaps been driven; and although she talked with me now and then, yet an under-current of thought was going on in my mind, which gave at last a result whereof it is not too much to say that I felt at once the importance. An electric circuit seemed to close; and a spark flashed forth, the herald (as I foresaw immediately) of many long years to come of definitely directed thought and work, by myself if spared, and at all events on the part of others, if I should ever be allowed to live long enough distinctly to communicate the discovery. I pulled out on the spot a pocket-book, which still exists, and made an entry there and then. Nor could I resist the impulse – unphilosophical as it may have been – to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k;
>
> $$i^2 = j^2 = k^2 = ijk = -1,$$
>
> which contains the solution of the Problem, but of course, as an inscription, has long since mouldered away. A more durable notice remains, however, on the Council Books of the Academy for that day (October 16th, 1843), which records the fact that I then asked for and obtained leave to read a paper on Quaternions, at the First General Meeting of the Session: which reading took place

accordingly on Monday the 13th of November following."[2]

The original vandalism ("long since mouldered away") has been replaced by a plaque on what is now called "Quaternion Bridge" in the Dublin suburb of Broombridge. The plaque gives the table as written above.

Essentially Hamilton solved the problem by admitting four dimensions, rather than restricting consideration to the impossible three. There are three square roots of -1, called i, j, k, so that

$$i^2 = j^2 = k^2 = -1.$$

Furthermore,

$$ij = k, \quad jk = i, \quad ki = j,$$

which formulae are neatly summarisable as $ijk = -1$, the result given on the plaque.

The algebra has one peculiar feature: if the order of the terms in a product is reversed, the value may alter. E.g. $ij = -ji$. This is readily proved:

$$-ji = (ii)ji = i(ij)i = iki = i(ki) = ij.$$

You may care to explore matters further yourself.

Hamilton would have been aware of algebras that do not allow automatic reversal of products (so-called non-commutative algebras[3]), so that his insight was essentially the admission of the extra dimension.

Hamilton, and later another mathematician, Tait, devoted much energy to the development of quaternions. The endeavour, indeed, occupied much of the rest of Hamilton's life. For Tait, the four-dimensionality of the result became the key to an intellectual development in which abstract results came to be seen as more important than the practical needs on which they were based.

The general quaternion, $a + bi + cj + dk$, was seen as composed of two parts (cf. the real and imaginary parts of a complex number): the real or *scalar* part a, and the *vector* part $bi + cj + dk$. This latter, being three-dimensional, was of most interest for applications.

[2]This quotation was taken from R.P. Graves' biography of Hamilton.
[3]An example known to you is the multiplication of matrices.

From this point of view, a vector is a quaternion of the special type $[0, b, c, d]$ or $bi + cj + dk$. If two such quaternions are multiplied together, there results:

$$\begin{aligned}(b_1 i + c_1 j + d_1 k) &\cdot (b_2 i + c_2 j + d_2 k) \\ &= -b_1 b_2 + b_1 c_2 k - b_1 d_2 j - c_1 b_2 k - c_1 c_2 \\ &\quad + c_1 d_2 i + d_1 b_2 j - d_1 c_2 i - d_1 d_2 \\ &= (-b_1 b_2 - c_1 c_2 - d_1 d_2) + (c_1 d_2 - c_2 d_1) i \\ &\quad + (d_1 b_2 - b_1 d_2) j + (b_1 c_2 - c_1 b_2) k \qquad (1)\end{aligned}$$

which is a full quaternion, so that multiplication of pure vectors does not result in a pure vector. (In much the same way, multiplication of pure imaginary numbers yields a real number.)

Two physicists, Gibbs and Heaviside, cut this Gordian knot by defining two products of the vectors (b_1, c_1, d_1) and (b_2, c_2, d_2). There was the *scalar* product (also known as 'dot' product)

$$(b_1, c_1, d_1) \cdot (b_2, c_2, d_2) = b_1 b_2 + c_1 c_2 + d_1 d_2$$

which, omitting the sign, corresponds to the first component in (1), and the *vector* product (also known as 'cross' product)

$$(b_1, c_1, d_1) \times (b_2, c_2, d_2) = (c_1 d_2 - c_2 d_1, d_1 b_2 - d_2 b_1, b_1 c_2 - b_2 c_1)$$

corresponding to the last three components in (1).

