Function

Founder Editor G. B. Preston

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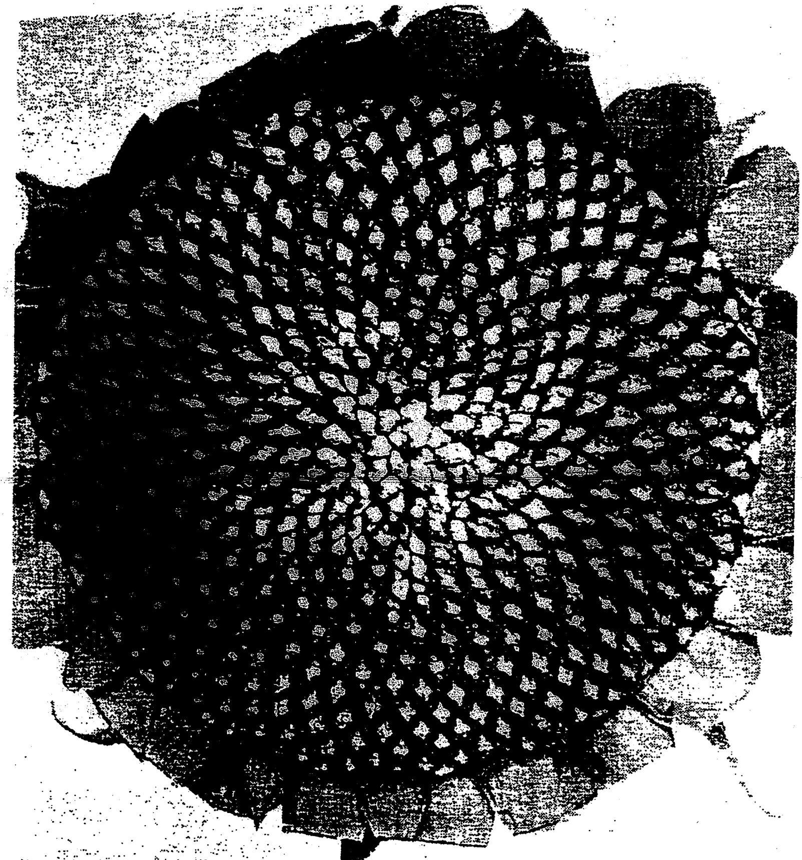

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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

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Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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FUNCTION

Volume 16 *Part 4*

(Founder editor: G.B. Preston)

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THE FRONT COVER

Michael A.B. Deakin, Monash University

Our front cover picture shows the head of a sunflower. It first appeared in a 1904 book by A.H. Church, entitled *On the Relation of Phyllotaxis to Mechanical Laws.* "Phyllotaxis" is a technical term used in Botany; strictly speaking, it means "leaf arrangement". It was observed last century that if one counted up leaves from the base of a plant, they wound their way round the stem. If we label one particular leaf as No. 0, then, counting up we find that the next leaf to lie vertically above this will often be No. 2, or No. 3, or No. 5, or No. 8, etc. where the numbers 2, 3, 5, 8, 13, ... form the so-called Fibonacci sequence (see *Function, Vol.* 1, *Part* 1; *Vol.* 12, *Part* 4; and Robyn Arianrhod's article in this issue).

This is the origin of the name "phyllotaxis". Later it was discovered that the patterns persisted with other botanical examples, but with larger Fibonacci numbers. Leaves in certain succulents, the facets of pine-cones or pineapples, the seeds of sunflowers, the petals of dahlias and the florescences of cauliflowers all show spiral patterns, based on the Fibonacci sequence. The arrangement of the seeds in the head of a sunflower is perhaps the easiest case to appreciate and to analyse.

Look at Figure 1 which reproduces the cover picture. The seeds in the sunflower head shown there appear to be arranged in spirals. Looking carefully at these, indeed, we can discern two predominant types: shortish spirals coming clockwise out of the centre and proceeding out to the edge of the flower, and longer ones coming anti-clockwise out of the centre and proceeding out to circles somewhat in from the edge. Figure 2 highlights these two types of spiral by showing the effect of removing some of the seeds to increase the contrast.

