A SCHOOL MATHEMATICS MAGAZINE

FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

> The Editors,
> FUNCTION,
> Department of Mathematics,
> Monash University,
> Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside front cover.

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*$8.50 for *bona fide* secondary or tertiary students.

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FUNCTION

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Note: A news update on the Monash University Sundial has had to be held over because
of lack of space. We apologize to those readers who had hoped to see this item
in the present issue.

THE FRONT COVER

Michael A.B. Deakin, Monash University

Our cover diagram for this issue shows two sets of *coaxal circles*. The solid circles all have their centres on the horizontal axis. The vertical axis is the so-called *radical axis* (see *Function, Vol. 15, Part 5*) for any pair of solid circles.
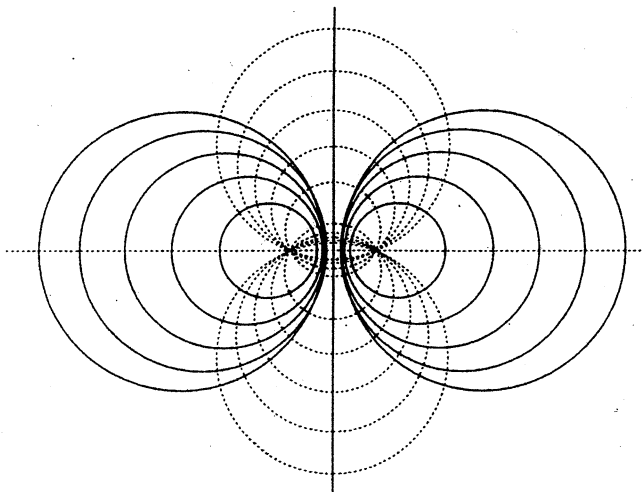
The second, dashed set of circles also forms a coaxal set. In this case, the circles intersect, whereas in the previous case they do not. The dashed circles all have their centres on the vertical axis and furthermore all intersect in the same two points. Call these points (the "eyeballs" of the figure) $(\pm a, 0)$. Any two circles in this second set have the horizontal axis as their radical axis.

Circles in the first set have equations

$$(x - k)^2 + y^2 = k^2 - a^2 \tag{1}$$

and each value of k corresponds uniquely to some one circle of the set, and *vice versa*. (Where we must have, for consistency, $k \geq a$ or $k \leq -a$.)

Circles in the second set have equations

$$x^2 + (y - b)^2 = a^2 + b^2 \tag{2}$$

and each value of b $(b \geq 0)$ corresponds uniquely to some one circle of this set, and *vice versa*.

As the diagram indicates, wherever a circle of one set intersects a circle of the other set, they intersect at right angles.

This underlies an important physical application. Suppose, e.g., a bar magnet is so placed that its poles occupy the points $(\pm a, 0)$. Then a magnetic field is set up and the two lots of circles give two different ways of visualising this. The dashed circles show the *lines of force*, the paths along which a minute magnetic monopole would travel. Commonly, and you may well have seen this in your science class, they are made visible by means of iron filings.

The solid circles show the contours of *magnetic potential* (similar to the more familiar electric potential). Lines of force are indeed always perpendicular to contours of potential, and *vice versa*.

As the radii of the solid circles get larger and larger, the value of k increases and the circles get flatter. It is usual to include as a member of the family the limiting case, which is the vertical axis, and to say that this corresponds to $k = \infty$.

Similarly with the dashed circles, where the horizontal axis corresponds, by convention, to $b = \infty$.

We are now in a position to notice an interesting fact. Every point of the plane corresponds uniquely to an intersection of a solid circle and a dashed circle (or *almost* every point; the special cases $(\pm a, 0)$ will be dealt with later). Each intersection of a solid with a dashed circle, however, corresponds to two points in the plane.

Let us see how this works. First suppose we are given a point (x, y). Then Equations (1), (2) both hold.

We thus have two equations in the unknowns b, k, given that we are supposing that x, y are known. We find, easily enough:

$$k = \frac{x^2+y^2+a^2}{2x}$$

$$b = \frac{x^2+y^2-a^2}{2y}$$

$$(3)$$

and thus, given (x, y), we may determine (k, b). The only difficulty arises when either x or $y = 0$ and this is usually solved by putting $k = \infty$ when $x = 0$, and $b = \infty$ when $y = 0$, in accordance with the convention outlined above.

Now consider the case in which k, b are known and we want to determine x, y. This means solving Equations (1), (2) as simultaneous quadratics. The result is:

$$x = \frac{(a^2+b^2)k^2 + b^2(k^2-a^2) \pm b\,k\sqrt{2b(k^2-a^2)}}{k(b^2+k^2)}$$

$$(4)$$

$$y = \frac{2b(k^2-a^2) \pm k\sqrt{2b(k^2-a^2)}}{b^2+k^2}$$

and as we have $b \geq 0$ and $k^2 \geq a^2$, the square roots exist. Notice that in most cases, there are indeed two values for (x, y) given b, k.

We now come back to the troublesome points $x = \pm a, y = 0$. Here Equations (3) yield $k = \pm a, b = 0/0$. Now $0/0$ is undefined – it may have any value (not necessarily ∞).

This corresponds to the fact that all the dashed circles, whatever the relevant value of *b*, pass through these points. Conversely, put $k = \pm a$ into Equation (4) to find the same result.

The correspondence between (x, y) on the one hand and (b, k) on the other may be used to construct an exotic coordinate system based on (b, k) rather than on (x, y). (Our cover diagram for *Vol.* 10, *Part* 5 was based on another, different, such exotic coordinate system.) Such coordinate systems can be useful for special purposes, giving simple forms to otherwise quite complicated expressions. For example, the complicated cubic expression

$$(x^2+y^2)(x-y) = a^2(x+y)$$

has, in these new coordinates, the simple equation

$$k = b.$$

You may care to see what curve this represents.

WINNING TATTSLOTTO - TWICE!

Malcolm Clark, Monash University

Introduction

In February 1991, Mr Ray Williams of Albury became the first person to win the top prize in Tattslotto twice. He won the First Division prize, by correctly choosing all six winning numbers out of the 45 numbers. He had the same success back in 1984.

Most people would regard winning Tattslotto twice as a one-in-a-billion chance. There are 8,145,060 ways of selecting 6 numbers from 45, and so the chance of winning Tattslotto just once in a single game is $1/8,145,060 \simeq 1.22 \times 10^{-7}$. Since successive Tattslotto draws are presumably independent in the probability sense, the chance of winning Tattslotto twice must surely be the above number squared, that is, 1.51×10^{-14} or about 1 in 66 million million.

This naive intuitive calculation is fundamentally flawed, on three grounds. First, many Tattslotto players have multiple entries or system entries, so that their chance of winning once is considerably higher than 1.22×10^{-7}. Secondly and more importantly, it is not valid here to multiply probabilities, as when finding the probability of two or more independent events. Such multiplication would only be valid if you wanted to find the probability that a particular person, *specified in advance*, will win twice.

Thirdly, we would be even more amazed to find several two-time winners, or even a person who had won more than twice. So it is more appropriate to consider the probability that *at least one* person will win more than once, over a given number of Tattslotto draws.

The Birthday Problem

Before answering this question, we consider the related and well-known "Birthday Problem".[†]   Suppose that N people are selected at random; what is the probability that two people in this group have the same birthday (day and month, ignoring year)?

Once again, we need to be more precise. Presumably, if we found, say, five people in our group of N with the same birthday, we would be even more surprised than if we found just two.   Hence what is of interest is the probability that *at least two* out of the N randomly selected people have the same birthday.

The easiest way to compute this probability is to consider the complementary event. If it is *not* the case that there are at least two birthdays the same, then all N birthdays must be different.   Ignoring leap-years, there are 365 possibilities for the birthday of the first person in our sample.   Once that is chosen, there are only 364 choices out of 365 for the next birthday, then 363 out of 365 for the next, and so on. Assuming that all 365 days are equally likely to be birthdays, then the probability that all N birthdays are different is

$$\frac{365 \times 364 \times 363 \times ... \times (365-N+1)}{365 \times 365 \times 365 \times ... \times 365} .$$

The numerator and denominator are both the product of N numbers.

Since this calculation refers to the complementary event, the probability we require is given by

$$P_N = 1 - \frac{365 \times 364 \times 363 \times ... \times (365-N+1)}{365 \times 365 \times 365 \times ... \times 365} .$$

Numerical values of this probability are much higher than what most people expect. For example, with $N = 23$,

$$P_{23} = 0.5073.$$

Hence in a group of just 23 randomly-chosen people, there is just over a 50:50 chance that at least two people will have the same birthday.   Note that we do not specify what day of the year that common birthday is.

Suppose, instead, that we had specified the day.   For example, what is the probability that in a random sample of N people, at least two would have been born on New Year's Day?

Once again we consider the complementary event, that at most one of the N people was born on New Year's Day.   There are two possibilities: either (i) nobody was born on that day, or (ii) just one of the N people was, but we don't know which person it was.