A roaring controversy ensued. Hamilton and Tait felt that an essential algebraic insight was being lost, that mathematical rigour was flying out the window, and that mere amateurs were attempting to take over the reins of advanced mathematics. Gibbs and Heaviside, for their part, saw their opponents as deliberate obscurantists. For them, vectors and scalars were quite distinct species, and even though you might mathematically justify their being added together, it showed lack of physical sense to do so. The eminent physicist Maxwell weighed in on their side with the grumble that he didn't see why he should yoke an ass to an ox to plough the furrow of Physics.

The participants indulged in marvellous invective, which makes splendid reading even today. Thus Heaviside:

> " 'Quaternion' was, I think, defined by an American school-girl to be an 'ancient religious ceremony'. This was, however, a

> complete mistake. The ancients – unlike Professor Tait – knew not, and did not worship Quaternions ... . A quaternion is neither a scalar, nor a vector, but a sort of combination of both. It has no physical representatives, but is a highly abstract mathematical concept."

Nowadays, vectors are quite mathematically respectable, and it takes some work to understand what all the fuss was about. The dispute was very largely over questions of notation, but other more metaphysical matters did keep intervening. The whole matter seems somehow so *passé*, so Victorian, that we tend to forget how recent it is. One of the lesser and later participants, C.E. Weatherburn, a professor at the University of Western Australia, died as recently as 1974.

The completeness of the victory won by Gibbs and Heaviside may be judged from the conversation at the start of this article and its dogmatic conclusion: *You can't add a scalar to a vector*!

For Hamilton and Tait, you not only could, but you *had to.*

* * * * *

LETTERS TO THE EDITOR

Inside the front cover of each issue of *Function* is an invitation to readers to send us letters, articles or problems for solution, and we are pleased when people respond. Recently, we received a contribution from one of our regular readers, Garnet J. Greenbury of Brisbane, in response to our front cover article in *Vol.* 18, *Part* 1 on Lockwood's Goldfish and negative pedals. He lists a number of examples of the negative pedals of various curves, and provides graphs showing how to construct them. Negative pedals of simple curves include some exotic species such as "Tschirnhausen's cubic" and the "cissoid of Diocles", as well as the more familiar ellipses, parabolas and hyperbolas. He also suggests a method for producing negative pedals by paper folding.

If you have comments to make on any of the articles that have appeared in *Function*, or just want to let us know what you think of the magazine, drop us a line!

COMPUTERS AND COMPUTING

Exact Arithmetic

Cristina Varsavsky

Performing basic arithmetic is a part of our everyday life for which we usually rely on calculators and computers. Useful as they are, they are limited in the size of the numbers they can handle. For example, if I multiply the numbers 123456789 and 9876 in my Canon F-800, it returns the approximate answer

$$1.219259248 \times 10^{12}$$

where the last three digits have been lost. Furthermore, I cannot enter integers longer than ten digits; I have to sacrifice precision when I operate with large integers.

The size of the integers that can be handled by calculators and computers is determined by their hardware. When we are dealing with larger integers, we need to use software rather than the in-built operations if we want to perform the calculations in exact form. As was mentioned in an earlier article (*Function*, *Vol.* 17, *Part* 4), computer algebra systems such as DERIVE, MATHEMATICA, THEORIST, MAPLE and others, have the capability of performing exact calculations; they are written in highly sophisticated programming languages to achieve this.

This article is about designing simple procedures to perform the four basic operations with integers: addition, subtraction, multiplication and division. We will apply algorithms we are familiar with, the same ones we use when we have to perform the calculations by hand, and we will implement them in the very simple programming language QuickBasic.

First, the digits of the numbers involved will be stored in arrays, which in our programs will be set to have dimension 20, but you can change this to any size or even modify the program to use \$DYNAMIC dimension. Since we always start with the unit digits when we add, subtract or multiply, the digits will be stored in reverse order, meaning that the first value in the array represents the units, the second the tens, and so on. For example, the number 12345 would be stored in the array named **a** as follows:

$$\mathbf{a}(1)=5,\ \mathbf{a}(2)=4,\ \mathbf{a}(3)=3,\ \mathbf{a}(4)=2,\ \mathbf{a}(5)=1,\ \mathbf{a}(6)=\mathbf{a}(7)=\ldots=\mathbf{a}(20)=0$$