Careful counting reveals that there are 34 long anti-clockwise spirals and 55 short clockwise ones. 34 and 55 are both Fibonacci numbers. In the case of other sunflowers studied by Church, the numbers 34 and 55 were replaced by 55 and 89 respectively – or even by 89 and 144 respectively. These numbers, too, are Fibonacci numbers.

Despite the efforts of Church (whose title indicates his intent to explain such phenomena) and others in researching the field, the whole area remains a contentious one. No widely accepted explanation has yet been found and it has even been doubted that the phenomena are quite as simple as I have indicated. For example, not all sunflower heads do exhibit such regularities. Church himself has cases of 18 spirals against 29, or 29 against 47. These can be generated from what Robyn Arianrhod in her article (pp. 107-112) calls the "Fibonacci-type" sequence

$$2, 1, 3, 4, 7, 11, 18, 29, 47, \dots .$$

But what significance are we to ascribe to this?

Coxeter in his *Introduction to Geometry* summed it up rather well: "... phyllotaxis is really not a universal *law* but only a fascinatingly prevalent *tendency*".

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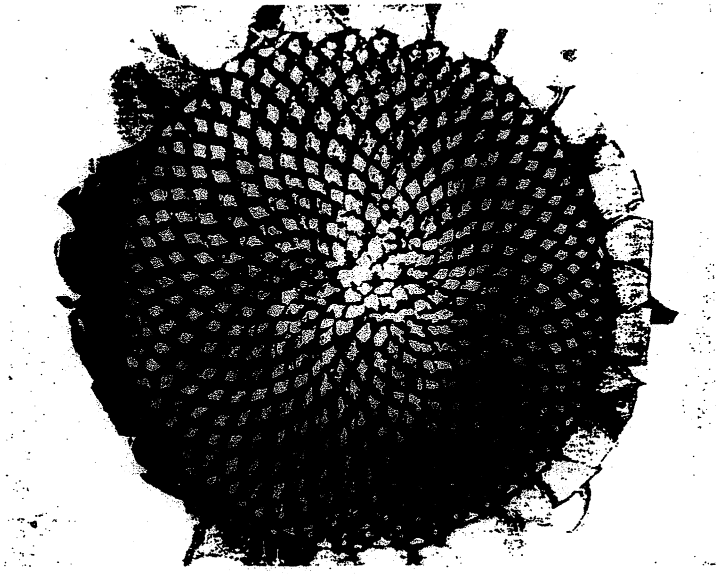


