Function

Founder Editor G. B. Preston

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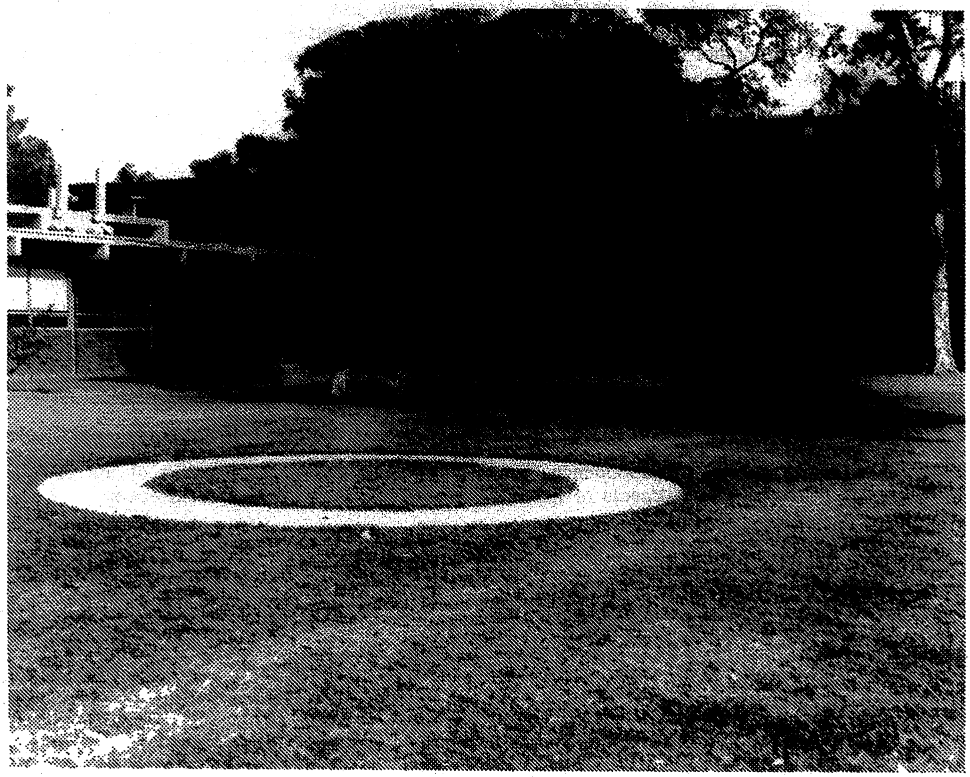

A SCHOOL MATHEMATICS MAGAZINE

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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FUNCTION

Volume 17 *Part 4*

(Founder editor: G.B. Preston)

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THE FRONT COVER†

Michael A.B. Deakin, Monash University

Our cover picture shows a new sculpture recently constructed on Monash University's Clayton campus, where it is to be found north of the first-year Chemistry laboratories and (appropriately) close to the new Mathematics Learning Centre. The sculpture comprises a concrete annulus and a turf mound in the shape of a paraboloid of revolution. (That is the shape formed by a parabola rotated about its axis. See Figure 1.)

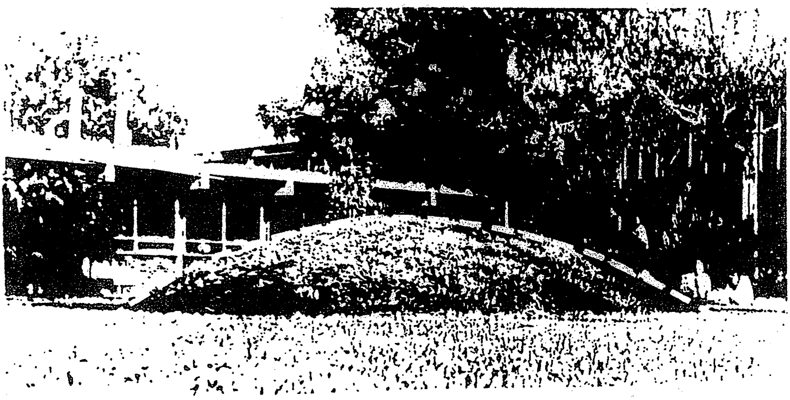

Figure 1

It is titled "Retrospective 1970-1993" (for reasons to be given below) and is the work of the British-born Melbourne sculptor Clive Murray-White.

In the late 1960s, there was much talk of having a major sculpture on the campus, and various plans were made and ideas mooted. In 1968, Murray-White produced a work, "Domes", which eventually was sited on the same lawn as the one where *Retrospective 1970-1993* now stands.

Domes comprised two aluminium shells in the shape of paraboloids of revolution. They caused great controversy when first produced. This arose mainly on two grounds:

(a) their supposed resemblance to a pair of breasts,
(b) they were *objets trouvés* (found objects) purchased by Murray-White from the (then) PMG, who used them as telecommunication antennae.††

† Thanks to Zora Stanhope of the Monash University Gallery for background details and to Steve Morton, who took the photographs.

†† The exhibiting of *objets trouvés* as art goes back to the anarchistic French artist Marcel Duchamp and the Dada movement.

Despite objections, *Domes* was installed in 1970, and graced the lawn for many years. Sadly the sculpture became a target for vandals until eventually repair became impractical and the paraboloids were removed in 1993. *Retrospective 1970-1993* evokes their memory and mourns their passing.

The paraboloid is used as the shape of an antenna because of a remarkable property: it concentrates parallel incoming rays into a single focal point. (See *Function, Vol.* 16, *Part* 4).

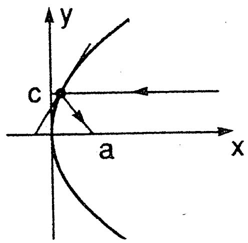

Figure 2

Figure 2 shows a cross-section of the paraboloid – i.e. a parabola. This parabola has been so aligned that its formula is

$$y^2 = 4ax.$$

The focal point is at $(a, 0)$. Imagine a ray entering the parabola at a height $y = c$ (say). This will encounter the parabola at the point $\left(\frac{c^2}{4a}, c\right)$. The slope at this point (the slope of the tangent) is $2a/c$ and thus the incident ray makes an angle θ with the tangent, where

$$\tan\theta = \frac{2a}{c}\,.$$

The reflected ray must also make an angle θ with the tangent and thus (see Figure 3) it makes an angle 2θ with the horizontal. Its slope is thus $\tan 2\theta$, i.e.

$$\frac{2\tan\theta}{1-\tan^2\theta}\,,$$

i.e. $4ac/(c^2 - 4a^2)$.

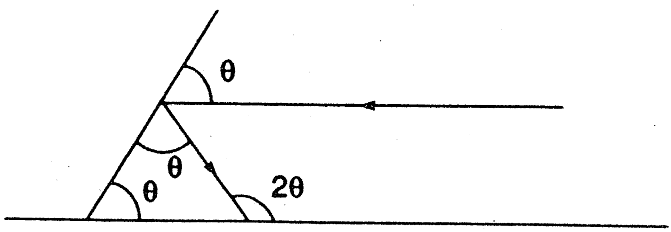

Figure 3

Thus the reflected ray has this slope and it passes through the point $\left(\frac{c^2}{4a}, c\right)$. Its equation is found to be

$$y = \frac{4acx}{c^2-4a^2} - \frac{4a^2c}{c^2-4a^2}$$
$$= \frac{4ac(x-a)}{c^2-4a^2}$$

and this (whatever the value of c) passes through the focal point $(a, 0)$.

Parabolic dish reflectors are used to gather and focus radio waves (in radio-astronomy, but also as receptors for, e.g., Sky Channel TV). Long cylinders with parabolic cross-sections are used in some solar heaters. We also use paraboloids to *send out* beams of light from a point source. Many torches incorporate them, as do car headlights. The cylindrical version was once very common in the old bar radiators.

Mr Murray-White's new sculpture serves to remind us of all this, standing as a monument to the earlier *Domes* whose connection to such a world was even more immediate.†

* * * * *

VIVE VIÈTE!

Richard Whitaker, 4 Gowrie Close, St. Ives, NSW

1. Introduction

1993 is a very special year on the mathematical calendar. It is the 400th Anniversary of the publication of Viète's equation [1]

[1] $$\frac{2^{1/2}}{2} \cdot \frac{(2^{1/2}+2)^{1/2}}{2} \cdot \frac{((2^{1/2}+2)^{1/2}+2)^{1/2}}{2} \cdots = \frac{2}{\pi}.$$

Representing an early example of an infinite product, it was used by Viète to calculate π to an accuracy of ten decimal places. François Viète, born in Fontenay-le-Compte, France, in 1540, was by profession a lawyer, but it was his prodigious talent as a mathematician which attracted the attention of the European academic community in the latter half of the 16th century. One of his more celebrated achievements was the discovery of the key to a Spanish cipher, consisting of more than 500 characters, which enabled the French to decode captured Spanish military communiqués. Phillip II of Spain was so outraged when he discovered that his secret mail was being perused, he formally complained to the Pope that the French were using "sorcery" against him.

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† For a history of the early controversy surrounding *Domes*, see F.W. Kent and D.D. Cuthbert (eds.), *Making Monash, A Twenty-Five Year History*, pp. 92-94.

Viète has been called the "father of modern algebraic notation" [2], and of his *ISAGOGE IN ARTEM ANALYTICUM* (1691), it has been said that the only significant difference between this work and a modern school algebra book is the absence of equality signs. In addition, he produced an advanced work on algebraic geometry (*RECENSIO CANONICA EFFECTIANUM GEOMETRICARUM*), published methods for the general resolution of equations of the second, third and fourth degree, and discovered the formula for the sine of a multiple angle.