In case (i), there are 364 choices for the first person's birthday, then 364 again for the second, 364 for the third, and so on.   Thus the probability that all N birthdays are different from New Year's Day is

$$\left[\frac{364}{365} \right]^N .$$

[†] See *Function, Vol. 6, Part 4*, pp. 12-16 (1982).

In case (ii), there must be $(N-1)$ birthdays different from New Year's Day, and there are N different ways of getting such an arrangement amongst the N people. So the probability for case (ii) is

$$N\left[\frac{1}{365}\right]\left[\frac{364}{365}\right]^{N-1}.$$

Hence the probability that at least two of the N persons were born on New Year's Day is

$$1 - \left[\frac{364}{365}\right]^{N} - N\left[\frac{1}{365}\right]\left[\frac{364}{365}\right]^{N-1}.$$

For $N = 23$, this probability is 0.00183. By specifying in advance *which* day is to be the common birthday, we have specified a relatively rare event.

Simplified Tattslotto

We return to the Tattslotto problem and start by considering a simplified version of the problem. Let us assume, for the moment, that there are N players in each of m successive Tattslotto draws, and each player has the same probability, p, of winning first prize on any draw. We are assuming that there is a fixed pool of N players, draw after draw. Our aim is to find a formula for the probability that at least one of N players will win more than once in the m draws.

As in the Birthday Problem, we consider first the complementary event, that is, the logical opposite of what we are interested in. This complementary event is that all N players win at most once.

We derive the subsequent formulae by first defining N random variables V_1, V_2, ..., V_N, where, for each $i = 1, 2, ..., N$

V_i = number of times that the i-th player has won first prize

in the m draws.

Clearly, in each week, the i-th player either has a "success" (he wins first prize) or a "failure". The probability of getting a "success" in any particular week is p, independently of what might have happened in all other weeks. In other words, if the i-th player happens to win in a particular week, his chance of winning again in the very next week is still p. This remains true, even if that player chooses the same 6 numbers week after week, since the Tattslotto machine presumably acts independently from draw to draw.

We now recognize V_i as the number of "successes" in m "trials", and so under our assumptions each V_i has the Binomial distribution, with parameters m and p.

Hence the probability that the i-th player wins at most once in m draws is

$$q_m = Pr(V_i \leq 1) = Pr(V_i = 0) + Pr(V_i = 1)$$
$$= (1 - p)^m + p(1 - p)^{m-1} \qquad (1)$$

Note that this probability is the same for every one of the N players.

We now make the additional assumption that at each draw the N players make their selections of 6 numbers out of 45 independently of one another. This means that it is valid to multiply probabilities when computing the probability of joint events. It also means that the V's are independent.

For example, suppose we wanted to compute the probability that both player 1 and player 2 win at most once in the m draws. In our previous notation, we require the probability of the event that both $V_1 \leq 1$ and $V_2 \leq 1$ simultaneously. Then

$$Pr\{(V_1 \leq 1) \cap (V_2 \leq 1)\} = Pr(V_1 \leq 1) \times Pr(V_2 \leq 1),$$

since the events $\{V_1 \leq 1\}$ and $\{V_2 \leq 1\}$ are independent.

With the above notation, our complementary event is that all N V_i's take values 0 or 1. The probability of this happening is

$$
\begin{aligned}
g_m &= Pr(V_1 \leq 1, V_2 \leq 1, ..., V_N \leq 1) \\
&= Pr(V_1 \leq 1)Pr(V_2 \leq 1) ... Pr(V_N \leq 1)
\end{aligned}
$$

(since the V's are independent)

$$= (Pr(V_i \leq 1))^N$$

(since the V's have the same probabilities)

$$= (q_m)^N. \tag{2}$$

Finally, the probability P_m that at least one player wins more than once in m draws is

$$P_m = 1 - g_m = 1 - (q_m)^N. \tag{3}$$

Since p is very small, m moderately large and N very large, it is possible to obtain simple but accurate approximations to the above formulae. By the Poisson approximation to the Binomial distribution,[†]

$$q_m \simeq (\lambda + 1)e^{-\lambda}, \text{ with } \lambda = mp. \tag{4}$$

Rearranging (3) and taking logarithms,

$$
\begin{aligned}
ln(1 - P_m) &= N \, ln(q_m) \simeq N \, ln((\lambda + 1)e^{-\lambda}) \\
&= N\{ln(\lambda + 1) - \lambda\}.
\end{aligned}
$$

It can be shown that for λ close to zero, $ln(\lambda + 1) \simeq \lambda - \frac{1}{2}\lambda^2$, taking just the first two terms of the logarithmic series. Hence

$$ln(1 - P_m) \simeq N\{(\lambda - \tfrac{1}{2}\lambda^2) - \lambda\} = -\tfrac{1}{2}N\lambda^2 = -\tfrac{1}{2}m^2Np^2,$$

[†] See *Function, Vol. 8, Part 5*, pp. 14-18 (1984).

leading to

$$P_m \simeq 1 - e^{-\frac{1}{2}m^2Np^2}.$$ (5)

Some Numerical Values

What are reasonable values of N and p to use in the above formulae? For simplicity, we will consider only Saturday draws of Tattslotto, in which typically 2.3 million tickets are sold, comprising a total turnover of about \$7.8 million. To enter Tattslotto, a player must make at least four selections of 6 numbers (and at most 12 selections) per ticket. Thus the chance that any regular Tattslotto ticket will be a winning ticket is at least 4 times the much-quoted 1 in 8,145,060.

Roughly 45% of Saturday Tattslotto entries are "regular" entries (in which the player makes up to 12 selections of 6 numbers), 33% are "Quick-Pick" entries (12 selections generated by computer), and 22% are "system" entries. The majority of regular entries contain 12 "games".

To get some idea of the order of magnitude of P_m, let us ignore system entries for the time being, and assume that there are 2.3 million players, each entering one regular entry of 12 "games" per week. So $N = 2.3 \times 10^6$, $p = 12/(8,145,060)$, and let us set $m = 260$, corresponding to five years of Saturday draws. With these very crude assumptions, we find
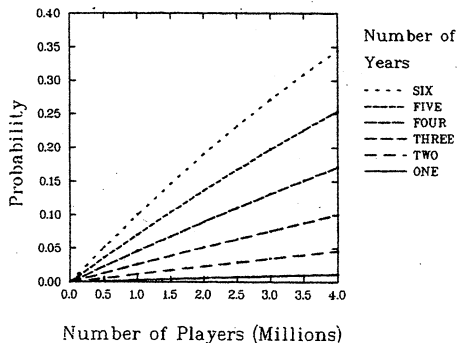
$$\lambda = 3.8305 \times 10^{-4}$$

and

$$P_{260} = 0.1557.$$

So with these assumed values of N and p, there is about a 1 in 6 chance that at least one person will win first prize in Saturday Tattslotto more than once in the next five years.

The following graph shows how the probability P_m depends on m, the number of weeks considered, and N, the number of players.

A more realistic approach

In the real-life Tattslotto, it is not the case that every player has the same probability of winning. Systems entries, comprising about 22% of all entries, are effectively multiple entries, and so have a much higher probability of winning. For example, in a System 9 entry, the player selects 9 numbers out of the 45 available. The Tattslotto computer automatically generates all $\begin{pmatrix} 9 \\ 6 \end{pmatrix} = 84$ possible choices of six numbers out of those nine. So a single System 9 entry is equivalent to 84 individual games, and has probability of 84/(8,145,060) of winning first prize. It is also about 84 times as expensive as a single game!

In fact, there are 21 different types of entry to Tattslotto, ranging from the minimum of four selections of six numbers to a System 20 entry costing $11,802. The corresponding probabilities of winning are easy to calculate using combinatorial methods, and range from 4 in 8,145,060 up to 38,760 in 8,145,060.

The preceding formulae can be readily adapted to this situation. We notionally subdivide the N players into 21 categories, corresponding to the 21 types of entry, and suppose that in category k, there are N_k players and each has known probability p_k of winning any given draw ($k = 1, 2, ..., 21$).

If there are no multiple winners in m successive draws, then there must be no such multiple winners in each category. To derive the relevant probabilities, we apply the argument leading to (2) to each category in turn. So combining (2) and (4) and writing $\lambda_k = mp_k$, the probability Q_k that no player in category k wins more than once in m draws is

$$Q_k = \left[\left[\lambda_k + 1 \right] e^{-\lambda_k} \right]^{N_k} \qquad \qquad .... \qquad (6)$$

Hence the probability that at least one player in at least one of the categories wins more than once in m draws is

$$P_m^* = 1 - (Q_1 \times Q_2 \times Q_3 ... Q_{21}). \qquad (7)$$

Applying exactly the same arguments which led to (5) as our approximation to (3), (7) can be closely approximated by the formula

$$P_m^* = 1 - e^{-S} \qquad (8)$$

where $S = \frac{1}{2} m^2 \left[N_1 p_1^2 + N_2 p_2^2 + ... + N_{21} p_{21}^2 \right]$.

Notice how (8) reduces to (5) when there is just one category.

Expression (8) shows clearly how the probability P_m^* depends on m, the number of draws, and the N_k's and p_k's. The p_k's are known, but the N_k's, the number of players in the different categories, must somehow be estimated.