Figure 1

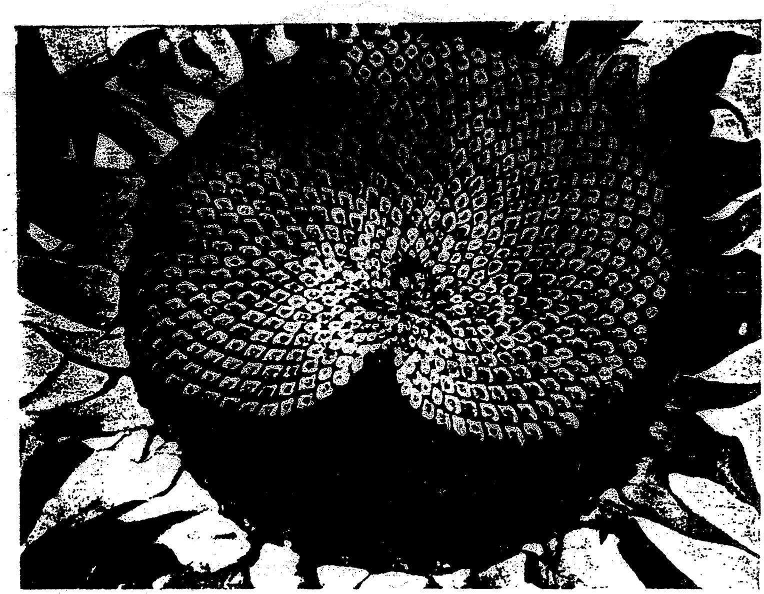


Figure 2

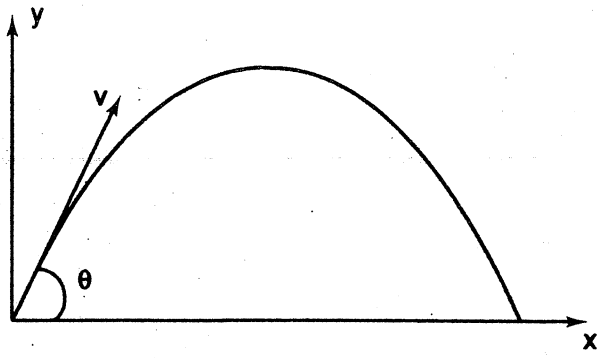
SEEING PARABOLAS

Rodney Carr, Deakin University, Warrnambool

I was walking through Bulga National Park in Gippsland a few weeks ago and on one of the walks is a suspension bridge going over a beautiful fern-covered gully. Of course, you're supposed to look at the ferns, but the bridge itself is nice, too. The shape of the curve traced out by the cables is approximately parabolic (I'll verify this for you at the end). Pretty curve, the parabola, and it pops up in a lot of places. I thought I'd try and think of all the places where I've "seen" a parabola lately.

First, the most obvious example occurred when my kids were out playing with the hose, squirting each other. As most people who have studied mechanics will know, the path traced out by the stream is, neglecting air resistance, a parabola. This is pretty easy to show.

Suppose that an object is thrown with an initial velocity v from the origin of a coordinate system at a fixed angle θ as shown.



Initially the vertical velocity is $v \sin \theta$ and the horizontal velocity is $v \cos \theta$.

Let $\underset{\sim}{r} = \underset{\sim}{i}x + \underset{\sim}{j}y$ be the position vector of a drop of water in the stream. Then, by Newton's second law, as the only force acting is gravity,

$$\ddot{\underset{\sim}{r}} = -g\underset{\sim}{j},$$

where the dots indicate a double differentiation with respect to time, t.

Integrating once gives

$$\dot{\underset{\sim}{r}} = -gt\underset{\sim}{j} + \underset{\sim}{u}$$

where $\underset{\sim}{u}$ is a constant vector. In fact $\underset{\sim}{u}$ is the velocity of the drop when $t = 0$, and so

$$\underset{\sim}{u} = (v \cos \theta)\underset{\sim}{i} + (v \sin \theta)\underset{\sim}{j}.$$

So

$$\dot{\underset{\sim}{r}} = (v \cos \theta)\underset{\sim}{i} + (v \sin \theta - gt)\underset{\sim}{j}.$$

Integrating again, find

$$\underset{\sim}{r} = (vt \cos \theta)\underset{\sim}{i} + (vt \sin \theta - \tfrac{1}{2}gt^2)\underset{\sim}{j} + \underset{\sim}{a},$$

where $\underset{\sim}{a}$ is a constant vector. However, when $t = 0$, $\underset{\sim}{r} = \underset{\sim}{0}$ and thus $\underset{\sim}{a} = \underset{\sim}{0}$. Thus

$$\underset{\sim}{r} = (vt \cos \theta)\underset{\sim}{i} + (vt \sin \theta - \tfrac{1}{2}gt^2)\underset{\sim}{j}.$$

It follows that

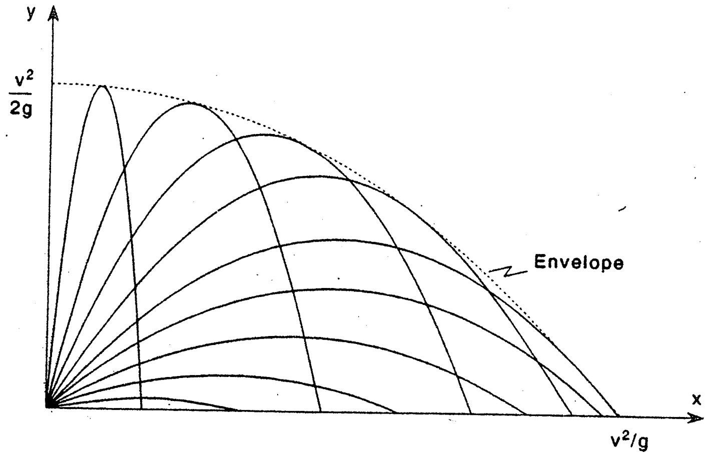
$$\left.\begin{aligned} x &= vt \cos \theta \\ y &= vt \sin \theta - \tfrac{1}{2}gt^2 \end{aligned}\right\}.$$

Eliminating t from these equations gives the relationship between y and x:

$$y = x \tan \theta - \left[\frac{g}{2v^2 \cos^2 \theta} \right] x^2. \qquad (*)$$

This is a parabola. The different drops of water in the stream all trace out this curve, so you see the entire parabola, all at once.