His most famous work, the so-called "Viète equation", would, from a mathematician's viewpoint, represent the best day's work ever produced by a lawyer, and gives a way to calculate π on a modern computer.

2. The Equations

Let $0 \le x_0 \le 1$; then we may find θ_0 $(0 \le \theta_0 \le \frac{\pi}{2})$ so that $x_0 = \cos\theta_0$. Let $\theta_1 = \frac{1}{2}\theta_0$ and define x_1 as $\cos\theta_1$ and so on. In general

$$x_n = \cos\theta_n = \cos(2^{-n}\theta_0). \tag{1}$$

From this it follows that

$$x_{n-1} = \cos\theta_{n-1} = \cos(2\theta_n) = 2\cos^2\theta_n - 1 = 2x_n^2 - 1. \tag{2}$$

Thus

$$x_n = \sqrt{\frac{1+x_{n-1}}{2}}\,. \tag{3}$$

Differentiating Equation (2), we find

$$\frac{dx_n}{dx_{n-1}} = \frac{1}{4x_n} \tag{4}$$

and so (applying the chain rule)

$$\begin{aligned}\frac{dx_n}{d\theta_0} &= \frac{dx_n}{dx_{n-1}}\cdot\frac{dx_{n-1}}{dx_{n-2}}\cdot\ldots\cdot\frac{dx_1}{dx_0}\cdot\frac{dx_0}{d\theta_0}\\ &= -\frac{1}{4x_n}\cdot\frac{1}{4x_{n-1}}\cdot\ldots\cdot\frac{1}{4x_1}\sin\theta_0\\ &= -\frac{1}{4x_n}\cdot\frac{1}{4x_{n-1}}\cdot\ldots\cdot\frac{1}{4x_1}\sqrt{1-x_0^2}.\end{aligned} \tag{5}$$

But we may also use Equation (1) to compute $\frac{dx_n}{d\theta_0}$ and find

$$\frac{dx_n}{d\theta_0} = -2^{-n}\sin(2^{-n}\theta_0)$$

and if n is large, $2^{-n}\theta_0$ will be small and so $\sin(2^{-n}\theta_0) \approx 2^{-n}\theta$. Thus

$$\frac{dx_n}{d\theta_0} \approx -2^{-2n}\theta_0. \tag{6}$$

Equations (5), (6) may now be combined to give

$$x_n \cdot x_{n-1} \cdot \ldots \cdot x_1 \approx \sqrt{1-x_0^2}/\theta_0. \tag{7}$$

The special case $x_0 = 0$ gives $\theta_0 = \frac{\pi}{2}$ and you may like to check via Equation (3) that

$$x_1 = \frac{\sqrt{2}}{2},\ x_2 = \frac{\sqrt{(2+\sqrt{2})}}{2},\ \text{etc.}$$

This special case is Viète's product.

You may like to write a computer program to calculate π in this way.

References

[1] Cadwell, J.H. (1966), *Topics in Recreational Mathematics*, Cambridge University Press, p.154.

[2] *ENCYCLOPAEDIA BRITANNICA* (1963), Vol. 23, p. 145.

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THE TANGENT RULE

We are all familiar with the sine rule for a triangle ABC. This states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \tag{1}$$

where a is the length of the side BC, b that of CA, c that of AB, and R is the radius of the circumcircle.

Similarly the cosine (or cos) rule

$$c^2 = a^2 + b^2 - 2ab\cos C \tag{2}$$

(and its permutations) is standard fare.

But there is also a tan rule, which is usually not taught. It says:

$$\frac{a+b}{a-b} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)} \tag{3}$$

(and again there are permutations).

The proof is not difficult. Here is one approach.
By Equation (1),

$$\frac{a+b}{a-b} = \frac{2R(\sin A + \sin B)}{2R(\sin A - \sin B)} = \frac{2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)}{2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)} = \frac{\tan\left(\frac{A+B}{2}\right)}{\tan\left(\frac{A-B}{2}\right)}$$

as required.

Other approaches are possible. You may like to construct a geometric proof. [One way to do this is to consider (*à la* Problem 14.4.6, solved in this issue) the angles formed at the incentre I of the triangle ABC and at the "ex-centres" – corresponding circles, tangent to all three sides, but lying outside the triangle.]

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More on the Monash Sundial

The Monash sundial on the north wall of the Union Building on the Clayton campus was the subject of earlier articles in *Function* (*Vol.* 5, *Part* 5 and *Vol.* 14, *Part* 4). The sundial is an intricate one that tells the viewers both the time and the date. It was designed by the late Carl Moppert and was officially "opened" in 1980. It was recalibrated in 1990 and gives remarkably accurate readings of time and reasonably accurate readings for date – the date is more difficult to read than is the time.

Sundials such as this are known as "analemmic sundials". Horizontal analemmic sundials were the subject of an article in *Scientific American* (Dec. 1980, pp. 174-180) but a vertically mounted one, such as that at Monash, is more complicated because its details depend not only on latitude and longitude, but also on the orientation of the wall.

We had long believed at Monash that ours was in fact the only vertical analemmic sundial in the world, although some sundials at Greenwich observatory were once partial realisations of the same idea.

A recent letter from Peter Ransom of the British Sundial Society sheds some more light on the matter. The Greenwich sundials (there were eight partial vertical analemmic dials) were all designed by Dr. Tadeus Przypkowski and erected in 1968 (in one case) and 1970 (for the others). They began to deteriorate, however, and eventually were removed and disposed of.

However, there are other sundials similar to that at Monash. Mr. Ransom has sent pictures of a number of others. They are in Paris, though none is as detailed as the Monash one. Four are so inaccurate that they really don't count. These are the ones on the Hôpital Val de Grace, the Musée de Cluny, the Église St. Louis de la Salpêtrière, and the Orangerie de l'Hôtel de Sully.

The remaining ones are to be found at the Lycée Louis le Grand and Mr Ransom supplied pictures of three of the eight sundials they have there. These are all partial vertical analemmic dials and one has a full set of analemmas (looped curves from which to read off the time of day). Others have rough indications of date. None can really be said to have both.

So we are still happy to claim that our Monash sundial is unique.

Since the 1990 recalibration, checks have continued on the accuracy of the Monash sundial. On May 11th, 1993, it was 105 seconds slow, which is quite acceptable. It will never be perfectly accurate for any length of time because of the variation between leap and non-leap years. Taking leap years as occurring 25% of the time, we have a discrepancy of up to $\frac{3}{4}$ of a day in the length of the year. So we can expect timing errors of up to

$$\frac{3}{4} \times \frac{24}{365} \times 3600 \text{ seconds}$$

in a day. This gives a figure of about 3 minutes, though normally the error would not be so great. Probably the sundial is about as accurate as we can make it.

We remain very proud of the accuracy and the uniqueness of Carl Moppert's Monash Sundial.

* * * * *

HISTORY OF MATHEMATICS

EDITOR: M.A.B. DEAKIN

Days of Eggs and Bunnies

The date of Easter, unlike (say) that of Christmas, is not fixed but differs from year to year. Its religious origins in the Christian church relate it to the Jewish feast of the Passover, but it also acquired other overtones, corresponding to the celebration of the coming of the Northern Hemisphere spring at the time of the equinox – March 21st or 22nd. (Our Easter eggs and Easter bunnies are non-Christian fertility symbols related to this aspect of the matter.)

Now the Jewish Passover is calculated by means of a complicated formula related to the phases of the moon and in the early days of Christianity there was much debate as to how the date of Easter should be set. Various rules were adopted and different dates were in use in different places in the same year.

In 325 A.D. the council of Nicea made an edict to the effect that:

> The vernal equinox was taken to be March 21st. The full moon falling either on or next after that date was the paschal full moon. The next Sunday after the paschal moon was Easter Sunday.

Since that date the Western branches of the Christian church have been guided by this rule. The Eastern branches adhere still to different guidelines, and this explains why most of my neighbours, in the Greek-speaking area where I live, celebrate Easter on a different date from that used by the Roman and Protestant churches.

Now the formula or rule laid down by the Council of Nicea requires considerable astronomical expertise before it can be turned into an algorithm (nowadays a computer program). Let us see what is involved.

The equinox was set by decree to be March 21st. This is actually not strictly correct, but it does reduce the difficulty of the calculation. March 21st may fall on any day of the week. It advances one day for each ordinary year and two days for each leap year. This gave a 28-year cycle in the older Julian calendar (see *Function, Vol.* 17, *Part* 2) and gives a 400-year cycle in the newer and more accurate (though considerably more complicated) Gregorian calendar.

Then there is the moon. In the earlier article, I mentioned that there are, on average, 12.368267 full moons per year. This number may be approximated quite accurately as $12\frac{7}{19}$, so that $19 \times 12.368267 = 234.997 \ldots \simeq 235$, an integer. Thus, to high accuracy, the phases of the moon mesh with the usual, solar based, calendar on a 19-year cycle.

There are thus three principal cycles to consider:

(a) a 4-cycle caused by the 4-year cycle of leap years,
(b) a 7-cycle caused by there being 7 days in a week,
(c) a 19-year cycle caused by the lunar periodicity.

However, (b) was corrected in the Gregorian calendar and is now much more complicated, and (c) is not entirely exact.

If, however, we ignore these complications for the moment, we see that the calculation need not be too difficult. The three numbers 7, 4 and 19 are all relatively prime, so in the Julian calendar, there was a grand cycle of $4 \times 7 \times 19 = 532$ years and the pattern of the dates of Easter repeated after this period had elapsed.