To simplify our notation we will write this as

$$(5,4,3,2,1,0,0,\ldots,0)$$

Let us work through the algorithm for the addition of two numbers which are stored in the arrays **a** and **b**, where

$$\mathbf{a} = (\mathbf{a}(1),\ \mathbf{a}(2),\ \mathbf{a}(3),\ \ldots,\ \mathbf{a}(20))$$
$$\mathbf{b} = (\mathbf{b}(1),\ \mathbf{b}(2),\ \mathbf{b}(3),\ \ldots,\ \mathbf{b}(20))$$

We start by adding the units, that is $\mathbf{a}(1)$ and $\mathbf{b}(1)$, carrying a 1 if the sum is greater than 10. Then we add the tens and the carry, carrying a 1 if necessary. We proceed in this fashion till we reach the last non-zero entry in the array representing the larger number.

The following QuickBasic program reads the two numbers, adds them, and displays the result on the screen:

```
REM Initialise the array to store 20 digits
size = 20
DIM a(size), b(size), x(size) AS INTEGER
a$ = "first": GOSUB reading: lengtha = length
FOR i = 1 TO size: a(i) = x(i) : NEXT i
a$ = "second": GOSUB reading: lengthb = length
FOR i = 1 TO size: b(i) = x(i) : NEXT i

GOSUB addition
GOSUB display
END

display:
REM This is to display the answer
PRINT : FOR i = length TO 1 STEP -1: PRINT x(i);
NEXT
RETURN

reading:
FOR j = 1 TO size: x(j) = 0: NEXT
PRINT "Enter the number of digits for the "; a$; " number: ";
INPUT length
```

```
PRINT : PRINT "Enter the "; a$; " number by
inputting a digit for each question mark."
FOR j = 1 TO length: INPUT x(length - j + 1):
NEXT
RETURN

addition:
IF lengtha >= lengthb THEN length = lengtha
ELSE length = lengthb
carry = 0
FOR i = 1 TO length
        sum = a(i) + b(i) + carry
        carry = INT(sum/ 10)
        x(i) = sum MOD 10
NEXT
IF carry = 1 THEN
        x(length + 1) = carry: length = length + 1
ENDIF
RETURN
```

Note that the array **x** is used as an auxiliary array and also to store the addition. The purpose of the **display** procedure is to output the result with the digits in the right order.

In the case of subtraction we do not carry a 1, but we may need to "borrow" a 1 which must be "paid back" in the following step. The implementation of this algorithm is left to the reader. Later, a modified version of it will be used as part of the division algorithm.

For multiplication you will have to recall the technique of long multiplication, which is illustrated in the following example where we multiply 623 by 97:

$$\begin{array}{r} 623 \\ \times 97 \\ \hline 4361 \\ 5607 \\ \hline 60431 \end{array}$$

The procedure could be described as follows. We start with the units of the second number, 7 in our case, and we multiply it by each digit in the first number. Now, the 3 contributes to the units of the product, the 2 to the tens, and the 6 to the hundreds. When we continue with the tens digit, 9, in the second number, the 3 will contribute to the tens of the product, the 2 to the hundreds, and the 6 to the thousands. In general, when we multiply **a**(i) and **b**(j), this product contributes to the position $i+j-1$ in the product **x**. Since we store only one-digit numbers in each position, we may need to carry to the next position, $i+j$.

The algorithm described above is implemented in the following subroutine called **product**, which you can insert and call in the previous program. Note that now it is important to initialise the array **x** where the product is going to be placed. Observe also that the upper limits in the FOR ... TO ... loops are one more than the length of each number. Can you see why?

```
product:
FOR i = 1 TO lengtha + lengthb: x(i) = 0: NEXT
FOR i = 1 TO lengtha + 1
          carry = 0
          FOR j = 1 TO lengthb + 1
               aux1 = a(i) * b(j) + carry
               aux2 = x(i + j - 1) + aux1 MOD 10
               x(i + j - 1) = aux2 MOD 10
               x(i + j) = x(i + j) + INT(aux2 / 10)
               carry = INT(aux1 / 10)
          NEXT j
NEXT i
IF x(lengtha + lengthb) = 0 THEN
     length = lengtha + lengthb - 1
ELSE
     length = lengtha + lengthb
RETURN
```

Now the program with this subroutine can give us an exact answer for the product of 123456789 and 9876: it is 1219259248164.