I wonder if my kids (aged 4 and 2) realized this. Probably not, but since they were interested in squirting each other they should have at least figured out the set of points (in space) that could be reached by the water from altering the angle of inclination of the hose's nozzle. The boundary curve is, not too surprisingly, another parabola. I could "see" this curve when one of the children was waving the hose madly up and down without moving his hand much. It's a little harder to see that this curve is a parabola, but it's not too bad.



From (*)

$$y = x \tan\theta - \frac{gx^2}{2v^2}(1 + \tan^2\theta)$$

so that

$$2v^2y = 2v^2x \tan\theta - gx^2 - gx^2 \tan^2\theta.$$

i.e.

$$gx^2 \tan^2\theta - 2v^2x \tan\theta + (2v^2y + gx^2) = 0.$$

This is a quadratic equation in $\tan\theta$, and thus

$$\tan\theta = \frac{2v^2x \pm \sqrt{4v^4x^2 - 4gx^2(2v^2y + gx^2)}}{2gx^2}.$$

This simplifies to

$$\tan\theta = \frac{v^2 \pm \sqrt{v^4 - 2gv^2y - g^2x^2}}{gx}.$$

This equation has precisely one solution for $\tan\theta$ if and only if

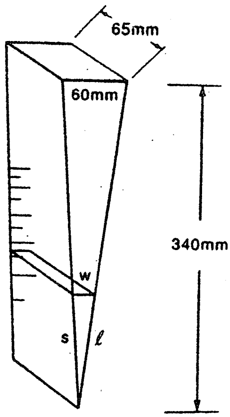
$$v^4 - 2gv^2y - g^2x^2 = 0$$

i.e.

$$y = \frac{v^2}{2g} - \left(\frac{g}{2v^2}\right)x^2$$

which is another parabola.

Then there was a thunderstorm and the kids came inside. It absolutely poured down for half an hour and then stopped. I then went outside to the rain gauge to see how much it had rained. And what did I see – another parabola! Well, not exactly, but a quadratic (i.e. parabolic) relationship was implied. Here is a picture of the rain-gauge.



Rainfall is measured as the depth that the water would cover a flat surface if it didn't run or drain away. So if it has rained an amount of R mm, the gauge would have captured

$$R \times 60 \times 65 \text{ mm}^3.$$

This amount of water reaches a distance s mm up the side of the gauge and forms a prism with the base being a right angled triangle with legs of length l and w mm, as shown. By similar triangles and Pythagoras' Theorem

$$\frac{l}{340} = \frac{s}{\sqrt{60^2+340^2}}, \quad \text{i.e.} \quad l = \frac{17}{\sqrt{298}}\, s;$$

$$\frac{w}{60} = \frac{s}{\sqrt{60^2+340^2}}, \quad \text{i.e.} \quad w = \frac{3}{\sqrt{298}}\, s.$$

The volume of water in the prism is

$$(\tfrac{1}{2}lw)65 = \frac{3315}{596}\, s^2$$

and so

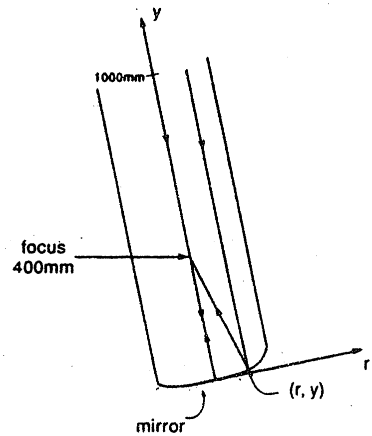
$$R \times 60 \times 65 = \frac{3315}{596}\, s^2,$$

$$\text{i.e.} \quad R = \frac{17}{11920}\, s^2 \approx 0.0014s^2,$$

and this is the quadratic relation. The marks up the side of the gauge show this calibration.