The method whereby Easter was calculated for the Julian calendar used the dominical letters (explained in the previous article) which followed a 28-year cycle, and also took account of the 19-year lunar cycle (known as the Metonic cycle) as well. To do this latter, each year Y (say) was assigned a "golden number" by dividing Y by 19 and adding 1 to the remainder.

In the year 1800, the very great German mathematician Carl Friedrich Gauss expressed the calculation in very straightforward mathematical terms. Here we give his algorithm:†

Divide Y by 4 and call the remainder a.
Divide Y by 7 and call the remainder b.
Divide Y by 19 and call the remainder c.
(*) Divide $19c + 15$ by 30 and call the remainder d.
(‡) Divide $2a + 4b + 6d + 6$ by 7 and call the remainder e.

Then if $d + e \leq 9$, Easter Sunday is March $(22 + d + c)$, while if $d + e > 9$, Easter Sunday is April $(d + e - 9)$.

However, there are two problems with this simple (or at least relatively simple) process. First, the Julian calendar is not accurate over long periods and was badly in error after two such cycles following the Council of Nicea. Second, the Metonic cycle is also inexact. We dealt with the first of these problems in the previous article. Now we address the second.

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† We may express these instructions in terms of modular (or "clock") arithmetic. See *Function, Vol.* 17, *Part* 1.

As the Julian year was 365.25 days long, the Metonic cycle was $19 \times 365.25 = 6939.75$ days. This is a slight, very slight, underestimate, but over the course of several centuries, it builds up to over a day. Thus the full moons began to appear a day earlier than the calculations predicted. Clavius, in his report on the new calendar (the report that gave rise to the Gregorian calendar) decided to tackle this problem, as well as adjusting the length of the year. The "golden number" of the Julian calendar was replaced by a much more complicated concept, referred to as the "epact".

Rather than go into all this, let us see how Gauss's computation is to be modified for the Gregorian calendar. Go back to the algorithm and focus on the statements (*) and (‡). The first contains a constant, 15, which now must be replaced by a number we will here call P. The second contains a constant, 6, which now must be replaced by a number we will here call Q.

Clavius' idea was that the corrections should be applied for each century and thus we compute the numbers P, Q to apply to any given century. I will give a (relatively) simple way to do this.[†] It will use the notation $[x]$:

$$[x] \text{ is the largest integer less than or equal to } x.$$

Now call the hundreds figure of the year K, i.e.

$$K = [Y/100].$$

Now put

$$L = [K/4] \qquad M = [(K - 17)/25] \qquad N = [(K - M)/3].$$

Next:

Divide $15 + K - L - M$ by 30; the remainder will be P,
Divide $4 + K - L$ by 7; the remainder will be Q.

Thus for our century, $K = 19$ and we find $L = 4, M = 0, N = 6$. This gives $P = 24$, $Q = 5$. These values will also apply next century.

Let us now use our algorithm to compute the date of Easter for 1994.

$$\begin{array}{ll} 1994 = 498 \times 4 + 2 & a = 2 \\ 1994 = 284 \times 7 + 6 & b = 6 \\ 1994 = 104 \times 19 + 18 & c = 18 \\ 19c + P = 366 = 12 \times 30 + 6 & d = 6 \\ 2a + 4b + 6d + Q = 69 = 9 \times 7 + 6 & e = 6. \end{array}$$

So $d + e = 12$ (> 9) and so Easter for 1994 will fall on April 3rd.

There are, however, further complications that take the form of two exceptions to the rules given so far.[††]

[†] In an article published in *Function, Vol.* 9, *Part* 3, S. Rowe gave an algorithm equivalent to the Gregorian one, but combining the calculation of P, Q with that of a, b, c, d, e in an apparently different way. The difference however *is* only apparent.

[††] Rowe's version of the algorithm incorporates these automatically.

1. If $d = 29$ and $e = 6$, the predicted date of Easter is April 26th, but in fact it is celebrated on April 19th.

2. If $d = 28$, $e = 6$ and $c > 10$, the predicted date of Easter is April 25th, but in fact it is celebrated on April 18th.

An example of such an exception occurred in 1954. For that year, the value of d was 23 and of e, 6. However, in that year $c = 16$ and so the date of Easter was April 18th, not April 25th. The exact (Universal, or Greenwich Mean) time for the first full moon after March 21st in 1954 was 5:48 a.m. on Sunday April 18th. One would therefore expect that this date would be assigned to the Paschal full moon and so the *next* Sunday, April 25th, would have been Easter. However, at the precise time of full moon, there were parts of the world (from the east coast of the Americas west to the International Date Line) where the date was still April 17th. Thus this date was assigned to the Paschal full moon and the following day, a Sunday, became Easter.

Such events come about because the Metonic (19-year) cycle is not exact.

The Julian Easter followed a cyclic pattern lasting 532 years, as we have seen. The Gregorian Easter also follows a cyclic pattern, but the period is much longer. First consider the calculation of P. It may readily be shown that if, in the calculation of P, we replace K by $K + 3000$, we reach the same value of P. It may also (fairly readily) be shown that 3000 is the smallest number that will do this, whatever the value of K. Thus P follows a cyclic pattern taking up 3000 *centuries*.

The regular Gregorian calendar – i.e. without reference to Easter – follows a 400-year cycle as we saw in the previous article. Thus P takes 750 such cycles to repeat.

Q takes 7 400-year cycles to repeat, but this turns out not to be important. This is because the basic 4-century Gregorian unit contains an exact number of weeks, so the 7-cycles involved in the calculation of b, Q do not affect the overall period.

The divisor 19, however, is a different story. 19 is relatively prime to 3000 and also to 146097, the number of days in 400 Gregorian years. The other divisors involved (4, 30 and 100) are all factors of 3000 and so need be considered no further. We thus have a cycle that lasts 3000×19 centuries, i.e. 5 700 000 years.

This is, of course, a purely notional figure. The Gregorian calendar itself is going to need adjustment long before this time has elapsed. But beyond this, it may well be that humankind itself will not last such a length of time.

Nor need we forever adhere to the Nicean formula which led to all this complication. From time to time, there are calls for a fixed date for Easter – or at least a fixed Sunday nearest some date or other. These reforms were given a limited endorsement by the Second Vatican Council, but their implementation is not an immediate prospect.

References : Again, the *Encyclopedia Britannica* and the *Astronomical Ephemeris* (referred to in *Vol.* 17, *Part* 2) are the most accessible sources. Since writing that column, I have had a chance to see Alexander Philip's *The Calendar: Its History, Structure and Improvement*. (Melbourne University's Baillieu Library holds a copy.) Chapter XV of this work gives a good account of the date of Easter, with further (though not very accessible) references. The version of the algorithm given here is taken from *The Argus Students' Practical Notebook*, *Vol.* 3 and it would seem to be based (possibly at some remove) on the discussion in an earlier edition of the *Britannica* (the 11th, published in 1910). The notation and terminology used there have been greatly simplified. Rowe's

Function article was reprinted from a local journal in the Belgrave area, but neither version provides a source for the modified algorithm he employs. It was reprinted in the 1990 anthology *Composite Function* (published by the Mathematical Association of Victoria). Rowe was the only author the editors of that work were unable to contact.

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PROBLEMS AND SOLUTIONS

In this issue, we continue the process of publishing solutions to long-outstanding problems.

SOLUTION TO PROBLEM 14.4.6

The problem (correcting a minor misprint) read: Let I be the incentre of triangle ABC, and let A', B', C' be the circumcentres of triangles IBC, ICA, IAB respectively. Prove that the circumcircles of ABC, $A'B'C'$ are concentric.

Solution. The problem may be viewed as a continuing exercise in the application of the sine rule. With the usual notation ($BC = a$, etc.) we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

where R is the circumradius of triangle ABC.

Now consult Figure 1.

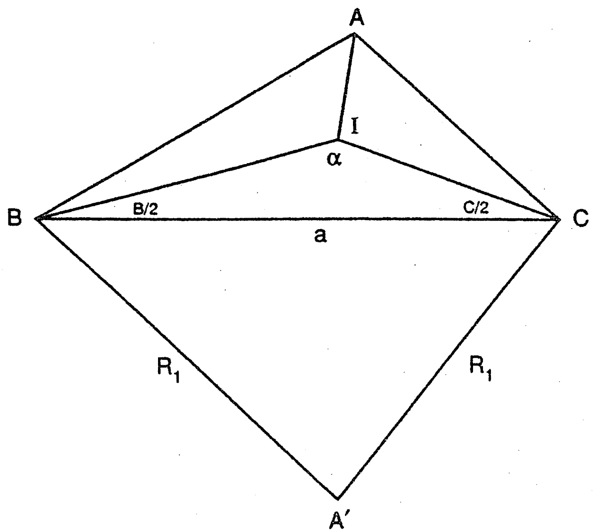

Figure 1

A' is the circumcentre of triangle IBC and so if R_1 is the circumradius

$$2R_1 = \frac{a}{\sin\alpha}$$

where $\alpha = \angle BIC$. Now I lies on the lines bisecting $\angle B$ and $\angle C$. Thus $\alpha = \pi - \frac{1}{2}(B + C)$ and so

$$2R_1 = \frac{a}{\sin[\frac{1}{2}(B+C)]}\,.$$

Further note that because I lies on the circular arc BIC (not drawn) and A' lies at its centre, the (reflex) angle $BA'C = 2\alpha$. Thus the (non-reflex) angle $BA'C$ is $\pi - 2\alpha = B + C$.

Now consult Figure 2.