Michaela Smale, a 1992 Honours student in Statistics at Monash University, derived various estimates of $N_1, N_2, ..., N_{21}$, using in part information given by Tattslotto (on

the numbers of winners in different categories in 1990-91 and numbers of transactions in different groups). By allowing in this way for different categories of player, the probability of there being a multiple winner increased dramatically. For example, Michaela estimated that the probability of at least one person winning more than once in 5 years of Saturday Tattslotto draws was 0.95!

Formula (8) is still a simplification, in that its derivation assumed that each player only made one entry per draw. Clearly, many people make multiple entries, possibly of different types, and do not do the same every week. Nevertheless, the probability given by (8) is likely to be of the right order of magnitude.

Coincidences (such as Mr Williams' double win) are not as remarkable as most people think. In fact, they *will* happen to someone, somewhere, some time. But if *you* expect to be the one to win Tattslotto twice, then the originally-given odds are more realistic. Don't hold your breath while you are waiting!

TRIGONOMETRIC SOLUTIONS TO QUADRATIC EQUATIONS

Richard Whitaker, 4 Gowrie Close, St. Ives, N.S.W.

Let

$$ax^2 + bx + c = 0 \tag{1}$$

be a quadratic equation whose roots A and B are real and have the same sign. We may write Equation (1) as

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \tag{2}$$

or as

$$(x - A)(x - B) = 0. \tag{3}$$

Multiply the factors in Equation (3) to find

$$x^2 - (A + B)x + AB = 0 \tag{4}$$

and compare this with Equation (2). We find

$$AB = \frac{c}{a} \tag{5}$$

and

$$A + B = -\frac{b}{a}. \tag{6}$$

Because the roots A, B have the same sign, AB must be positive. Thus $\frac{c}{a}$ is positive, which means that a, c are either both positive or both negative. In either case, we must also have

$$ac > 0. \tag{7}$$

Equation (6) suggests that we set

$$A = -\frac{b}{a} \cos^2\varphi \left.\begin{array}{c} \\ \\ \end{array}\right\}$$

$$B = -\frac{b}{a} \sin^2\varphi$$

(8)

as this will automatically satisfy Equation (6). If φ can be determined, A, B will be found.

To determine φ, substitute from Equations (8) into Equation (5). This gives

$$\frac{b^2}{a^2} \cos^2\varphi \sin^2\varphi = \frac{c}{a}$$

(9)

which may be written

$$b^2(2 \sin \varphi \cos \varphi)^2 = 4ac,$$

that is to say

$$\sin^2 2\varphi = \frac{4ac}{b^2} .$$

(10)

But now

$$\cos 4\varphi = 1 - 2 \sin^2 2\varphi$$

and so Equation (10) becomes

$$\cos 4\varphi = 1 - 2\left[\frac{4ac}{b^2}\right]$$

i.e.

$$\cos 4\varphi = \frac{b^2 - 8ac}{b^2} .$$

(11)

This equation will have solutions if

$$-b^2 \leq b^2 - 8ac \leq b^2$$

and this reduces to two inequalities:

$$b^2 \geq 4ac,$$

which is the requirement for the roots of Equation (1) to be real, and

$$ac \geq 0$$

which is Inequality (7) with the further possibility that one or other root is zero.

Thus Equation (11) may be solved and the roots expressed in the form given by Equations (8).

[If Equation (1) has real roots of opposite signs, then Equations (8) do not apply. In this case, it is possible to use functions akin to the trigonometric functions and write

$$A = - \frac{b}{a} \cosh^2 \varphi \left.\right\}$$

$$B = + \frac{b}{a} \sinh^2 \varphi \left.\right\}$$

$$(12)$$

where cosh and sinh are so-called *hyperbolic functions*. This, however, lies outside the scope of *Function*.]

Consider as an example

$$x^2 - 4x + 3 = 0.$$

Here $a = 1$, $b = -4$, $c = 3$. Equation (11) has a solution $\varphi = \frac{\pi}{6}$ and this gives (check as an exercise!) the roots $A = 3$, $B = 1$.

COMPUTERS AND COMPUTING

EDITOR: CRISTINA VARSAVSKY

Solving Mathematical Problems with a Spreadsheet

Computers are becoming more and more accessible these days. If you do not own a computer you certainly have access to a PC in your school. Perhaps you find them attractive because of the many interesting games you can play (some of them very challenging indeed). Of many other computer applications you have most certainly seen a spreadsheet. It would have been one of the many available in the market: LOTUS 123, EXCEL, QUATTRO, VP-PLANNER, VC_CALC AND OTHERS.

A spreadsheet is like a big electronic table which is used for presenting and manipulating data. In one sense, it is a programming language. It is usually considered a business tool because it is very useful for the manipulation of numbers organised in a table: adding columns or rows of numbers, working out the average, combining contents of two or more columns, graphs, etc.

I have noticed that too few students (and teachers) appreciate its potential as a tool to explore mathematics. Spreadsheets are very useful when solving some mathematical problems, specially those involving a repetitive task. This article will show some problems that could be solved smartly and quickly with a spreadsheet.

Let us first have a short introduction to spreadsheets (or revision) because you need to be able to use them effectively before you actually start solving problems with them.

As I mentioned before, a spreadsheet is a big electronic table consisting of many cells (many more than we usually use). The address of each cell is determined by a letter (or two) followed by a number, indicating the column and the row respectively. Text, a number, or a formula can be stored in each cell. To enter or change the content of a cell you need to move the cursor to that cell (using either the mouse or the arrow keys), type the content, and hit the ENTER key. For example, enter the number 234.5 in *A*1 and the number 672.1 in *A*2. In *A*3, enter

$$+ A1 + 3* A2$$

(the "+" sign at the front means that what follows is a formula, not text). Your spreadsheet should look like Figure 1.

	A	B	C
1	234.5		
2	672.1		
3	2250.8		
4			
5			
6			

	A	B	C
1	234.5		23
2	672.1		10
3	2250.8		53
4			
5			
6			

Figure 1   Figure 2

Notice that if you now change 178.45 to 234.5 in $A1$ the content of the cell $A3$ changes accordingly.

One of the most powerful features of a spreadsheet is the copying of formulae containing references to other cells, like the formula we stored in $A3$ which adds to the content in $A1$ three times the content of $A3$. This formula is actually interpreted as follows:

add to the content of the cell two rows above three times the content of the cell immediately above.

Before actually copying the formula, let us store the number 23 in $C1$ and the number 10 in $C2$. Now copy the formula you have in $A3$ to the cell $C3$:

from the main menu select COPY, then highlight the cell $A3$ (range to be copied) and hit ENTER; move the cursor to $C3$ (range to be copied to) and press ENTER.

What happened? The cell $C3$ now contains 53, which is the content of $C1$ (two rows above) added to the triple of the number in $C2$ (immediately above). See Figure 2. This is why the references to $A1$ and $A2$ in the formula we have in $A3$ are called *relative references*. If we wanted to copy that formula to keep the references to $A1$ and $A2$ regardless of the position to be stored to, we would have used the dollar sign ($) in front of the column and the row, meaning absolute reference to that cell.

Let us now go to some examples where we use this concept of copying formulae containing relative references to other cells.

Generating sequences

Do you remember the Fibonacci sequence? The first two terms are 1, and then the subsequent terms are generated by adding the two previous terms:

1st term = 1
2nd term = 1
3rd term = 1st term + 2nd term = 2
4th term = 2nd term + 3rd term = 3
5th term = 3rd term + 4th term = 7
etc. ...

A spreadsheet is an ideal tool for generating this sequence, as we only need to enter the first two terms, then a formula for the third which is to be copied as far as we want to go. So, start with a blank spreadsheet[†] and store 1 in $A1$ and another 1 in $A2$. Now, the content for $A3$ must be the sum of the two previous ones, that is:

$$+ A1 + A2 \quad \text{(Don't forget the + sign at the front!)}$$

You should see 2 in cell $A3$. The formula in cell $A3$ is the one to be copied as many times as sequence terms we want to generate. This is done as follows:

select COPY, highlight the cell $A3$ and hit ENTER, then go to cell $A4$ and highlight the column A as many rows down as you want, say up to row 20; hit ENTER.[††]

In just a blink you have the first 20 terms of the sequence on the screen as in Figure 3.

	A	B
1	1	
2	1	
3	2	
4	3	
5	5	
6	8	
7	13	
8	21	
9	34	
10	55	
11	89	
12	144	
13	233	
14	377	
15	610	
16	987	
17	1597	
18	2584	
19	4181	
20	6765	
21		
22		

Figure 3

	A	B	C
1	1		
2	1		1
3	2		2
4	3		1.5
5	5		1.666667
6	8		1.6
7	13		1.625
8	21		1.615385
9	34		1.619048
10	55		1.617647
11	89		1.618182
12	144		1.617978
13	233		1.618056
14	377		1.618026
15	610		1.618037
16	987		1.618033
17	1597		1.618034
18	2584		1.618034
19	4181		1.618034
20	6765		1.618034
21			
22			

Figure 4

You may remember that the quotient of two consecutive terms had an important characteristic: it converges to the *golden ratio*. Let us check this by starting a new column, say C. In $C2$ we enter the formula

$$+ A2/A1$$

[†] Call the menu (in most spreadsheets, the "/" key) and select WORKSHEET, then ERASE, or BLANK.