(Note: you can see why rain gauges are shaped the way they are – they greatly magnify the depth of the water. For example, if it rains enough so that the water in the gauge reaches a distance of 10 mm up the slanted side, the actual amount of rainfall is only about $(0.0014)(10^2)$ mm, i.e. 0.14 mm, not much!

That evening, after the clouds had gone away, I got out my telescope to look at the moons of Jupiter. They are always nice to look at – did you know that you can actually see them quite clearly with a reasonable pair of binoculars? And there was another parabola! Through my telescope the moons of Jupiter are approximately points and are so far away that the light coming from them to the telescope is very nearly parallel. The mirror of the telescope reflects the light so that it ends up at a point (the focus) about 400 mm away. The mirror shape that achieves this is a paraboloid, i.e. a shape achieved by rotating a parabola. It's easy to see why:



According to a powerful principle in Optics, light travels by the shortest available path between two points. So if two photons enter the end of the telescope together and end up together at the focus, they must travel the same distance.

A photon travelling through the point (r, y) from $(r, 1000)$ and ending at the focus $(0, 400)$ travels a distance (in mm)

$$1000 - y + \sqrt{r^2+(400-y)^2},$$

while an axial photon travels 1400 mm.

Thus

$$1400 = 1000 - y + \sqrt{r^2+(400-y)^2}$$

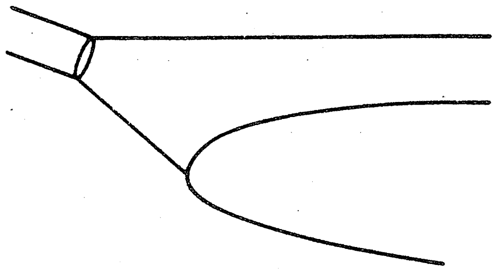
and this simplifies to give

$$y = 0.000625r^2$$

which is, of course, a (very "flat") parabola.

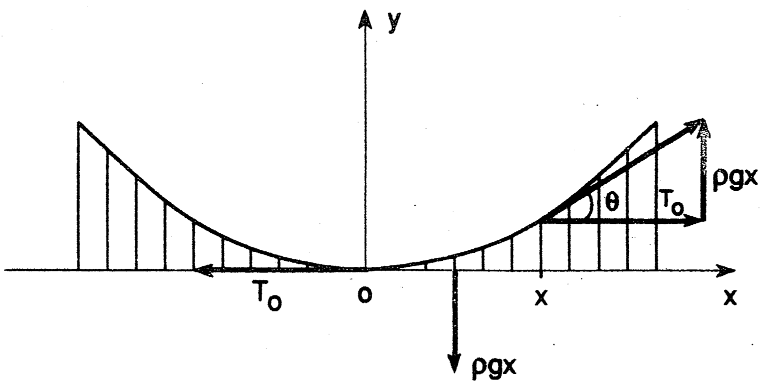
Note that this calculation also works in reverse: spotlights and car headlights reflect light from a focus into a beam of parallel rays. The radiator warming my feet does much the same with radiant heat. All these devices have parabolic cross-sections.

So the moons were still there, looking just as beautiful as ever. When I had finished I turned on my torch while I packed things away. (I didn't turn it on before, of course – didn't want to ruin my night vision.) The beam struck the ground with the top ray of the outer cone (not the bright inner beam) shining parallel to the ground, as shown.



Blow me down, there was yet another one of the devils – a perfect parabola! In fact this is one of the original definitions of a parabola – a parabola is formed when a plane cuts a cone parallel to one of its generators† . In our case the outer rays of the light beam generate the cone; the ground forms a plane that lies parallel to a generator. (Note: other curves can be formed this way. If the torch is pointed down so that the whole of the beam shines on the ground an ellipse is formed; if it is pointed up so that the ground is not parallel to the edge of the light cone an hyperbola is formed. Together these curves are called the "conic sections".)

I said I'ld verify that the shape of the cables on a suspension bridge is parabolic. Here goes!



† The *generators* of a cone are the straight lines that may be drawn on its surface. They connect the apex to the points on a circular cross-section of the cone. Rotating any generator so that this latter end goes around this circle, produces, or generates, the conical surface.