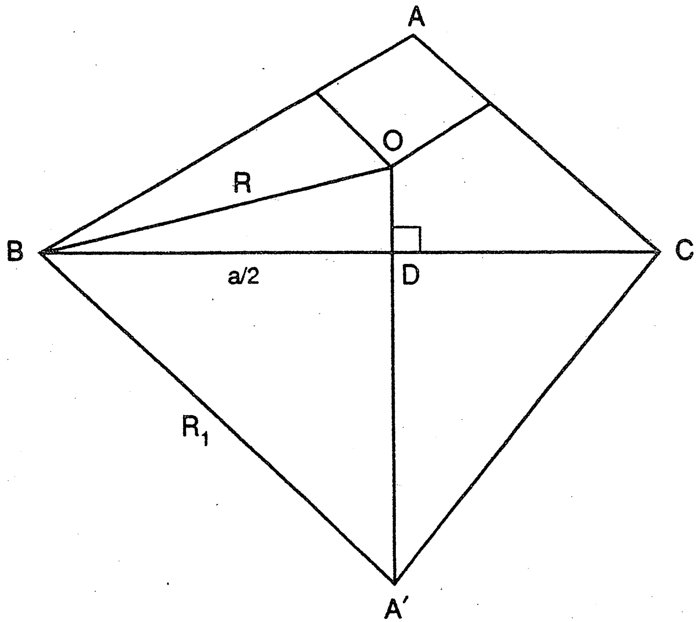

Figure 2

O is the circumcentre of triangle ABC and because $A'B = A'C$, A' lies on the perpendicular bisector OD of side BC. From previous deductions, $\angle BA'D = \frac{1}{2}(B + C)$. Thus

$$A'D = R_1 \cos[\tfrac{1}{2}(B + C)] = \tfrac{a}{2} \cot[\tfrac{1}{2}(B + C)]$$

$$= \frac{a}{2} \tan \frac{A}{2} \qquad \text{as } A + B + C = \pi$$

$$= R \sin A \tan \frac{A}{2}$$

$$= 2R \sin^2 \frac{A}{2} = R(1 - \cos A). \tag{1}$$

But now, by applying Pythagoras' Theorem to the triangle OBD, we find

$$OD = \sqrt{(R^2 - \frac{a^2}{4})} = R \cos A. \tag{2}$$

Thus we find

$$A'O = R' \text{ (say)} = AD + OD = R.$$

Similarly $B'O = C'O = R$.

Thus A', B', C' all lie on a circle, centre O and radius R. So triangle $A'B'C'$ has the same circumcircle as triangle ABC.

Note that this proves a slightly stronger result than was asked for.

SOLUTION TO PROBLEM 14.5.2

The problem read: Given a stack of pancakes of varying diameters, rearrange them into a stack with decreasing diameter (as you move up the stack) using only "spatula flips". With a spatula flip you insert the spatula and invert (i.e., turn upside down) the (sub)stack of pancakes above the spatula. Design an algorithm that correctly solves the pancake problem for a stack of n pancakes with at most $2n$ flips. Count exactly how many flips your algorithm uses in the worst case.

Solution (from Peter Grossman, Monash University): Let $f(n)$ denote the smallest number with the property that any stack of n pancakes can be put into the desired order with $f(n)$ or fewer flips. Then it is not difficult to see that $f(n) \leq f(n-1) + 2$, since we can flip a stack of n pancakes by first flipping with the spatula under the largest pancake (1 flip), then inverting the entire stack (1 flip), and finally rearranging the top $n-1$ pancakes into the required order ($f(n-1)$ flips).

It is obvious that $f(1) = 0$ and $f(2) = 1$, and this is sufficient to ensure that n pancakes can be rearranged in $2n-3$ flips (when $n > 1$) by applying the procedure described above. This answers the problem as posed, but it is interesting to try to discover whether the result can be improved upon.

It turns out that the inequality is the best result possible for some values of n but not for others. We can see that $f(3) = 3$ by observing that, if the stack of three pancakes starts with the smallest pancake on top and the largest in the middle, then it cannot be arranged into the required order in fewer than three flips. It may similarly be shown that $f(4) = 4$.

SOLUTION TO PROBLEM 15.1.10

It was required to show that no integers a, b, c and k can satisfy the equation

$$a^2 + b^2 + c^2 = 8k + 7.$$

Solution. Because the right-hand side of the equation is odd, at least one of a, b, c is odd. Suppose c is odd. Then $c = 2d + 1$ and we have

$$a^2 + b^2 + 4d^2 + 4d = 8k + 6 \qquad (*)$$

Now a, b must be either both even or both odd. If both are even, put $a = 2e$, $b = 2f$ and then (*) becomes

$$4e^2 + 4f^2 + 4d^2 + 4d = 8k + 6$$

which is impossible as 4 does not divide 6. Thus try $a = 2e + 1$, $b = 2f + 1$. In this case (*) becomes

$$e(e+1) + f(f+1) + d(d+1) = 2k + 1.$$

But now $e(e+1)$ is the product of two consecutive numbers and so is even. Similarly for $f(f+1)$, $d(d+1)$. We thus have a contradiction as $2k + 1$ is odd.

There are therefore no solutions.

SOLUTION TO PROBLEM 15.2.4

This problem asked, in essence, how many "derangements" there are of n elements – in how many ways can n elements be permuted so as to leave *no* element in its original position.

The number is U_n where

$$U_n = n![1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n\frac{1}{n!}].$$

Essentially one considers all permutations ($n!$) and then subtracts out those leaving one element intact, $(n-1)!n$, but this is an over-estimate because it counts twice those that leave two elements intact, etc.

The problem is a standard one and is to be found in (e.g.) *Combinatorial Theory: An Introduction* by A.P. Street and W.D. Wallis (p. 114).

We have $U_2 = 1$, $U_3 = 2$, $U_4 = 9$, etc.

SOLUTION TO PROBLEM 15.2.5

The problem considered n points on a circle, joined in all possible ways. How many regions is the circle divided into?

Again the problem is a standard one. The answer is $n + \binom{n}{4} + \binom{n-1}{2} = f(n)$, say, and this follows from an analysis which may be found in (e.g.) Ross Honsberger's *Mathematical Gems*, Chapter 9. For small values of n, we need the interpretation $\binom{m}{r} = 0$ if $m < r$. We then find

n	1	2	3	4	5	6	7	...
$f(n)$	1	2	4	8	16	31	57	... ,

a rather surprising pattern, because many people would expect $f(6) = 32$, etc.

The matter has been discussed before in *Function* – see *Vol.* 6, *Part* 5, p. 7 and *Vol.* 7, *Part* 1, pp. 24-25.

SOLUTION TO PROBLEM 15.2.6

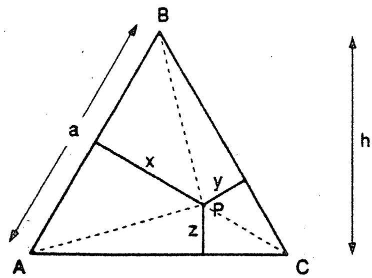

Figure 1

ABC (Figure 1) is an equilateral triangle and P a point inside it. x, y, z are the perpendicular distances from P to the sides. If a is the length of the side of the original triangle, then $\frac{1}{2}ax$ is the area of the triangle ABP.

Thus this total area is $\frac{1}{2}a(x+y+z)$, but as the total area is also $\frac{1}{2}ah$, we have

$$x + y + z = h$$

which is the relation the problem asked for.

The problem also asked about generalisations. A number are possible. The method of proof used is that employed in the proof of a standard theorem: the inradius of a triangle is equal to the area divided by the semi-perimeter. x, y, z may also be related to *barycentric coordinates*, useful for certain specialised purposes. (See the cover story of *Function*, *Vol.* 6, *Part* 5.)

SOLUTION TO PROBLEM 15.2.7

The problem read: A triangle T is drawn on a square grid (graph paper) with vertices at points of the grid, but no other grid-points are in or on it. Show that the area of T is $\frac{1}{2}$.

Solution. The result follows readily from Pick's theorem (*Function*, *Vol.* 8, *Part* 1, pp. 4-10). According to this theorem, the area A of a figure drawn on a grid is $i + \frac{b}{2} - 1$ where i is the number of internal gridpoints and b the number on the boundary. Here we have $i = 0$, $b = 3$ and the result follows.

SOLUTION TO PROBLEM 15.2.8

This problem read: Let S be a set of n points in the plane such that any two of them are at most 1 unit apart. Find the radius of the smallest circular disc which will cover S.

Solution. We may so place the first three points that they form an equilateral triangle of side 1. See Figure 2. Let A, B, C be the vertices of the triangle. With centre A and radius 1 swing a circle. The arc of this circle that lies between B, C is shown in the figure. Similar arcs (not shown) may be drawn with B, C as centres. The figure formed by the overlap of the three circles (i.e. with the three arcs as boundaries) gives the region within which all other points of the set must lie.†

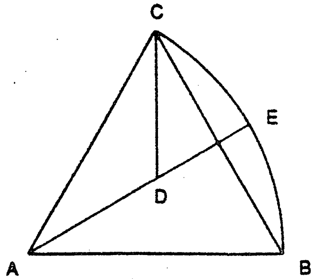

Figure 2

Now let D be the centre of the triangle ABC. With centre D and radius DA, swing a circle. The radius DA has length $\frac{1}{\sqrt{3}}$. Let E be the mid-point of the arc BC. The distance DE is $1 - \frac{1}{\sqrt{3}} \approx 0.422...$, which is less than DA. Thus a circle of radius $\frac{1}{\sqrt{3}}$ covers all the points we can put in the set.