[††] Some spreadsheets may require to "anchor" the first corner of the range meaning you need to type the period (.) before moving down the column.

Copy that formula from column C down to row 20. What do you see? Two consecutive terms of the new sequence are closer as we move down the column C, indicating that the sequence of consecutive quotients converges approximately to 1.618034, known as the golden ratio. (See Figure 4.) Although this is not a rigorous proof, it certainly gives a good approximation to the answer.

Here you have two exercises along the same lines:

Exercise 1: Change the values for the two first terms of the Fibonacci sequence. What happens to the sequence of the quotients?

Exercise 2: Generate some terms of the sequence defined as follows:

$$x_1 = 1 \; ; \; x_2 = \tfrac{1}{2} \; ; \; x_{n+1} = x_n^2.$$

Make a conjecture about its convergence.

Finding limits

Spreadsheets are also very useful to investigate the limiting values of functions. Take for example $lim_{x \to 0} \frac{sin\ x}{x}$. The function $y = \frac{sin\ x}{x}$ cannot be evaluated at $x = 0$ because both denominator and numerator are zero. We can see the behaviour of that function near the origin by evaluating it at $x = 0$. So we need to create a column containing values close to zero and another column with the evaluation of the function at those values. We can start with 0.1 and then move down with a step of 0.01. Start with a blank spreadsheet, and enter the number 0.1 in $A1$. In $A2$ enter the formula

$$+ A1 - 0.01$$

and in $B2$ the formula

$$@sin(A2)/A2.$$

(The "@" symbol indicates that what follows is an internally defined function, not text.) Next copy the two formulae, namely $A2$ and $B2$, down 8 rows. You produced a table (Figure 5) and from it you can see that the function $y = \frac{sin\ x}{x}$ approaches 1 as x gets closer to 0. You can do the same approaching zero from the negative side.

	A	B
1	0.1	
2	0.09	0.998651
3	0.08	0.998934
4	0.07	0.999184
5	0.06	0.9994
6	0.05	0.999583
7	0.04	0.999733
8	0.03	0.99985
9	0.02	0.999933
10	0.01	0.999983
11		
12		

Figure 5

Solving algebraic equations

You certainly know how to solve linear and quadratic equations. Although there are formulae for the cubic and the quartic, they are too complicated. Usually you end up guessing a solution, and if you are lucky you may find one.

Guessing would certainly not help in equations like $x = cos\ x$ or $x + tan\ x = 2$. There are some practical ways of getting an approximate solution, and I will show here two of them.

Tabulating the function

Solving an equation means finding the value of the variable such that both sides are equal, or where the left-hand side minus the right-hand side is zero. The idea is then to produce a table with two columns: in the first one we list values for the variable, and in the second we evaluate the difference between the left- and right-hand sides at those values to identify where it is close to zero. Usually we can get some bounds for the solution, for example by looking at the graphs of $y = cos\ x$ and $y = x$ we can see that the solution is somewhere between 0 and 1. So let us create the table: start with a blank spreadsheet and enter 0 in $A1$. Move to $A2$ and enter the formula

$$+ A1 + 0.1.$$

In $B2$ enter the function

$$@cos(A2) - A2.$$

Then copy the formulae in $A2$ and $B2$ down to row 10. If you investigate the table (Figure 6, columns A and B) you can conclude that $cos\ x - x$ is equal to 0 somewhere between 0.7 and 0.8. If we need more precision we can create another table, but taking a smaller step: start with 0.7 and go up to 0.8 with the step 0.01. (Figure 6, columns D and E.) By doing that we can now see that the solution lies between 0.73 and 0.74. If we want more accuracy for the solution we should look closer at the values between 0.73 and 0.74 by using a smaller step, say 0.001.

	A	B	C	D	E
1	0			0.7	
2	0.1	0.895004		0.71	0.048362
3	0.2	0.780067		0.72	0.031806
4	0.3	0.655336		0.73	0.015174
5	0.4	0.521061		0.74	-0.00153
6	0.5	0.377583		0.75	-0.01831
7	0.6	0.225336		0.76	-0.03516
8	0.7	0.064842		0.77	-0.05209
9	0.8	-0.10329		0.78	-0.06909
10	0.9	-0.27839		0.79	-0.08615
11					
12					

Figure 6

A similar procedure may be followed to solve the equation $ln\ x + x = 2$ which could be rewritten as $ln\ x + x - 2 = 0$. Again, we tabulate the function $ln\ x + x - 2$ in steps of 0.5 between 0.5 and 5 (logarithm is not defined at 0!). This is done in Figure 7: the first column is for the values of the variable, the second column contains the formula. In $A1$ we enter the value 0.5. In $A2$, we have the formula $+ A1 + 0.5$ and in $B2$ the formula $@ln(A2) + A2 - 2$. We can see that the values jump from positive to negative between 1.5 and 2, so this is the range blown up in the third column. From the fourth column we can conclude that the solution is between 1.55 and 1.6 and we gain one decimal in the precision. You can complete the spreadsheet to increase the accuracy (columns G and H).

	A	B	C	D	E	F	G	H
1	0.5			1.5			1.55	
2	1	-1		1.55	-0.01175		1.555	-0.00352
3	1.5	-0.09453		1.6	0.070004		1.56	0.004686
4	2	0.693147		1.65	0.150775		1.565	0.012886
5	2.5	1.416291		1.7	0.230628		1.57	0.021076
6	3	2.098612		1.75	0.309616		1.575	0.029255
7	3.5	2.752763		1.8	0.387787		1.58	0.037425
8	4	3.386294		1.85	0.465186		1.585	0.045584
9	4.5	4.004077		1.9	0.541854		1.59	0.053734
10	5	4.609438		1.95	0.617829		1.595	0.061874
11								
12								

Figure 7

Iterative methods

In *Function*, *Vol. 16*, *Part 3*, you can find an iterative technique to be used for the solution of equations of the form $f(x) = x$ and it consists basically in getting a first approximation x_0 and then generating the sequence by the iterative formula

$$x_{n+1} = f(x_n).$$

What better tool than a spreadsheet for an iterative task? Let us illustrate it with the two equations we used above. Start with $x = cos\ x$. We already determined $x = 1$ as a first approximation, therefore we enter the number 1 in *A1* (I assume you started with a blank spreadsheet). Now in *A2* we enter the iterative formula

@cos(A1)   (Notice that now $f(x) = cos\ x$.)

This formula needs to be copied about twenty rows down, generating in this way a sequence that would ultimately converge to the exact solution of the equation. This appears in Figure 8. As you can see in that table, the subsequent terms are closer and closer as we move down the column. The 7 in the first decimal position appears in the fourth iteration and it does not change from then on, meaning that it is correct. The 3 in the second decimal place remains unchanged from the row 16 on. So we can confidently say that an approximate solution is $x = 0.73$.

Let us now go to the second equation, $ln\ x + x = 2$. We first need to rewrite it in the form $f(x) = x$, that is, $2 - ln\ x = x$. Now, start with the approximation $x = 1.5$. Enter 1.5 in *A1* or any other blank area of your spreadsheet. In the cell immediately below, enter the iterative formula

2 − @ln(A1)

and copy it down several rows. Figure 9 shows the result. The first three digits remain unchanged from row 14 on, meaning that the solution to the equation, accurate to three decimal places, is $x = 1.557$.

	A	B
1	1	
2	0.540302	
3	0.857553	
4	0.65429	
5	0.79348	
6	0.701369	
7	0.76396	
8	0.722102	
9	0.750418	
10	0.731404	
11	0.744237	
12	0.735605	
13	0.741425	
14	0.737507	
15	0.740147	
16	0.738369	
17	0.739567	
18	0.73876	
19	0.739304	
20	0.738938	
21		

	A	B
1	1.5	
2	1.594535	
3	1.533418	
4	1.572501	
5	1.547333	
6	1.563467	
7	1.553094	
8	1.559751	
9	1.555474	
10	1.55822	
11	1.556456	
12	1.557589	
13	1.556861	
14	1.557328	
15	1.557028	
16	1.557221	
17	1.557097	
18	1.557177	
19	1.557126	
20	1.557158	
21		

Figure 8   **Figure 9**

Although this is not the best technique to solve equations iteratively[†] , it is a good example to show that spreadsheets are not only good for business, but also to solve a wide range of mathematical problems, specially those involving iterative formulae.

There are many other possible applications but it is impossible to show them in just a few pages. I hope this small sample is enough to encourage you to think of the spreadsheet as a useful tool when solving problems.

To conclude, I suggest you investigate the iteration

$$x_{n+1} = Cx_n(1 - x_n).$$

Start with the constant $C = 1$ and $x_1 = 0.5$. Generate a sequence using the iterative formula (use at least 200 iterations). Then find out what happens when you systematically increase C from 1 to 4. You may also use graphs the better to understand it. By doing this exercise you are entering in the field of chaos. Good luck!