Set up co-ordinates so that the middle of the bridge is taken as the origin, the x-axis is horizontal and the y-axis vertical. Suppose the bridge has a linear density of ρ kg m^{-1} and that the density of the cables and other structures may be neglected.

Consider a section of the bridge from $(0, 0)$ to $(x, 0)$. This will have a weight $W = \rho g x$ and this will act in the downward direction. The portion of cable supporting this section will pass through $(0, 0)$ at its left-hand end and (x, y) at its right-hand end. At $(0, 0)$ there will be a horizontal tensile force T_0 as shown. At the right, there will be a force T acting in a direction tangential to the cable.

The horizontal component of T must equal T_0 to counteract the tension on the left. The vertical component must equal W to hold up the bridge. Thus, if θ is the angle shown,

$$\tan\theta = \frac{dy}{dx} = \frac{\rho g x}{T_0} .$$

Thus

$$y = \left[\frac{\rho g}{2T_0}\right]x^2 + c$$

where c is a constant. In fact, as $(0, 0)$ is on the curve, $c = 0$ and

$$y = \left[\frac{\rho g}{2T_0}\right]x^2,$$

which is the equation of a parabola.

(Note: If we can't neglect the weight of the cables the curve is not a parabola – it is more complicated. In the case that the cables hang freely with no additional load (like overhead power cables) the curve assumed is called a catenary. Galileo, who figured out that the curve followed by a projectile is a parabola also thought that this shape was a parabola, but he was wrong.)†

Well, that's just about it, though I'm sure that I've seen other parabolas without knowing it. See if you can find some. And see if you can find other curves as well in your daily travels – you'll be very surprised at just how many there are around!

* * * * *

† See *Function, Vol.* 16, *Part* 2, p.43.

FIBONACCI SEQUENCES

Robyn Arianrhod, Monash University

I'm sure you've heard of the Fibonacci† sequence, whose first two terms are both 1 and whose subsequent terms are each obtained by adding the immediately preceding two terms:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \dots \qquad (1)$$

This sequence seems to appear in certain plant growth patterns. For example, the head of a sunflower (or similar member of the daisy family) contains embryonic seeds which are arranged in sets of clockwise and anti-clockwise spirals.†† There is a relationship between the number of clockwise spirals and the number of anti-clockwise spirals in any sunflower: these two numbers are adjacent terms in the Fibonacci sequence. This type of spiral growth can also be seen, for example, in pine cones and in the growth of spines on some cacti.

In an article in a recent *New Scientist* magazine (18 April 1992, p.18), Christine Sutton reported on the work of two French scientists, S. Douady and Y. Couder, who observed the same Fibonacci-related spiral pattern in a *non-botanical* dynamical system (consisting of droplets of oil in a magnetic field). After analysing their system and results, they concluded that these Fibonacci-related spiral patterns emerge in certain dynamical systems (including some botanical ones such as sunflower heads!) in such a way that the energy of the system is kept to a minimum. Thus, for example, when the embryonic seeds in the sunflower are laid down in the most energy-efficient way possible for the plant, they must form Fibonacci-related spirals.

These conclusions make the Fibonacci sequence quite fascinating. In this article, I want to show you some mathematical properties of the Fibonacci sequence.

The Golden Ratio

The terms of the Fibonacci sequence can be denoted by $t_1, t_2, \dots, t_n, \dots$, where

$$t_1 = t_2 = 1, \quad t_n = t_{n-1} + t_{n-2}, \quad n > 2. \qquad (2)$$

What happens to the ratio of successive terms, $r_n = \dfrac{t_{n+1}}{t_n}$, as n approaches infinity?

Does this ratio approach some particular number? With some examples on a calculator, you can see that it seems to do so.

† Leonardo di Pisa (nicknamed Fibonacci) (1175-1250) published his work in 1202 in a book titled *Liber Abaci.*

†† See the front cover of this issue and the article on pp. 98-99.