SOLUTION TO PROBLEM 15.3.1

This problem came in two versions, one easy, one hard. The hard version asked for a proof that $4^x + x^4$ was not prime if x was an integer greater than 1. We here solve this.

Solution (from H. Lausch, Monash University). Clearly the expression cannot be prime if x is even. If x is odd, then $4^x + x^4$ may be factorised as

$$(x^2 + 2^x + x.2^{(x+1)/2})(x^2 + 2^x - x.2^{(x+1)/2}).$$

The second factor may equal 1 only in the case $x = 1$.

* * * * *

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† This curve is interesting in its own right as a "curve of constant breadth". See (e.g.) E. Northrop's *Riddles in Mathematics*.

NUMBERS AND COMPUTERS

Cristina Varsavsky, Monash University

Natural numbers were created by humans to count objects. Later people learnt to add, subtract, multiply and divide pairs of natural numbers. Although some divisions were possible, new numbers had to be invented to give meanings to expressions like 7 ÷ 3. The invention of fractions was a major step in the development of mathematics. In the early days many strange practices were followed. The Babylonians considered only fractions whose denominators were 60 and the Romans only those with denominators 12. The Egyptians insisted that numerators must be one, writing $\frac{1}{2}+\frac{1}{4}$ instead of $\frac{3}{4}$. Negative numbers were used to some extent by the Arabs, Chinese and Indians, but they were fully accepted by all mathematicians at the beginning of the seventeenth century. The surprising discovery that rational numbers are not entirely sufficient for all practical purposes was a scientific event of the highest importance. The theory of irrational numbers had its origins in Euclid's *Elements* but became fully appreciated only in the late nineteenth century, after Cantor, Dedekind and Weierstrass had constructed a rigorous theory of irrational numbers.

Early systems of numeration were based on an "additive" principle. In the Roman symbolism, for example, they wrote LXXVIII to represent

fifty + ten + ten + five + one + one + one.

This system has the disadvantage that more and more symbols are needed to represent big numbers.

Today we use the *positional system* with base 10, where only ten symbols are used to represent any number, large or small. The invention of positional systems is attributed to the Sumerians and Babylonians and was developed by the Hindus. The use of ten as the base goes back to the beginning of civilisation, and is undoubtedly due to the fact that humans have ten fingers with which to count. The meaning of digits depends on their position. For example, in the number 375, 3 is in the position of the hundreds, 7 in the position of the tens and 5 is in the position of the unit. We can express this in expanded form as follows.

$$375 = 3\times10^2 + 7\times10^1 + 5\times10^0.$$

The numbers 5, 7 and 3 are the remainders left after successive divisions by 10:

$$375 = 37\times10 + 5$$

$$37 = 3\times10 + 7$$

$$3 = 0\times10 + 3.$$

The same applies to digits after the decimal point, but now they will be in the positions of tenths, hundredths, etc. For example, the expanded form of 0.1738 is

$$0.1378 = 1\times10^{-1} + 3\times10^{-2} + 7\times10^{-3} + 8\times10^{-4}.$$

The numbers 1, 3, 7 and 8 are the integer parts of successive multiplications of the decimal part by 10:

$$10\times0.1378 = 1 + 0.378$$

$$10\times0.378 = 3 + 0.78$$

$$10\times0.78 = 7 + 0.8$$

$$10\times0.8 = 8.$$

The same procedure could be followed to represent numbers in any base. For example, 3462 in the base 7 system, in which only the digits 0 to 6 are used, represents

$$3\times7^3 + 4\times7^2 + 6\times7^1 + 2\times7^0 = 1269 \text{ in base 10.}$$

We express this using subindices for the base:

$$3462_7 = 1269_{10}.$$

Similarly, the numbers 2, 6, 4 and 3 are the remainders of successive divisions by 7, starting with 1269:

$$1269 = 181\times7 + 2$$

$$181 = 25\times7 + 6$$

$$25 = 3\times7 + 4$$

$$3 = 0\times7 + 3.$$

Gottfried Leibniz (1646-1716) was fond of the binary system in which the base is 2 and which uses only the digits 1 and 0. He imagined that unity represented God and zero the void; that the Supreme Being drew all beings from the void, just as unity and zero express all numbers in his system of numeration.

When computers were invented, the binary system was the best one with which to represent numbers.† The reason is that the devices for storing data in a digital computer consist of memory elements called bits with two states, on and off, which correspond to 1 and 0. So numbers are stored as strings of ones and zeros. The number of bits available to store data in a computer may be huge, but it is finite, and not every number can be represented. The common lengths for a computer word are 16 or 32 bits, in which case the largest integers that can be stored are those represented by 15 ones and 31 ones†† respectively, which are equivalent to $2^{16} - 1$ and $2^{32} - 1$ in the decimal system. Therefore computers treat integers as a finite set.

The most widely implemented way to store real numbers in a computer is to approximate them with the finite set of floating-point numbers. To do this, each number is first represented in the binary system in normalised form. For example, 10101.11 is the equivalent, in binary, to the decimal number 21.75. The normalised form is obtained by

† The so-called "related bases" are also used: 8 for the octal system, and 16 for the hexadecimal.

†† One bit is usually used for the sign, 0 for positive numbers and 1 for negative numbers.

moving the "binary" point five places to the left and multiplying by 2^5:

$$10101.11 = 0.1010111\times2^5.$$

Each number is represented using a fixed number of bits for the exponent (5 in our case) and for the precision (1010111 in our case). In single precision, 8 bits are usually used for the exponent† with the remaining 23 bits for the precision. So the computer treats real numbers as a finite set and not every real number within that range can be represented.

Let us illustrate this with a toy computer which uses only three bits for the precision and 3 bits for the exponent. Since the exponent needs 1 bit for the sign, the smallest exponent is -3 $(= -11_2)$, and the largest is 3 $(= 11_2)$. The following table lists all the positive real numbers this computer can handle.

Normalised Binary Form	Decimal Expression	Normalised Binary Form	Decimal Expression
0.100×2^{-3}	0.0625	0.110×2^{0}	0.75
0.101×2^{-3}	0.078125	0.111×2^{0}	0.875
0.110×2^{-3}	0.09375	0.100×2^{1}	1
0.111×2^{-3}	0.109375	0.101×2^{1}	1.25
0.100×2^{-2}	0.125	0.110×2^{1}	1.5
0.101×2^{-2}	0.15625	0.111×2^{1}	1.75
0.110×2^{-2}	0.1875	0.100×2^{2}	2
0.111×2^{-2}	0.21875	0.101×2^{2}	2.5
0.100×2^{-1}	0.25	0.110×2^{2}	3
0.101×2^{-1}	0.3125	0.111×2^{2}	3.5
0.110×2^{-1}	0.375	0.100×2^{3}	4
0.111×2^{-1}	0.4375	0.101×2^{3}	5
0.100×2^{0}	0.5	0.110×2^{3}	6
0.101×2^{0}	0.625	0.111×2^{3}	7

Observe that these numbers are not equally spaced throughout the range.

† It is not the exponent that is stored in the computer, but the characteristic, which is calculated by adding $2^7 - 1$ (called the exponent bias) to the exponent.

The operations of addition, subtraction, multiplication and division are hardware implemented, that is, they are included in the computer circuitry to perform calculations faster. Given this discrete representation, computer arithmetic does not always coincide with standard arithmetic.

Since the set of numbers in computer arithmetic is finite, not every pair of numbers can be added together. Take (in our simple computer case) the numbers 3 and 5. As you can see from the table above, both numbers belong to the system but their addition does not, causing what is called "overflow". Furthermore, the associative and distributive laws for addition and multiplication fail. Consider, for example, the addition

$$0.75 + (0.875 + 0.875) = 0.75 + 1.75 = 2.5.$$

If we apply the associative law

$$(0.75 + 0.875) + 0.875 = 1.625 + 0.875 \neq 2.5$$

because 1.625 does not belong to the system and has to be replaced by either 1.5 or 1.75.

This non-standard arithmetic in the computer number system was a major problem to be sorted out by the developers of computer algebra packages. You may have already been exposed to some of these systems: DERIVE and MATHEMATICA seem to be the most widely used; MAPLE, REDUCE and others are used in research environments. These systems need to work with, and represent, the true numbers, not their approximations. A good test consists of trying to find a big integer, such as 60!. For example, while my calculator (which stores numbers and operates with them using the same principles as a computer) gives the approximate value 8.320987113×10^{81}, DERIVE outputs the exact value using 82 digits:

832098711274139014427634118322336438075417260636124595244927769640960000000000
00000

The maximum factorial that I can calculate with my calculator is 69! In DERIVE it will depend very much on the available memory. For example, the factorial of 100 is

933262154439441526816992388562667004907159682643816214685929638
952175999932299156089414639761565182862536979208272237582511852109
168640000000000000000000000000.

Another example is that of trying to represent the number π with as many digits as possible. A computer or a calculator will be limited by the range of numbers it can handle. So we get $\pi = 3.141592654$ with a calculator, but could get

3.1415926535897932384626433832795028841971693993751058209749445923078164062862
089

if we ask DERIVE to write it with 80 digits (and we could ask for many more!).

It is important to note that it is not only the final answer that matters, but also the intermediate calculations. This was seen in the article "Handling polynomials with a computer algebra system" (*Function* 16, *Part* 1) in which it was shown that, although the final greatest common divisor of two polynomials was simple, the intermediate steps to calculate it involved huge numbers.