[†] The Newton-Raphson method converges much more rapidly. This may be a topic for another *Function* article!

HISTORY OF MATHEMATICS

The Wonderful Deduction

Mathematics has, as one of its most important components, the concept of proof. It is necessary in Mathematics, not only that one's assertion be true, but that one also be able to *demonstrate* their truth.   [See, for example, the article by Jim Mackenzie in *Function, Vol. 17, Part 1*.]   Proofs may take many different forms, but some follow standard patterns, and this article will examine one of the least-known of these.

But before we get to that, let us begin with another: the so-called *reductio ad absurdum*. Suppose we want to prove a proposition P to be false. We begin by supposing it to be true.   That is to say, we wish to prove the negative of P, normally written not–P, or $\sim P$, and to this end, we entertain the possibility P.

The proof then proceeds by showing that P implies some false proposition, Q say. This we write as

$$P \Rightarrow Q. \qquad (1)$$

Deduction (1) is known to be equivalent to

$$\sim Q \Rightarrow \sim P, \qquad (2)$$

and, since Q is false, $\sim Q$ is true and thus, by Deduction (2), $\sim P$ is established.

Consider as an example the theorem:

> *If two angles of a triangle are equal to one another, then the sides which are opposite to the equal angles are equal to one another.*

This is Proposition 6 of Book I of Euclid's *Elements*. It is the *converse*[†] of Proposition 5 of Book I: the theorem known as the *pons asinorum*, stating that the angles at the base of an isosceles triangle are equal. (For more on this, see *Function, Vol. 3, Part 3*.)

The proof of I.6 makes use of Figure 1 at right. We are told that

$$\angle ABC = \angle ACB$$
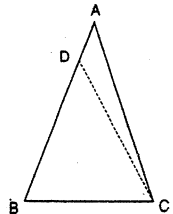
and must prove that, in consequence, $AB = AC$.

Figure 1

[†] Many theorems are themselves statements of the form $P \Rightarrow Q$. The *converse* of such a theorem is the statement $Q \Rightarrow P$. The converse may be true (as in this case) or false. An example of the latter: If the series $a_1 + a_2 + ... + a_n + ...$ converges, then $a_n \to 0$ as $n \to \infty$.   The converse (if $a_n \to 0$, then the series converges) is false.   See Peter Grossman's article in *Function, Vol. 17, Part 2*.

To do this, suppose $AB \neq AC$; then one or other of these lengths, AB say, will be the greater. On AB then, measure off DB such that $DB = AC$. Then join CD as shown.

Now consider the triangles DBC and ACB. We have:

    (1) $DB = AC$, by our hypothesis and construction,
    (2) $BC = CB$, obviously,
    (3) $\angle DBC = \angle ACB$, also by hypothesis.

Thus these triangles are congruent and so have equal areas, which is, of course, a nonsense.

Thus our hypothesis of different lengths has led to an absurdity and so we must abandon it. Let P represent the statement $AB \neq AC$ and let Q be the statement that the whole is equal to one of its parts. We have found

$$P \Rightarrow Q$$

and as Q is false, $\sim P$ is established, i.e. P is false, and so $AB = AC$.

An even starker form of *reductio ad absurdum* is to be found in the event that $Q = \sim P$. That is to say, we assume P and find that

$$P \Rightarrow \sim P. \tag{3}$$

(If P is true, then it is also false!) Clearly P is then false, i.e. $\sim P$ is true. Logicians write this form of the *reductio ad absurdum* argument as

$$(P \Rightarrow \sim P) \Rightarrow \sim P \tag{4}$$

(a proposition that implies its own falsehood must be false).

An example of this mode of argument is to be found in the proof of I.19 of Euclid's *Elements*. This states:

> *If one angle of a triangle is greater than another, then the side opposite the greater angle is greater than the side opposite the less.*

This theorem is the converse of the preceding theorem (I.18) in the *Elements*:

> *If one side of a triangle is greater than another, then the angle opposite the greater side is greater than the angle opposite the less.*

The proof of I.19 runs as follows.

In a triangle ABC (Figure 2 at right), suppose $\angle B > \angle C$. We wish to show that AC (opposite $\angle B$) exceeds AB (opposite $\angle C$).

Begin by assuming that AC is *not* greater than AB. Then:

    either (a) $AC = AB$
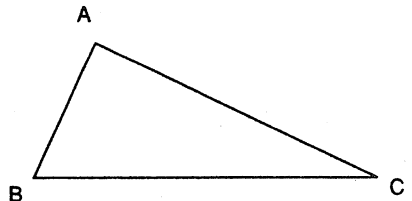   or (b) $AC < AB$.

Figure 2

If (a), then by the *pons asinorum* (I.5) $\angle B = \angle C$, which contradicts the data; if (b), $AB > AC$ and so by I.19 $\angle C > \angle B$, again contradicting the data. We thus cannot entertain either possibility (a) or possibility (b) and so must have

$$(c) \quad AC > AB.$$

The argument has the structure

$$(AC \leq AB) \Rightarrow \sim (AC \leq AB) \tag{5}$$

which is of the form (3) and so our argument is established by Deduction (4).

We may however write Deduction (5) in another way. As I have outlined it, comparing (3) with (5), we have

$$P = (AC \leq AB).$$

But I could instead have written P for $(AC > AB)$, the direct statement I wished to prove. This statement is the negation of $(AC \leq AB)$ and then Deduction (5) would have read

$$\sim(AC > AB) \Rightarrow (AC > AB). \tag{6}$$

This has the same form as Deduction (3), except that direct and negated statements have changed places. Using this notation in Deduction (4) we find

$$(\sim P \Rightarrow P) \Rightarrow P. \tag{7}$$

This is the standard form of a mode of argument known as the *consequentia mirabilis.*

The name is Latin and probably derives from the medieval schoolmen. It translates literally as the "consequence miraculous", but the words "wonderful deduction" give a better feel for the original. Its character of quaint paradox has continued to fascinate logicians through the ages.

The Euclidean proof of I.19 is probably the earliest example of its use in Mathematics, but it was also used, probably about the same time, in philosophical contexts by the Stoics (followers of a school of thought whose origins were roughly contemporary with Euclid).

The clearest example of the use of the *consequentia mirabilis* in Mathematics is probably IX.12 of the *Elements.* This is a proposition in Number Theory – i.e. the study of the properties of the natural numbers (positive integers). Here, Euclid was concerned to show that if p is a prime and a, n are natural numbers and if p divides a^n exactly, then p divides a exactly. In symbols,

$$(p \,|\, a^n) \Rightarrow (p \,|\, a).$$

I will present Euclid's proof here, but not in its original order. The *consequentia mirabilis* appears in its starkest form in the case $n = 2$ and I will begin with that.

First we need a preliminary result:

If $(p \,|\, ab)$ and $(p \nmid a)$, then $(p \,|\, b)$.

(Here $\nmid$ means "does not divide".) This result I will not stop to prove, but it should be fairly obvious.

Now suppose $(p \mid a^2)$ and apply this theorem in the case $a = b$. If we deny $(p \mid a)$, we have

$$(p \nmid a) \implies (p \mid a).$$

So, setting $P = (p \mid a)$, we have Deduction (7), the *consequentia mirabilis*, and have proved that $(p \mid a)$.

This argument may easily be extended to the cases $n = 3, 4, \ldots$. Consider for example the case $n = 3$ and assume $(p \mid a^3)$ but $(p \nmid a)$. Then $(p \mid a^2)$ and the above argument then shows $(p \mid a)$. Similarly for $n = 4, 5, \ldots$. [This is an example of

another standard pattern of proof, known as *mathematical induction*.]

The Euclidean proofs using the *consequentia mirabilis*, particularly to my mind the case $n = 2$ of IX.12, remain perhaps the best illustrations of its use. It was revived by the medieval schoolmen and their successors, mostly in philosophical and theological contexts that lie outside the scope of *Function*. However, in 1967 Girolamo Saccheri (1667-1733) published a work entitled *Logica Demonstrativa* (now very rare). This contains the following interesting passage.

> "It is now my intention to follow another and, as I think, a very beautiful way of proving these same truths without the help of any assumption. I shall proceed as follows: I take the contradictory of the proposition to be proved and elicit the required result from this by straightforward demonstration. This method of proof has been applied by Euclid (IX.12), by Theodosius (*Spherica*, I.12), and by Cardan (*De Proportionibus* V.201), whom Clavius reproves (in his Scholium to IX.12) for boasting that he was the first to discover this kind of proof."

Saccheri goes on to discuss rather technical propositions (here omitted) and to prove them by application of his principle – which is of course the *consequentia mirabilis*. Of the various people mentioned in the passage above, Euclid is familiar, the others perhaps less so.

Theodosius was a Greek mathematician who was probably active around 180 BC. The *Spherica* (or *Sphaerics*) is his principal work and it concerns the geometry of the sphere. Spherical geometry is important in many practical contexts, e.g. navigation. (It was discussed in this context in *Function, Vol.* 14, *Part* 1 and more generally in *Vol.* 6, *Parts* 4 and 5). Proposition 12 of Book I of the *Spherica* considers two circles drawn on the surface of the sphere and supposes that each bisects the other. It shows that in this case the two circles must both be great circles – i.e. circles whose centres coincide with the centre of the sphere.