We have

$$r_1 = 1, \; r_2 = 2, \; r_3 = 1{\cdot}5, \; r_4 = 1{\cdot}666\ldots, \ldots,$$

$$r_9 = 1{\cdot}6176\ldots, \; r_{10} = 1{\cdot}6181\ldots, \; r_{11} = 1{\cdot}6179\ldots, \text{ etc.}$$

In fact, as n approaches infinity, the ratio of successive terms of the Fibonacci sequence approaches a number which turns out to be the one which the ancient Greeks called the *golden ratio*, because they felt that a rectangle whose sides had this particular ratio was particularly aesthetic. (Hence such rectangles can often be seen in Classical architecture.) The following is a derivation of this ratio.

$$r_n = \frac{t_{n+1}}{t_n} = \frac{t_n + t_{n-1}}{t_n} = 1 + \frac{t_{n-1}}{t_n} \qquad (3)$$

Now, as n approaches infinity, the ratio of successive terms approaches a constant, which we will call R; thus, in this limit, $\frac{t_{n-1}}{t_n}$ approaches $\frac{1}{R}$. Equation (3) then becomes

$$R = 1 + \frac{1}{R} \qquad (4)$$

or, equivalently,

$$R^2 = R + 1. \qquad (5)$$

The solution to Equation (5) is

$$R = \frac{1 \pm \sqrt{5}}{2}. \qquad (6)$$

Since all the Fibonacci terms are positive, choose the positive root

$$R = \frac{1 \pm \sqrt{5}}{2}. \qquad (7)$$

This is the golden ratio.

The sum of the terms of a Fibonacci sequence

Can you find an expression for the sum of the first n terms in the Fibonacci sequence, $t_1 + t_2 + t_3 + \ldots + t_n$? One way of doing so is the following. Keep in mind the definition given in Equation (2).

$$t_3 = t_2 + t_1 \quad \text{therefore} \quad t_1 = t_3 - t_2$$

$$t_4 = t_3 + t_2 \quad \text{therefore} \quad t_2 = t_4 - t_3$$

$$t_5 = t_4 + t_3 \quad \text{therefore} \quad t_3 = t_5 - t_4$$

$$\cdots\cdots\cdots\cdots$$

$$t_{n+1} = t_n + t_{n-1} \quad \text{therefore} \quad t_{n-1} = t_{n+1} - t_n$$

$$t_{n+2} = t_{n+1} + t_n \quad \text{therefore} \quad t_n = t_{n+2} - t_{n+1}.$$

Adding the terms in the right-hand sides of the above list of expressions, we see that

$$t_1 + t_2 + \ldots + t_n = t_{n+2} - t_2 = t_{n+2} - 1. \tag{8}$$

Thus, we can express the sum of the first n terms of the Fibonacci sequence in terms of the $(n+2)$nd term! In fact, we have a compact formula (known as *Binet's formula*, a proof of which is given below) for the nth term t_n:

$$t_n = \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]/\sqrt{5} \tag{9}$$

Thus, $t_1 + t_2 + \ldots + t_n = t_{n+2} - 1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+2}\right] - 1.$ Notice the presence of the golden ratio!

A Proof of Binet's Formula

The Fibonacci sequence is one of many possible sequences $t_1, t_2, \ldots, t_n, \ldots,$ where the rule for obtaining t_n is

$$t_n = t_{n-1} + t_{n-2}. \tag{10}$$

Let us refer to sequences whose terms obey Equation (10) as "Fibonacci-type" sequences. (The distinguishing feature of the Fibonacci sequence is that its first two terms are both 1: $t_1 = t_2 = 1$ (see Equation (2)); for the moment, however, we will consider only certain "Fibonacci-type" sequences).

Let us start with a "Fibonacci-type" sequence (whose terms are given by the rule (10)) which is *also a geometric sequence*

$$1, p, p^2, p^3, \ldots . \tag{11}$$

(Recall that p is the ratio of successive terms in a geometric sequence: $t_{n+1}/t_n = p$.)

Clearly, the "Fibonacci-type" condition, $t_3 = t_2 + t_1$, yields the equation $p^2 = p + 1$ (which is Equation (5)!). In fact, in general we can write $t_n = t_{n-1} + t_{n-2}$ as $p^2 = p^{n-1} + p^{n-2}$; on dividing both sides of this second equation by p^{n-2}, we also obtain Equation (5).