Finally, we arrive at division, which we will perform using the long division algorithm. Do you still remember it? Here is an example:

Divide 9562 by 23

```
     415
 23 |9562
    92
    --
     36
     23
    ---
     132
     115
    ---
      17
```

So, what is the procedure we followed to arrive at the quotient 415 and the remainder 17? We start by taking as many digits of the dividend, 9562, as the number of digits in the divisor 23. Then we work out the number of times 23 goes into 95. This gives us 4, the first digit of the quotient, and a remainder of 3. To obtain the second digit of the quotient we multiply this remainder by ten and add the next unused digit of the dividend, namely 6. We repeat the procedure with this new number 36 and the divisor 23 till all the digits of the dividend have been used.

The algorithm is implemented in the following subroutine which you can append to the original program, making the appropriate adjustments. The dividend is placed in the array **a** and the divisor in the array **b**. The quotient and remainder are stored in the arrays named **q** and **r** respectively. Initially, the remainder contains the first digits of the dividend, as many as the length of the divisor. The inner WHILE ... WEND structure counts, through the variable **counter**, the number of times we need to subtract the divisor **b** from **r** to get a negative number; then **b** is added back to **r**. This is followed by a shift of one place up in the array **r** and the addition of one more digit from **a**. These steps are repeated till all digits from **a** have been used. The outer WHILE ... WEND structure controls this through the variable **number**.

```
division:

FOR i = 0 TO lengthb - 1
    r(lengthb - i) = a(lengtha - i)
NEXT

number = lengthb
```

```
WHILE number < lengtha + 1
    counter = 0: borrow = 0

    REM Subtract the divisor repeatedly
    WHILE borrow = 0
        counter = counter + 1: borrow = 0
        FOR i = 1 TO lengthb + 1
            aux1 = r(i) – b(i) + borrow
            borrow = 0
            IF aux1 < 0 THEN
                borrow = -1
                aux1 = aux1 + 10
            END IF
            r(i) = aux1 MOD 10
        NEXT

    WEND

    q(lengtha – number + 1) = counter – 1

    REM Add the divisor back to the remainder
    aux2 = 0
    FOR i = 1 TO lengthb + 1
        aux1 = r(i) + b(i) + aux2
        aux2 = INT(aux1 / 10)
        r(i) = aux1 – aux2 * 10
    NEXT
    REM Shift values in the remainder one place up
    FOR i = 0 TO lengthb – 1
        r(lengthb – i + 1) = r(lengthb – i)
    NEXT
    REM and take one more digit from the dividend
    r(1) = a(lengtha – number)
    number = number + 1

WEND

PRINT : PRINT " The quotient is = ";
```

```
FOR i = lengtha – lengthb + 1 TO 1 STEP -1
    PRINT q(i);
NEXT
PRINT : PRINT " The remainder is = ";
FOR i = lengthb + 1 TO 2 STEP -1:
    PRINT r(i);
NEXT

RETURN
```

As has already been pointed out, the algorithms presented here are the ones we use when performing calculations by hand. Although good and fast enough for our use, these are not the most efficient algorithms for performing integer arithmetic. The efficiency is measured by the relationship between the input size, that is, the length of the numbers involved, and the number of operations and memory used to perform the desired calculation. One efficient way of handling big integers was described in the article "Modular Arithmetic Keeps the Numbers Small", *Function*, *Vol.* 17, *Part* 1, where modular or "clock" arithmetic plays a crucial rôle.

* * * * *

MATHEMATICAL LIMERICK

Integral vee squared dee vee
From one to the cube root of three,
Times the cosine
Of three pi on nine
Is the log of the cube root of e.

Or, in symbols:

$$\left(\int_1^{\sqrt[3]{3}} \nu^2 \, d\nu\right) \cos \frac{3\pi}{9} = \ln \sqrt[3]{e}.$$

* * * * *

PROBLEM CORNER

SOLUTIONS

PROBLEM 18.1.1 by K.R.S. Sastry, Addis Ababa, Ethiopia

Through a fixed point K a variable line is drawn to cut the parabola $y = x^2$ in the points P and Q. Let R be the midpoint of the chord PQ of the parabola. Find the locus of R.