Cardan (or Cardano) was an Italian algebraist who lived from 1501-1576. He is best remembered for his work on cubic equations, contained in his major work *Ars Magna*. *De Proportionibus* is a rarer and lesser-known book and unavailable to me. Several authors credit Cardan with the first modern use of the *consequentia mirabilis* (in Mathematics, at least) and cite V.201. However, they give no details.

Cristoph Clavius (1537-1612) was the German mathematician we met in *Function, Vol.* 17, *Part* 2 in connection with the Gregorian calendar. Like Saccheri, he was a member of the Jesuit order and likely acquainted with earlier work by the schoolmen. It is thought that it was the Jesuits who coined the name *consequentia mirabilis*.

Clavius wrote commentaries on both Euclid's *Elements* and Theodosius' *Spherica*. Regrettably neither of these nor any edition of the *Spherica* has been available to me. The use of the *consequentia mirabilis* in the proof of I.12 has been variously attributed to Theodosius himself (as in the passage quoted above) or to Clavius as editor. Neither original is available to me, but I have seen a summary of Clavius' version. This summary would seem to show that the proof is inadequate (though the theorem itself is true).

Saccheri himself went on to use the *consequentia mirabilis* in a context that was much more interesting – though his proof was also flawed. Saccheri was one of a number of mathematicians who sought to prove Euclid's "Fifth Postulate" and it is in this context that he is best remembered. The postulates in Euclid, with the exception of the Fifth, are basic assumptions whose truth we may readily accept. For example, the Third Postulate states that we may construct a circle about any centre and with arbitrary radius.

The Fifth, however concerns parallels and, in essence, states that, in Figure 3, the lines l_1 and l_2 are parallel if and only if $\alpha + \beta = 180°$. This postulate seems much less obvious than the others and it struck many people, from Euclid's time till quite recently, that it was the sort of thing that really should be proved.

The suspicion arose that this postulate could in fact be deduced from the others. [In the case of the Fourth Postulate (All right angles are equal) such proof *is* possible.] It is *now* known that the Fifth Postulate *cannot* be so proved, for we may assert its opposite and reach consistent non-Euclidean geometries (see *Function, Vol. 3, Parts* 2 and 4, and *Vol. 12, Part* 4).
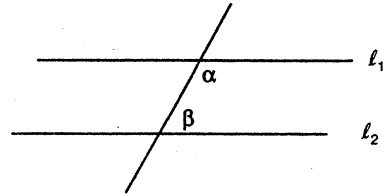

Figure 3

However, before this was known, there were many attempts to prove the Fifth Postulate. One of the best-known and most thorough was Saccheri's. He began by denying the postulate – by supposing that there were cases in which $\alpha + \beta \neq 180°$, but nevertheless l_1 and l_2 were parallel. This is the geometry later invented by Gauss, Bolyai and Lobachevsky and referred to nowadays as Lobachevskian geometry. The consistency of Lobachevskian geometry, the fact that no contradictions may arise in it, was later proved by Poincaré.

Saccheri investigated the consequences of denying the Fifth Postulate and did so in great detail. In the course of this work, he discovered many of the theorems of Lobachevskian geometry. But ultimately he made a mistake and, working from this incorrect basis, he deduced that, after all, the Fifth Postulate was true. What he thought he had was (with P representing the Fifth Postulate):

$$\sim P \Rightarrow P$$

and so, using the *consequentia mirabilis* (7), he argued

$$(\sim P \Rightarrow P) \Rightarrow P$$

and therefore P. So, Euclid was vindicated – his Fifth Postulate was proved.

In a sense, it's sad that he was wrong; it was a genuinely heroic effort, marred by a single, but critical, mistake.

The non-mathematical uses of the *consequentia mirabilis* lie mainly outside the scope of *Function*, but perhaps one deserves mention. This is the proof of the statement (*P*):

> *Some proposition is true.*

To prove this, consider the negation of *P* (~ *P*):

> *No proposition is true.*

But if *P* is false, then ~ *P* is true and ~ *P* is a proposition! So at least one proposition (~ *P*) is true and so *P* is true after all. We may put things thus: *P* is true, because its negation, ~ *P*, is self-contradictory.

[This form of the example has been attributed to the Belgian Arnold Geulincx (1624-1669), but its roots are much older. The Stoics, mentioned briefly above, were at odds with the Sceptics, those who denied the possibility of knowledge. The Sceptics then "knew" that you could know nothing. The irrationality of universal doubt also played a major rôle in the philosophical thought of Descartes (1596-1650) after whom cartesian (co-ordinate) geometry is named.]

To conclude this article, I give a really beautiful demonstration of the following theorem (*P*):

> *An irrational number, raised to an irrational power, may yield a rational answer.*

To prove this, suppose (~ *P*):

> *An irrational number, raised to an irrational power, always yields an irrational answer.*

Now put $\alpha = \sqrt{2}^{\sqrt{2}}$. Then ~ *P* implies that α is irrational. Now form $\alpha^{\sqrt{2}}$,

$$\alpha^{\sqrt{2}} = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}.\sqrt{2}} = \sqrt{2}^{2} = 2,$$

which is rational, so ~ *P* is false and *P* therefore is true.

It should be noted that the above argument, beginning from a premiss which later proves to be false, actually says nothing about the irrationality or otherwise of α. α actually is irrational, but the above argument does not prove this. It does, however, prove the theorem *P* most elegantly and efficiently.

Further Reading

A recent article in the learned journal *History and Philosophy of Logic* drew much of this material to my attention. The most useful of the references given there is *The Development of Logic*, by W. and M. Kneale. Łukasiewicz's *Aristotle's Syllogistic* has a brief account (it calls the *consequentia mirabilis* "The Law of Clavius", presumably because of Clavius' use of it) and other works of logic refer to it, but not always by name. The translation from Saccheri's *Logica Demonstrativa* is taken from the Kneales' book.

PROBLEMS AND SOLUTIONS

We continue in this issue to publish solutions to long-outstanding problems.

SOLUTION TO PROBLEM 14.1.6

See the figure at right depicting a convex pentagon whose perimeter is u $(= AB + BC + CD + DE + EA)$ and with $p = AD + AC + BD + BE + CE$. We want to show that

$$1 < \frac{p}{u} < 2$$

and that the number 1 may not be increased and that the number 2 may not be decreased.

Solution (from Hans Lausch, Monash University) :

(a) Note that $AB + BC > AC$, $BC + CD > BD$, etc.
By addition, find $\frac{p}{u} < 2$.

(b) Next note that $AF + FB > AB$, $BG + GC > BC$, $CH + HD > CD$, $DJ + JE > DE$, $EK + KA > EA$.
By addition, find

$p - (FG + GH + HJ + JK + KF) > u$

so that $p > u$, i.e. $1 < \frac{p}{u}$.

(c) Let $CE = 2$. If D is chosen sufficiently close to the midpoint of the line segment CE, A sufficiently close to E and B sufficiently close to c, then the ratio $\frac{p}{u}$ may be as close to 2 as we like.
Now let $AD = 2$. If B, C and E are chosen sufficiently close to the midpoint of the line segment AD, then the ration $\frac{p}{u}$ will be as close to 1 as we like.
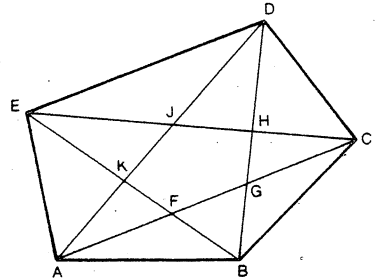
SOLUTION TO PROBLEM 14.3.1

The problem read: $ABCD$ is a rhombus and E the intersection point of its diagonals, AC and BD. Let r_1, r_2, r_3 and r denote the radii of the incircles of the triangles ABE, ABC, ABD and the rhombus $ABCD$ respectively. Prove that

$$\frac{1}{r} + \frac{1}{r_1} = \frac{1}{r_2} + \frac{1}{r_3} .$$

Solution:

In the diagram at right, let $BA = s$. Then $BC = CD = DA = s$ also. Let $\angle BAD = \alpha$. So $\angle CAD = \frac{\alpha}{2}$. Let F be the foot of the perpendicular from E to AD. Then $r = EF$. Thus $r = AE \sin \frac{\alpha}{2}$.
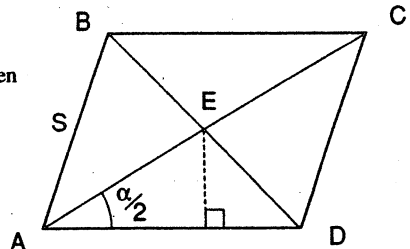
But now it is a theorem that $\angle AED$ is a right angle. Therefore $AE = AD \cos \frac{\alpha}{2} = s \cos \frac{\alpha}{2}$. Thus

$$r = s \sin \frac{\alpha}{2} \cos \frac{\alpha}{2} = \frac{1}{2} s \sin \alpha. \tag{1}$$

To find r_1, r_2, r_3 we need to know a formula. The in-radius of a triangle is equal to its area divided by its semi-perimeter. As $BE = s \sin \frac{\alpha}{2}$, we readily find

$$r_1 = \frac{s}{2} (\sin \alpha)/(1 + \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}) \tag{2}$$

$$r_2 = \frac{s}{2} (\sin \alpha)/(1 + \cos \frac{\alpha}{2}) \tag{3}$$

$$r_3 = \frac{s}{2} (\sin \alpha)/(1 + \sin \frac{\alpha}{2}). \tag{4}$$

The required result now follows, after a little algebra, from Equations (1), (2), (3) and (4).