So, how do computer algebra systems handle numbers when the discrete and limited computer representation is not good enough? They treat them as "lists" of digits, and when the length of the computer word is not enough, many lists are chained to represent the number. For example, the number 12379843529 is stored in a computer with word length 3 in 4 words:

→ | + 12 | → | 379 | → | 843 | → | 529 |

where the arrows mean that each list points to the following one. Addition, subtraction, multiplication and division of integers are software implemented and apply the same principles one learns in primary school. Fractions are represented as two integers, denominator and numerator, and they are, not replaced by floating numbers for that could lead to false results.

It seems the great mathematician Kronecker was right when he said: "God created the integers, the rest is the work of man".

* * * * *

LETTER TO THE EDITOR

A Standard Cubic

Richard Whitaker's article "Trigonometric Solutions to Quadratic Equations" (*Function, Vol.* 17, *Part* 3) and J.B. Henry's "A Classification of Cubic Polynomials" (*Vol.* 17, *Part* 2) reminded me of some algebraic theory I learned in school. To find the zeroes of a cubic polynomial it was first put into a standard form

$$x^3 + 3Hx + G = 0 \tag{1}$$

by suppressing the quadratic term in the way described by Henry.

The theory then considered the discriminant $G^2 + 4H^3$. If this was positive, then we formed

$$p^3 = \tfrac{1}{2}\{G + \sqrt{(G^2+4H^3)}\} \tag{2}$$

$$q^3 = \tfrac{1}{2}\{G - \sqrt{(G^2+4H^3)}\} \tag{3}$$

and p^3, q^3 were both real quantities. Thus p, q could be determined. In this event, the cubic (1) had one real root, whose value was $-(p + q)$.

However, if $G^2 + 4H^3$ was negative, we still had (2), (3) but p^3, q^3 were complex numbers and the problem arose as to how to find their cube roots. My school text (Durell and Robson's *Advanced Algebra, Volume 2*) wrote

$$p^3 = A + iB = r(\cos\theta + i\sin\theta) \tag{4}$$

$$q^3 = A - iB = r(\cos\theta - i\sin\theta) \tag{5}$$

and found the roots to be

$$-2r^{1/3}\cos\left(\frac{\theta}{3}\right),\ -2r^{1/3}\cos\left(\frac{\theta+2\pi}{3}\right),\ -2r^{1/3}\cos\left(\frac{\theta-2\pi}{3}\right),$$

all three of which are real. This seems similar to Whitaker's approach to the quadratic.

I tried to see that would happen if I avoided the trigonometric approach and wrote $G^2 + 4H^3 = -K^2$ so that

$$p^3,\ q^3 = \tfrac{1}{2}(G \pm iK). \qquad (6)$$

Then I looked for numbers $\alpha,\ \beta$ so that

$$(\alpha \pm i\beta)^3 = \tfrac{1}{2}(G \pm iK). \qquad (7)$$

Equating real and imaginary parts I found

$$\alpha^3 - 3\alpha\beta^2 = \tfrac{1}{2}G \qquad (8)$$

$$3\alpha^2\beta - \beta^3 = \tfrac{1}{2}K. \qquad (9)$$

Next I put $\alpha = \beta\gamma$ and $G = LK$. This gave a new cubic

$$\gamma^3 - 3L\gamma^2 - 3\gamma + L = 0 \qquad (10)$$

and I found that if I put $\gamma = L + \delta$, then a standard form resulted, namely

$$\delta^3 - 3(1 + L^2)\delta - 2L(1 + L^2) = 0. \qquad (11)$$

This was a new standard cubic and so I went about solving it as already described for (1). After going through all the algebra, I ended up with

$$y^3 - 3(1 + L^2)y + 2L(1 + L^2) = 0, \qquad (12)$$

which is the same equation as (11), except for the sign of L. There was no point continuing, because clearly each time the process was repeated either (11) or (12) would result.

This then is a standard form whose solution would enable the solution of all cubics whose discriminants are negative.

Kim Dean,
Union College, Windsor.

[*The fact that the determination of three* real *roots of a cubic requires* complex *algebra is what led to the introduction of the complex numbers in the first place. (See* Function, Vol. 3, Part 5, p. 11 *and* Vol. 5, Part 3, p. 3.*) Mr Dean's analysis would seem to show that the general case also necessarily involves either trigonometry or numerical analysis. Eds.*]

* * * * *

THE LAW OF CUBIC PROPORTIONS IN ELECTIONS

Ravi Phatarfod and Mary Constantinou, Monash University

Federal Election night, Saturday 13 March, 1993. We were in a group of people watching the election result telecast, mainly on Channel 2 but occasionally switching to Channel 9. By about 10.30 p.m. it was obvious that Labor had won the election; the question was, by how many seats? The Channel 2 computer expert mentioned that, at that stage, Labor had 51.5% of the two-party preferred vote and the Coalition 48.5%. Those amongst us who didn't take much interest in politics, and therefore were hopelessly ill-informed about elections, thought that the seats distribution in the House of Representatives would follow on the same lines as the two-party preferred vote. Thus, of the 144 seats available (of the 147 seats, election for one seat, Dickson, was postponed by five weeks, and two seats were won by Independents, Mack in North-Sydney and Cleary in Wills), the breakdown would be Labor: 74 ($144 \times 0.515 = 74.16$), Coalition: 70 ($144 \times 0.485 = 69.84$). There were others in the group who, having noted that in past elections a small difference in the two-party preferred vote had resulted in a much greater difference in the number of seats won, gave various estimates for the number of seats Labor would win, ranging from 75 to 85. None was aware of the statistical law connecting the proportion of seats won and the proportion of votes obtained by a political party. This law, called the Law of Cubic Proportion in election results, is as follows:

> *In a reasonably homogeneous country with a two-party political system, the number of seats won by a party is approximately proportional to the cube of the total number of votes cast for that party, i.e. the ratio of the number of seats won by the two parties is approximately equal to the cube of the ratio of the total number of votes cast for the two parties.*

Applying this law to the present case, we see that the number of Labor and Coalition seats won would be in the ratio $(51.5/48.5)^3 = 1.197$, yielding the result, Labor: 78 ($144 \times 1.197/2.197 = 78.46$), Coalition: 66 ($144 \times 1/2.197 = 65.54$). The postponed election for the seat of Dickson eventually resulted in a Labor win, and the final count of the 145 seats, ignoring Independents, was Labor: 80, Coalition: 65. On the other hand, the final vote-split at the end of the counting was 51.4% : 48.6% so that the law would give the ratio of the number of seats won as 1.1830, making the seat-split 79 : 66. Thus the law resulted in an error of one seat.

It should be mentioned here that, although technically Australia does not have a two-party system, the system of preferential voting effectively makes it so – barring some rare cases such as that of the election of two Independents in 1993. A vote which has, say, Australian Democrats 1, Labor 2 is effectively a vote for Labor, only temporarily postponed.

The law of cubic proportion was first suggested by the Rt. Hon. James Parker Smith in evidence before the Royal Commission on Systems of Elections, in 1909. This was in connection with elections to the House of Commons in the United Kingdom. Now, the Minutes of Evidence before a Royal Commission are not exactly favourite reading material for statisticians, or for that matter for anybody else, and this law wouldn't have become known to statisticians and others, had it not been for an article in the *Economist* newspaper of 7 January 1950. Subsequently, two eminent statisticians, M.G. Kendall and A. Stuart (1950), wrote at length on the phenomenon, gave the conditions for its validity, and showed that the law held for the U.K. House of Commons elections for 1935, 1945 and 1950, when the U.K. had effectively a two-party political system. In the present article, we apply the law to Australian Federal Elections.

The rationale behind the law is as follows. Suppose, for the election of 1993, the two-party preferred split was 51.4 : 48.6. In each electorate (the population of voters within the boundaries of the seat), there are Labor voters and Coalition voters, with the proportion of Labor voters varying from electorate to electorate. At one extreme, we could have the situation where each electorate is either full of Labor voters or full of Coalition voters. Since each electorate contains roughly the same number of voters, then 51.4% of the electorates would contain only Labor voters. Hence there would be 51.4% Labor seats and 48.6% Coalition seats, and out of 145 seats, we would have the result Labor 75, Coalition 70.

At the other extreme, we could have the case where in *every* electorate 51.4% of the voters were Labor voters. In this case, Labor would have won *every* seat, giving the election result Labor: 145, Coalition: 0.

The above two are extreme cases, and the true situation is somewhere in the middle. The proportion of Labor voters varies from electorate to electorate, mainly because voters with similar political views tend to live near one another. For example, for the 1990 Federal election, the proportion of Labor voters ranged from 70.8% in Chifley (NSW), to 26.2% in Mallee (Vic.). Now, it so happens that the proportion of Labor voters (per seat), say, over the total population of the seats (148 seats in the 1990 election) has an approximately normal distribution. This may not come as much of a surprise; however, what is interesting is that the standard deviation σ of this normal distribution is around 0.11 – 0.13 and remains in that region from election to election. Of course, the mean of the distribution changes from election to election, and for a particular election, it is nearly equal to the final two-party preferred proportion. Figure 1 gives the histogram of the proportion of Labor voters in the 147 seats at the election of 1984.

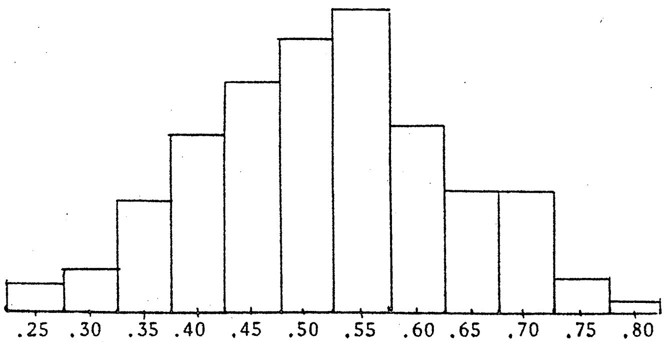


Figure 1

The above phenomenon has been observed for Australian federal elections since 1949, for five New Zealand elections before 1950 and, as mentioned before, for the U.K. elections of 1935, 1945 and 1950. Kendall and Stuart have given the details for these elections – the value of σ observed was 0.133, 0.135, 0.138 for the elections of 1935, 1945 and 1950 respectively. They also state that if the results of the 1944 House of Representatives elections in the USA are modified to take into account certain peculiarities of USA politics (thus removing the 49 seats of the ten states of the South), we find the proportion of Democrat votes has a normal distribution with $\sigma = 0.132$.