SOLUTION by K.R.S. Sastry (modified by the editors)

Let the co-ordinates of K be (m, n). Let the equation of the line through K, P and Q be $y = bx + c$. Then $n = bm + c$, so the equation of the line can be written $y = bx + n - bm$. Let the co-ordinates of P and Q be (x_1, x_1^2) and (x_2, x_2^2) respectively. Then x_1 and x_2 are the two (real) roots of the quadratic equation $x^2 - bx + (bm - n) = 0$. Upon writing $x^2 - bx + (bm - n) = (x - x_1)(x - x_2)$, expanding the factors, and equating the coefficients of x, we obtain:

$$x_1 + x_2 = b.$$

Let R be the midpoint of PQ, with co-ordinates (x, y). Then:

$$\begin{aligned} x &= \frac{x_1 + x_2}{2} = \frac{b}{2} \\ y &= \frac{x_1^2 + x_2^2}{2} \\ &= \frac{1}{2}(bx_1 + n - bm + bx_2 + n - bm) \\ &= \frac{1}{2}b(x_1 + x_2) - bm + n \\ &= \frac{1}{2}b^2 - bm + n \end{aligned}$$

Eliminating b to obtain an equation connecting x and y, we obtain the equation of the locus:

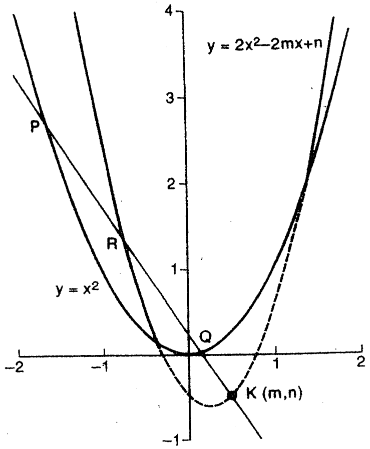
$$y = 2x^2 - 2mx + n \tag{1}$$

Equation (1) suggests that the locus is a parabola, but we need to be more careful. If K is *inside* the parabola $y = x^2$, then the locus is the complete parabola described by Equation (1). If K is *outside* the parabola

$y = x^2$, however, the locus is just the part of the parabola in Equation (1) that falls inside the parabola $y = x^2$. In order to ensure that the condition $y > x^2$ is satisfied in this case, we must impose the restriction:

$$x < m - \sqrt{m^2 - n} \quad \text{or} \quad x > m + \sqrt{m^2 - n}$$

Geometrically, the locus has two branches, and R jumps from one branch to the other as the line passes through the vertical. Each branch starts at a point of contact of a tangent from K to the original parabola $y = x^2$, and extends to infinity, as shown in the diagram.



It is interesting to note that K is a point on the parabola described by Equation (1).

PROBLEM 18.1.2. The polynomial factorisation shown below was written down so hastily that most of the digits are illegible.

$$x^2 + *x - *1 = (x + **)(x - *)$$

(Each asterisk denotes an illegible digit.) How should it read?

SOLUTION

Write

$$x^2 + *x - *1 = (x + a)(x - b)$$

where the integers a and b are such that:

(i) a has two digits, b has one;

(ii) the difference $(a - b)$ is a single-digit number;

(iii) the product ab is a two-digit number ending in 1.

The case $b = 1$ is easily ruled out, since (ii) could not be satisfied. It is now a simple matter to list the non-trivial factorisations ab of each two-digit number ending in 1. Of these, only $91 = 13 \times 7$ satisfies (i) and (ii), so the answer must be $a = 13$, $b = 7$, giving

$$x^2 + 6x - 91 = (x + 13)(x - 7).$$

PROBLEMS

Just for starters, here's a quick and easy problem to test your ability to visualise in 3-D. See if you can answer it without resorting to experiment!

PROBLEM 18.3.1 A man's shirt is normally buttoned up with the left side overlapping the right side. If a man puts his shirt on inside-out and buttons it up, which side will be outermost?

And now for one more mathematical problem ...

PROBLEM 18.3.2 Find the unique 5-digit number which, when multiplied by 4, yields the number formed by writing the digits of the original number in the reverse order.

* * * * *

Correction

We aim to make each issue of *Function* as free of errors as is humanly possible. Nevertheless, despite our best efforts, mistakes and misprints still occur. In the previous issue of *Function*, the name of the Polish mathematician Sierpinski was wrongly spelt in three places, leaving readers in doubt as to the correct spelling. So, in case anyone is still wondering, the spelling shown above is the correct one.

* * * * *

Published by Monash University Mathematics Department