SOLUTION TO PROBLEM 14.3.2

The problem read: ABC is a triangle right-angled at A, and D is the foot of the altitude from A. Let X and Y be the incentres of triangles ABD and ADC respectively. Determine the angles of triangle AXY in terms of triangle ABC.

Solution: Consult the diagram at right. The dotted lines bisect the angles at A, B, C, D as shown. Thus if $\angle ABC = \beta$ and $\angle ACB = \gamma$, then $\beta + \gamma = \frac{\pi}{2}$. $\angle BAD = \gamma$ and $\angle CAD = \beta$. Thus $\angle XAY = \frac{1}{2}(\beta + \gamma) = \frac{\pi}{4}$. Let $AB = b$. Then $AC = b \tan \beta$.

Apply the sine rule to the triangle ABX. Note that $\angle AXB = \pi - \left[\frac{\beta+\gamma}{2} \right] = \frac{3\pi}{4}$. Thus

$$AX = (\sin \frac{\beta}{2})b/(\sin \frac{3\pi}{4}) = \sqrt{2}\, b \sin \frac{\beta}{2}.$$

Similarly,

$$AY = \sqrt{2}\, b \tan \beta \sin \frac{\gamma}{2}.$$

Put $\angle AXY = \theta$. Then $\angle AYX = \pi - \theta - \frac{\pi}{4} = \frac{3\pi}{4} - \theta$. Now apply the sine rule to the triangle AYX.
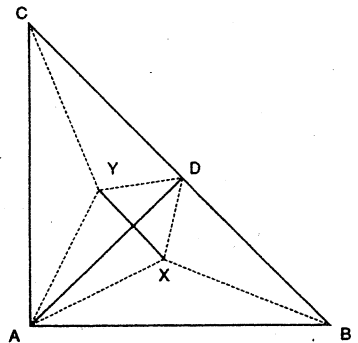
$$\frac{\sqrt{2}\, b \tan \beta \sin \gamma/2}{\sin \theta} = \frac{\sqrt{2}\, b \sin \beta/2}{\sin(\frac{3\pi}{4} - \theta)}.$$

After a large number of somewhat tedious manipulations (here omitted in the interest of brevity), this equation reduces to

$$\tan \theta = \tan \left[\frac{\pi}{2} + \frac{\beta}{2} \right]$$

so $\theta = \frac{\pi}{2} + \frac{\beta}{2}$ and $\frac{3\pi}{4} - \theta = \frac{\pi}{4} - \frac{\beta}{2}$.

The angles are thus $\frac{\pi}{4}$, $\frac{\pi}{2} + \frac{\beta}{2}$, $\frac{\pi}{4} - \frac{\beta}{2}$.

SOLUTION TO PROBLEM 14.3.3

The figure at right comes from a work by the 17th-Century astronomer Longomontanus. $\angle ABC$ is a right angle and thus the arc ABC is a semi-circle. The length of this arc is therefore πa.

We seek to determine the length BG.

By Pythagoras' Theorem, $BC = a\sqrt{3}$, and by similar triangles $CG = \frac{1}{2}b\sqrt{3}$. Thus $BG = a\sqrt{3} + \frac{1}{2}b\sqrt{3}$.

Longomontanus had $a = 43$, $b = 70$ and so reached $BG = 135.09996$. The arc ABC measures 135.08848 which is very close. We thus have a quite remarkable approximation

$$\pi \simeq \sqrt{3}\left(1 + \frac{35}{43}\right) = \frac{78}{43}\sqrt{3} = 3.1418... \ .$$

It is now known that no such approximation can yield an *exact* value for π. This is referred to as the impossibility of "squaring the circle".

SOLUTION TO PROBLEM 14.4.1

Simplify $(x-a)(x-b)(x-c) \ ... \ (x-z)$.
Answer: 0.

Try this one on your friends!

SOLUTION TO PROBLEM 14.4.2

The problem read: The bisectors of the angles C and D of a convex quadrilateral $ABCD$ meet at a point P on AB such that $\angle CPD = \angle DAB$. Prove that P is the mid-point of AB.

Solution: Consult the diagram at right, and label the angles α, β, γ, in accordance with the data, as shown. Then $\alpha + \beta + \gamma = \pi$, and so

$$\angle APD = \beta, \ \angle BPC = \alpha, \ \angle PBC = \gamma.$$

Then the triangles PCB, DCP and DPA are all similar. Because PCB and DCP are similar:

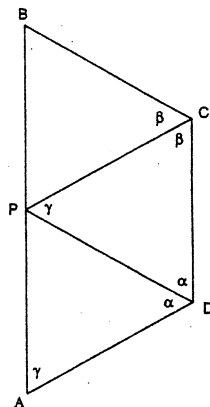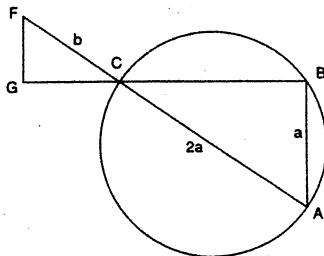$$\frac{BC}{BP} = \frac{CP}{PD} \quad \text{or} \quad BP = BC.AD/CP.$$

Because PCB and DPA are similar:

$$\frac{BC}{CP} = \frac{AP}{PD} \quad \text{or} \quad AP = BC.PD/CP.$$

Thus $AP = BP$, which is the required result.

SOLUTION TO PROBLEM 14.4.3

The problem read: ABC is a triangle, right-angled at C. Let CD be perpendicular to AB. The bisector of $\angle CDB$ meets CB in X, and the bisector of $\angle ADC$ meets AC in Y. Prove $CX = CY$.

Solution: A coordinate geometry approach works well. Let C be $(0,0)$, A be $(0,a)$ and B be $(b,0)$. Let $\angle BCD$ be α, where $\tan \alpha = b/a$. [This last is readily proved.] D is the point

$$\left[\frac{ab^2}{a^2+b^2} , \frac{a^2b}{a^2+b^2} \right]$$

and DX makes an angle in radians[†] of $(\frac{\pi}{4} + \alpha)$ with CB. Thus DX has slope $\tan(\frac{\pi}{4} + \alpha)$ and this is $(b+a)/(b-a)$. We may thus obtain the equation of the line DX and so find the x-coordinate of x to be $ab/(a+b)$. I.e.

$$CX = ab/(a+b).$$

The symmetry of this expression now allows us to conclude that CY also equals $ab/(a+b)$. Hence the result.

SOLUTION TO PROBLEM 14.4.4

The problem read: $\angle BAC$ is an obtuse angle. A circle through A cuts AB at P and AC at Q. The bisectors of angles $\angle QPB$ and $\angle PQC$ cut the circle at X and Y respectively. Prove that XY is perpendicular to the bisector of $\angle BAC$.

Solution: Let AZ bisect $\angle BAC$ and meet XY at Z. (See diagram.) Extend PX and AZ to meet at D. Let AZ, PQ cut at E.
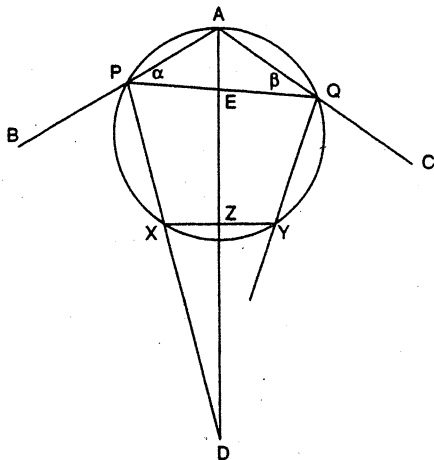
[†] In elementary geometry, it matters little what angular units are used. However, in a calculus context, one must employ radians, so it is a good idea to get into the habit early.

For simplicity, put $\angle QPA = \alpha$, $\angle PQA = \beta$. Then, working in radians, $\angle BPQ = \pi - \alpha$ and so $\angle XPQ = \frac{1}{2}(\pi - \alpha)$. Similarly $\angle YQP = \frac{1}{2}(\pi - \beta)$. But $\angle PAQ = \pi - \alpha - \beta$, so $\angle PAZ = \frac{1}{2}(\pi - \alpha - \beta)$.