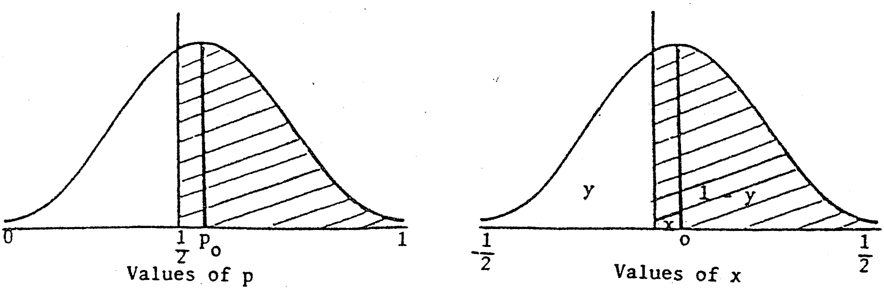
Why the distribution of the proportion of votes over the seats is approximately normal with a standard deviation around 0.11 - 0.13 is not known, and there is no reason why this should be true universally, or indeed that it should continue to hold for Australian elections in the future. Already there is a slight trend towards a reduced value of the standard deviation – values of 0.134 and 0.132 in 1958 and 1961, to 0.1054 and 0.1021 for 1987 and 1990. This may be due to the fact that the two parties are getting closer in policy.

We shall now show mathematically that if the Law of Cubic Proportion holds, then the proportion of votes per seat for a party has approximately a normal distribution with a standard deviation of 0.137.

Let us assume that at an election the proportion P of Labor voters per seat varies randomly, with a certain probability distribution, with mean value p_0, the overall two-party preferred proportion. The proportion of seats won by Labor is then given by $Pr(P > \frac{1}{2})$, i.e. the area under the probability density curve of P between $\frac{1}{2}$ and 1. According to the Law of Cubic Proportion,

$$\frac{Pr(P > \frac{1}{2})}{Pr(P < \frac{1}{2})} = \left[\frac{p_0}{1-p_0} \right]^3 \qquad (1)$$

The question is: what probability density function (for P) has this property?



Distribution of P Distribution of X

Figure 2

To answer this question, we first make a change of origin to the mean value p_0, and work with a new random variable $X = P - p_0$, which has mean value zero. Since X is just a "shifted" or translated version of P, deriving the distribution of P is equivalent to deriving that of X. (See Figure 2.)

Let $x = \frac{1}{2} - p_0$. Then $P < \frac{1}{2}$ is equivalent to $X \le x$. Let $y = Pr[X \le x] = F(x)$, i.e. the area under the density function of X to the left of x. Then $Pr[P \ge \frac{1}{2}] = Pr[P \ge x] = 1 - y$, and (1) reduces to the condition

$$\frac{1-y}{y} = \left[\frac{\frac{1}{2} - x}{\frac{1}{2} + x}\right]^3 = \frac{1-F(x)}{F(x)} \tag{2}$$

By making $F(x)$ the subject of equation (2), it can be shown (using the fact that density $f(x) = dF(x)/dx$) that the density of X which has the property (2) is given by

$$f(x) = \frac{3(1-4x^2)^2}{(1+12x^2)^2}, \quad -0.5 \le x \le 0.5 \tag{3}$$

The density (3) is symmetrical about the line with equation $x = 0$, and hence the mean of X must be zero. It can be shown that the standard derivation of X is 0.137. The density curve (3) is symmetrical and bell-shaped, and is virtually indistinguishable from the Normal distribution, with mean zero and standard deviation 0.137, as Figure 3 shows. The graph of $f(x)$ is the one with a slightly higher value at zero.

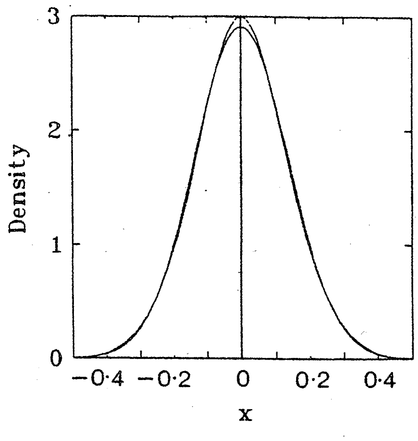


Figure 3

It then follows that if (1) is true, the proportion P of Labor votes per seat is very close to a normal distribution, with mean p_0 and standard deviation 0.137.

Let us now see the application of the law to Australian federal elections. The following table (taken from Constantinou (1992)) gives the results of applying the law to all the 18 elections during the period 1949-1990.

Table. Results of Australian federal elections 1949-1990
(Actual and those predicted by the law of cubic proportions)

Election Year	Total Seats	Labor vote %	Coalition vote %	Actual No. of Labor Seats	Predicted No. of Labor Seats	Actual No. -Predicted No.
1949	123	49.0	51.0	48	58	–10
1951	123	49.3	50.7	54	59	–5
1954	123	50.7	49.3	59	64	–5
1955	124	45.8	54.2	49	47	2
1958	124	45.9	54.1	47	47	0
1961	124	50.5	49.5	61	64	–3
1963	124	47.4	52.6	52	52	0
1966	124	43.1	56.9	41	38	3
1969	125	50.2	49.8	59	63	–4
1972	125	52.7	47.3	67	73	–6
1974	127	51.7	48.3	66	70	–4
1975	127	44.3	55.7	36	43	–7
1977	124	45.4	54.6	38	45	–7
1980	125	49.6	50.4	51	61	–10
1983	125	53.2	46.8	75	74	1
1984	148	51.8	48.2	82	82	0
1987	148	50.8	49.2	86	78	8
1990	148	49.9	50.1	78	74	4

By and large, the predicted number of seats is close to the actual result, except for the elections of 1949, 1980 and 1987. However, the following points are worth noting.

1. The figures for Labor and Coalition percentage votes in columns 3 and 4 are, for elections up to 1983, only the result of guess-work by scrutineers and need not be the true percentages obtained. This is because until 1983, the counting of second and later preferences (if required at all) for a seat stopped when one candidate obtained 50% + 1 of the total vote. In such situations the split of the two-party preferred vote for that seat was only guessed at. From 1984, however, counting of preferences is carried to the bitter end, i.e. even if a certain candidate has won a seat by obtaining 50% + 1 votes, all the preferences of all the candidates are counted, to finally give a two-party preferred split for that seat, and hence for the whole country.

2. If the two-party preferred split is close, as for example for the 1980, 1987 and 1990 elections, then obviously there are bound to be discrepancies between the actual result and that predicted by the law.

It is interesting to speculate as to why the Law of Cubic Proportion is as accurate a predictor as it is. You may also like to consider other aspects of this intriguing law, such as whether or not it is a consequence of an assumption that the distribution of votes in a two-party-preferred system is normal, and whether (or why) such an assumption is valid.

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SING ANOTHER SONG

Malcolm Clark, Monash University

Over the past 50 years or so, there must have been thousands of popular songs written. Most of these songs would have been sung only a few times, and then forgotten or discarded as unworthy. Some songs would immediately catch on, and make it to the "Top 40", but most of these would be forgotten after a few months. Only a few songs, out of the thousands sung and written, endure to become "classics".

This is not the place to debate the merits of songs produced by such songsmiths as Cole Porter, Andrew Lloyd Webber, or even M.C. Hammer. Instead, we ask the question: how many *more* songs is it possible to write?

Any song is made up of four elements: melody, harmony, rhythm and words. In the following calculations, we ignore the choice of words, since the essential musicality of a song does not change with changing words. All those different versions, serious and profane, of *John Brown's Body* are instantly recognisable as the same tune. Instead, we concentrate initially on evaluating the number of distinct combinations of rhythm and melody.

To fix ideas, let us consider just one bar of music in 4/4 time, and assume that all notes in this bar of music are multiples of eighth notes (*quavers*). Hence at one extreme, the rhythm pattern for that bar of music could consist of eight 8th-notes. At the other extreme, there could be just one note (a *semi-breve*), held for the duration of the bar. These are just two of the 128 (= 2^7)† possible rhythm patterns for such a bar of music.

Let us look at those cases where there are *four* distinct notes. There are many different rhythm patterns which can be made in such a case. Three examples are as follows:


(1)


(2)


(3)

† For a derivation of this, see later in the article.

How many more such patterns, comprising four distinct notes, are there? Fortunately, rather than trying to list all possibilities one by one, we can derive a simple formula for the total number of patterns.

To do this, we represent the eight beats of the bar by eight asterisks, and the beginning and end of each note by a dot. So rhythm pattern (1) is represented by the diagram

$$* \; * \; \bullet \; * \; * \; \bullet \; * \; * \; \bullet \; * \; * \tag{1'}$$

Similarly, patterns (2) and (3) are represented by

$$* \; * \; * \; * \; \bullet \; * \; \bullet \; * \; \bullet \; * \; * \tag{2'}$$

$$* \; \bullet \; * \; * \; \bullet \; * \; \bullet \; * \; * \; * \; * \tag{3'}$$

If you were playing this pattern on the piano, each dot represents the instant when you move your finger from one key to the next, or possibly repeat the same note.