Now $\angle PEZ = \angle QPA + \angle PAZ = \frac{1}{2}(\pi + \alpha - \beta)$ and similarly $\angle QEZ (= \angle PEA) = \frac{1}{2}(\pi - \alpha + \beta)$. But now

$$\angle PDA = \angle PEA - \angle XPQ = \tfrac{1}{2}\beta.$$

Furthermore, $PQYX$ is a quadrilateral inscribed in a circle, so $\angle PXY = \pi - \angle YQP = \frac{1}{2}(\pi + \beta)$. So $\angle DXZ = \frac{1}{2}(\pi - \beta)$.

Now $\angle AZX = \angle DXZ + \angle PDA$
$$= \tfrac{1}{2}(\pi - \beta) + \tfrac{1}{2}(\pi + \beta) = \frac{\pi}{2}$$

as required.

SOLUTION TO PROBLEM 14.4.5

The problem stated: $ABCD$ is a square and P a point on the circumcircle and lying between A and B. The distances from P to A, B, C and D are denoted by a, b, c and d respectively. Show $(\sqrt{2}+1)(a+b) = d + c$ and that $a - b = (\sqrt{2}+1)(d-c)$.

Solution. The diagram below illustrates the situation.
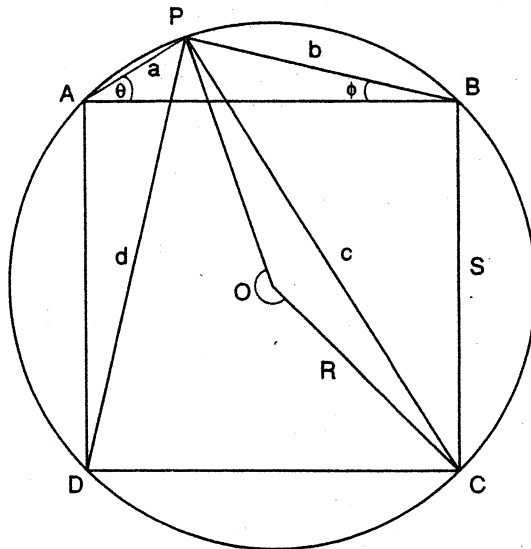
Let $\angle PAB = \theta$, $\angle PBA = \varphi$; let s be the length of the side of the square and R be the radius of the circle. Let O be the centre of the circle.

Apply the sine rule to the triangle ABC. Then

$$a = 2R \sin \varphi, \quad b = 2R \sin \theta, \quad s = 2R \sin(\theta+\varphi). \tag{1}$$

Further $\angle COP$ (the reflex angle indicated) is twice $\angle CBP$, i.e. $\pi + 2\varphi$. Thus the angle COP (the one in the triangle COP) is $\pi - 2\varphi$. Now apply the cosine rule to the triangle COP. $c^2 = R^2 + R^2 - 2R^2\cos(\pi-2\varphi) = 2R^2[1 + \cos 2\varphi] = 4R^2\cos^2\varphi$.
Thus

$$c = 2R \cos \varphi. \tag{2}$$

Similarly

$$d = 2R \cos \theta. \tag{3}$$

Now

$$\frac{c+d}{a+b} = \frac{\cos \varphi + \cos \theta}{\sin \varphi + \sin \theta} = \frac{2\cos(\frac{\theta+\varphi}{2})\cos(\frac{\theta-\varphi}{2})}{2\sin(\frac{\theta+\varphi}{2})\cos(\frac{\theta-\varphi}{2})} = \frac{1}{\tan(\frac{\theta+\varphi}{2})}. \tag{4}$$

But $s = R\sqrt{2}$ and so, by (1),

$$\sin(\theta + \varphi) = 1/\sqrt{2}$$

and

$$\theta + \varphi = \frac{\pi}{4}.$$

Thus

$$\frac{\theta+\varphi}{2} = \frac{\pi}{8}$$

and

$$\tan(\frac{\theta+\varphi}{2}) = \frac{1}{\sqrt{2}+1}.$$

[The reader may prove this as an exercise.]

The first part of the problem is thus proved. The second follows likewise.

SOLUTION TO PROBLEM 14.4.10

The problem asked for a proof that $2x + 3y$ and $9x + 5y$ are divisible by 17 for the same set of integral values of x and y.

Solution: Suppose $2x + 3y$ is divisible by 17. Then $2x + 3y = 17m$ for some integer m. Then, multiplying by 9, we find

$$18x + 27y = 9 \times 17m$$

or

$$18x + 10y = 9 \times 17m - 17y.$$

The right-hand side is clearly divisible by 17 and so

$$18x + 10y = 17n$$

for some integer n, which also clearly must be even. Therefore $9x + 5y$ is divisible by 17.

A similar argument shows that if $9x + 5y$ is divisible by 17, then so is $2x + 3y$.

SOLUTION TO PROBLEM 14.4.11

The problem asked for a proof that all integers, positive or negative, can be expressed in the form

$$a_p(-2)^0 + a_1(-2)^1 + a_2(-2)^2 + \ldots + a_n(-2)^n,$$

where each of the a_m is either 0 or 1.

Solution: Consider first positive integers. These may be expressed in binary form, that is to say in the form

$$b_0 2^0 + b_1 2^1 + b_2 2^2 + \ldots + b_n 2^n$$

where each of the b_m is either 0 or 1.

The first two columns in the table below list the first few numbers in decimal and in binary.

0	0	0		
1	1	1	⟵	$1 + 0 = +1$
2	10	−2	⟵	$-2 + 0 = -2$
3	11	−1		$-2 + 1 = -1$
4	100	4	⟵	$4 + 0 = +4$
5	101	5		$4 + 1 = +5$
6	110	2		$4 - 2 = +2$
7	111	3		$4 - 1 = +3$
8	1000	−8	⟵	$-8 + 0 = +8$
9	1001	−7		$-8 + 1 = -7$
10	1010	−10		$-8 - 2 = -10$
11	1011	−9		$-8 - 1 = -9$
12	1100	−4		$-8 + 4 = -4$
13	1101	−3		$-8 + 5 = -3$
14	1110	−6		$-8 + 2 = -6$
15	1111	−5		$-8 + 3 = -5$
16	1000	16	⟵	$16 + 0 = 16$

The 1's and 0's of Column 2 are the b_m described above. However, if we interpret them as a_m, we find using base −2, the decimal numbers of Column 3.

Note the structure of this column. The arrows (Column 4) pick out powers of −2, alternately positive and negative. They separate vertical blocks of numbers. The first block (0, 1) contains two numbers. Starting at −2, add first 0 and then 1 to produce −2, −1 and so a block of 4 (0, 1, −2, −1). The next power is 4 and to this add successively the four numbers just mentioned. This produces a block of 8 (0, 1, −2, −1, 4, 5, 2, 3) and the next power of −2 is −8 and we produce 8 more

entries in the table. After this we will have 16 entries and so on.

Before 4, we have generated the integers −2, −1, 0, 1.
Before −8, we have generated the integers −2, ..., 5.
Before 16, we have generated the integers −10, ..., 5.

Continuing this pattern, we see that before 32, we will have generated a run of 32 integers starting with −10, i.e. −10, ..., 21; before 64, a run of 64 integers ending with 21, i.e. −42, ..., 21.

Thus all positive and all negative integers are eventually generated.

This analysis also answers PROBLEM 14.5.1 for if the number of digits is even, the number is negative, and yes, the representations are clearly unique.

SOLUTION TO PROBLEM 14.4.12

The problem is historically based and may be stated in terms of the diagram below. DBL is the diameter of a circle and BZ is a radius perpendicular to it. We are told that $ST = BD$ and are asked to prove that $\angle\,TBL = \frac{1}{3}\angle\,DBE$.
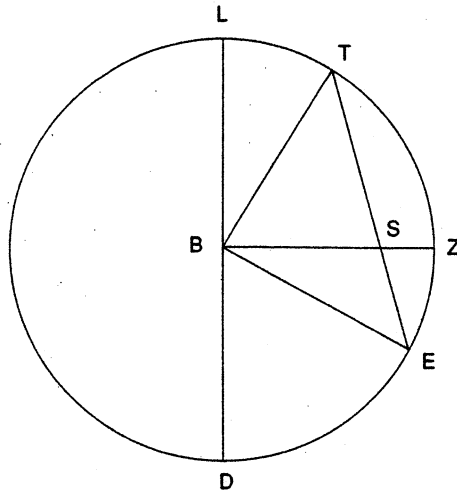
Solution: Let $\angle\,TBL = \theta$. Then $\angle\,TBS = \frac{\pi}{2} - \theta$ and since $ST = DB = BT$, $\angle\,TSB = \angle\,TBS = \frac{\pi}{2} - \theta$. Thus $\angle\,BTS = 2\theta$. But now in the triangle TBE, $TB = EB$ and so $\angle\,TEB = 2\theta$ also. Thus $\angle\,TBE = \pi - 4\theta$, whence $\angle\,LBE = \pi - 3\theta$ and so $\angle\,DBE = 3\theta$.

The problem raises the question as to whether we have successfully trisected the angle DBE. We have, but it is not possible to construct the point T with ruler and compass alone. No such construction is possible.

For more on angle trisection, see *Function, Vol. 3, Part 3.*

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