Each such pattern of dots and asterisks represents, in non-musical notation, a possible rhythm pattern with, in this case, four distinct notes. Notice that in (1′), (2′) and (3′) there are three dots, and there are 7 possible positions for each dot, namely the 7 spaces between the 8 asterisks. Hence the number of possible rhythm patterns in this case is the number of ways of choosing 3 of the 7 spaces to be occupied by a dot. This number is the combinatorial coefficient†

$$\left[\begin{matrix}7\\3\end{matrix}\right] = \frac{7\times6\times5}{3!} = 35.$$

So much for the rhythm pattern; what about the melody? Let us assume that the melody is restricted to the 8 notes of an octave of a major scale. Hence there are 8 choices of pitch for each of the four notes in the bar.

The total number of distinct combinations of rhythm and melody, with four notes to the bar, is therefore

$$\left[\begin{matrix}7\\3\end{matrix}\right]8^4 = 143{,}360.$$

Proceeding in this way, if there are x distinct notes in the bar, the number of distinct rhythm patterns is equal to the number of choices for $(x-1)$ dots in the 7 possible spaces between asterisks. For each such pattern, there are 8^x possible melodic patterns. Hence with x notes, the total number of combinations of rhythm and melody is

$$\left[\begin{matrix}7\\x-1\end{matrix}\right]8^x, \quad x = 1, 2, \ldots, 8.$$

Adding up over all possible values of x, the total number of distinct "songs" is

$$N = \left[\begin{matrix}7\\0\end{matrix}\right]8^1 + \left[\begin{matrix}7\\1\end{matrix}\right]8^2 + \left[\begin{matrix}7\\2\end{matrix}\right]8^3 + \ldots + \left[\begin{matrix}7\\7\end{matrix}\right]8^8 = 38{,}263{,}752. \tag{4}$$

† This is also written in the form ${}^nC_r$.

This is just from one bar of music, in effectively 8/8 time, with 8 choices of pitch for each note. There is no guarantee that any of these songs will be pleasant to listen to, or even easy to sing!

Let us now consider the general case where our song comprises several bars of music, with up to n distinct notes, and p choices of pitch for each note. We will now derive a simple formula for $N(n, p)$, the corresponding number of distinct combinations of rhythm and melody.

Suppose our general "song" has x distinct notes. To compute the number of rhythm patterns, we now have a line of n asterisks, and we must insert a dot in $(x-1)$ of the $(n-1)$ spaces between these asterisks. The number of ways of doing this is

$$\binom{n-1}{x-1}.$$

There are p possible choices of pitch for each of these x notes, so the total number of combinations of rhythm and melody with x distinct notes is

$$\binom{n-1}{x-1} p^x, \qquad x = 1, 2, \ldots, n. \tag{5}$$

Summing (5) over all values of x, we find

$$N(n, p) = \binom{n-1}{0} p + \binom{n-1}{1} p^2 + \ldots + \binom{n-1}{x-1} p^x + \ldots + \binom{n-1}{n-1} p^n. \tag{6}$$

Setting $m = (n-1)$ we can re-write this as

$$N(n, p) = p\left[\binom{m}{0} + \binom{m}{1} p + \binom{m}{2} p^2 + \ldots + \binom{m}{m} p^m\right]$$

$$= p\left[\binom{m}{0} p^0 1^m + \binom{m}{1} p^1 1^{m-1} + \binom{m}{2} p^2 1^{m-2} + \ldots + \binom{m}{m} p^m\right]$$

$$= p(p+1)^{n-1} \tag{7}$$

noting that the last summation is just the Binomial expansion for $(p+1)^m$.†

In the special case considered at the start of this article, $n = 8$ and $p = 8$, giving

$$N(8, 8) = 8 \times 9^7 = 38{,}263{,}752.$$

This is the quick way of doing the summation in (4)!

If we set $p = 1$ in (7), we are effectively ignoring melody, and so obtain the total number of rhythm patterns. In this example, this number is

$$N(8, 1) = 2^7 = 128,$$

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† Equation (7) can also be obtained using a direct argument. The first quaver (or whatever basic rhythmic unit we are using) may be any one of the p pitches, while each of the other $n-1$ quavers is either a new note with one of the p pitches, or a continuation of the previous note ($p+1$ choices altogether).

as already stated.

Many popular songs essentially comprise 12 bars of a 4/4 rhythm pattern. If we assume that only quarter-notes or multiples thereof are used, then $n = 48$. Restricting the range of the melody to an octave $(p = 8)$, formula (7) gives, for the number of distinct popular songs,

$$N(48,\ 8) = 8 \times 9^{47} = 5.656 \times 10^{45}.$$

To get some idea how enormous this number is, imagine we had a super-computer which could list these "songs" at the rate of one billion per second. Then the time taken for this super-computer to list all possibilities would be approximately 40 million million million times the age of the Universe!

If we allow up to an octave and a half, then $p = 12$, giving

$$N(48,\ 12) = 12 \times 13^{47} = 2.720 \times 10^{53}.$$

So far we have ignored two other features of popular songs, namely repetition and harmony. In most songs there is some repetition in both the rhythm pattern and the melody. Secondly, in each bar only a subset of the p possible choices of pitch is permissible, depending on the chord progression or harmony being used, so it would be more realistic to take into account this structure in our calculation.

If we denote rhythm patterns per bar by A, B, C, ..., and harmony or chords by 1, 2, 3, ..., then a typical pattern for a modern popular song, based on "12-bar blues", would be

A1 A1 A1 C1
A2 A2 A1 C1
B3 B2 A1 C1

Now suppose for the moment that the number of notes in the patterns A, B, C are a, b, c respectively, and let p_i denote the number of permissible choices of pitch under chord i, for $i = 1, 2, 3$. Further, let us assume that each bar is written in what might be called "m/m time", that is, the bar is notionally subdivided into m units, and all notes in that bar are integer multiples of these $1/m$-th units. With this notation the number of arrangements for the first bar in this song is, by the preceding arguments,

$$\binom{m-1}{a-1} p_i^a.$$

Remembering that the rhythm patterns are repeated as indicated in the above structure, but the choice of the pitch of notes is independent, then for fixed values of a, b, c, the total number of arrangements is

$$\binom{m-1}{a-1}\binom{m-1}{b-1}\binom{m-1}{c-1} p_1^{5a} p_2^{2a} p_2^{b} p_3^{b} p_1^{3c}. \tag{8}$$

To understand this formula, notice that according to our 12-bar blues pattern, there are seven bars with rhythm pattern A, five with chord 1 and two with chord 2, and a notes per bar. Of this total of $7a$ notes, $5a$ are with chord 1, and each of these has p_1 choices of pitch. Similarly, the $2a$ notes with chord 2 each have p_2 choices of pitch. While there are $\begin{bmatrix} m-1 \\ a-1 \end{bmatrix}$ choices for the rhythm pattern, once such a choice is made, the chosen pattern is repeated in all 7 bars. Hence the contribution from the 7 bars with rhythm pattern A is

$$\begin{bmatrix} m-1 \\ a-1 \end{bmatrix} p_1^{5a} p_2^{2a}.$$

We now have to add expression (8) over all possible values of a, b and c. Applying esssentially the same argument that reduced (6) to (7), it can be shown that this sum reduces to

$$N = p_1^8 p_2^3 p_3 \{(p_1^5 p_2^2 + 1)(p_2 p_3 + 1)(p_1^3 + 1)\}^{m-1}. \qquad (9)$$

Assuming, for definiteness, that $p_1 = 6$, $p_2 = 5$, $p_3 = 5$ and $m = 4$ (so that the total number of quarter-notes is still 48), we find that

$$N = 1.385 \times 10^{36}.$$

This calculation confirms that, by imposing constraints or restrictions on the rhythmic and melodic patterns, the total number of possibilities is drastically reduced, in this case by a factor of about 4 billion, relative to the comparable $N(48, 8)$.

The calculation also indicates that there is still a huge number of "12-bar blues" songs yet to be written. How many of these 10^{36} possibilities will be worth listening to is a moot point.

* * * * *

Fermat's Last Theorem Solved!

You may have heard on the news or read in the paper (*e.g.* the *Australian*, 25.6.1993) that the famous three-and-a-half-century-old problem known as **Fermat's Last Theorem** has apparently been solved by English mathematician Andrew Wiles. The announcement of Wiles's result has excited mathematicians all around the world. In *Function*, *Vol*. 15, *Part* 4, Michael Deakin outlined the history of Fermat's Last Theorem and the various attempts to solve it throughout the intervening centuries. The Theorem states that

There are no positive integers **a, b, c** *and* **n** *(where* $n > 2$*) for which*

$$a^n + b^n = c^n. \qquad (0.1)$$

Recall that Pythagoras' Theorem satisfies the case where $n = 2$; recall, too, that the equation in Fermat's Last Theorem is called a Diophantine equation. Indeed, Fermat had written in the margin of his copy of Diophantus' *Arithmetic* that he had a "truly wonderful" proof of his Theorem, but that the margin was too small to contain it. Now Wiles's proof apparently is hundreds of pages long. Mathematicians are excited and fascinated by it, particularly its use of geometry (elliptic curves) and the fact that it does not rely on limit arguments for proofs of cases where n is huge but finite, *cf.* those discussed in the previous *Function* article. Many mathematicians, however, are also still wondering whether Fermat actually had a proof which was only a little larger than the margin of his book! Regardless, congratulations to Dr Wiles!

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