Equation (5) has two solutions: $R_1 = (1 + \sqrt{5})/2$ and $R_2 = (1 - \sqrt{5})/2$. So if the geometric sequence (11) is also to be a "Fibonacci-type" sequence, then p must take either of the values $p = R_1$ or $p = R_2$. In other words, for these two values of p, (11) is a geometric "Fibonacci-type" sequence. Clearly, however, neither of these values of p gives $t_2 = p = 1$ as required for the Fibonacci sequence. Thus, (11) is not the Fibonacci sequence.

Consider then the "Fibonacci-type" sequence whose terms are

$$k,\ kp,\ kp^2,\ \ldots \qquad (12)$$

where k is a constant. These terms also satisfy Equation (10) if $p = R_1$ or $p = R_2$; thus, *any* multiple of R_1 or R_2 is also a solution of Equation (11). In fact, any linear combination $c_1R_1 + c_1R_2$ (where c_1 and c_2 are constants) is also a solution of (10). Thus, our "Fibonacci-type" sequence can be written more generally as

$$c_1 + c_2\,,\ c_1R_1 + c_2R_2\,,\ c_1R_1^{\,2} + c_2R_2^{\,2}\,,\ \ldots\,. \qquad (13)$$

The sequence (13) will be the Fibonacci sequence if the first two terms are 1, that is, if both the equations $c_1 + c_2 = 1$ and $c_1R_1 + c_2R_2 = 1$ are true. We have already solved for R_1 and R_2, so the latter of these two simultaneous equations can be written $c_1(1 + \sqrt{5})/2 + c_2(1 - \sqrt{5})/2 = 1$. The solution of this equation simultaneously with $c_1 + c_2 = 1$ is

$$c_1 = \frac{1 + \sqrt{5}}{2\sqrt{5}}\,, \quad c_2 = \frac{-1 + \sqrt{5}}{2\sqrt{5}}\,. \qquad (14)$$

Then the nth term of the Fibonacci sequence, $t_n = c_1r_1^{\,n-1} + c_2r_2^{\,n-1}$, is

$$t_2 = \left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right]/\sqrt{5}. \qquad (15)$$

This proves the result.

The Conway sequence

In an article in *The New York Times* (30.8.88), Malcolm Browne reported on "an intellectual duel" involving the Conway sequence, which is described as a cousin of the Fibonacci sequence. John Conway, a Princeton mathematician, had devised a sequence whose first two terms are 1, and whose subsequent terms, like those of the Fibonacci sequence, are found by adding together previous terms. The rule for adding terms in the Conway sequence is more complicated than that for the Fibonacci sequence, however:

$$t_n = t_{t_{n-1}} + t_{n-t_{n-1}}, \tag{16}$$

with $t_1 = t_2 = 1$.

Formula (16) requires some interpretation. Put $n = 3$; then

$$t_3 = t_{t_2} + t_{3-t_2}$$

$$= t_1 + t_{3-1} \qquad \text{as } t_2 = 1$$

$$= t_1 + t_2$$

$$= 1 + 1 = 2.$$

Continuing in this way, we may build up the sequence

n	t_n									
1 – 10	1	1	2	2	3	4	4	4	5	6
11 – 20	7	7	8	8	8	8	9	10	11	12
21 – 30	12	13	14	14	15	15	15	15	16	16
31 – 40	16	16	17	18	19	20	21	21	22	23
41 – 50	24	24	25	26	26	27	27	27	28	29
51 – 60	29	30	30	30	31	31	31	31	32	32
61 – 70	32	32	32	32	33	34	35	36	37	38
71 – 80	38	39	40	41	42	42	43	44	45	45
81 – 90	46	47	47	48	48	48	49	50	51	51
91 – 100	52	53	53	54	54	54	55	56	56	57

and so on. To compute a new term in the sequence, use the table constructed so far. Thus

$$t_{101} = t_{t_{100}} + t_{101-t_{100}}$$

$$= t_{57} + t_{101-57} \qquad \text{as } t_{100} = 57 \text{ (from the table)}$$

$$= t_{57} + t_{44}$$

$$= 31 + 26 \qquad \text{again from the table}$$

$$= 57.$$

This sequence has some interesting properties, such as the fact that, as n approaches infinity, t_n/n approaches 1/2. Dr Conway and his wife (who is also a mathematician, but the New York Times didn't mention her name) proved this result after an enormous amount of hard work. Then Dr Conway went on to wonder what value of n, say N, was required for the ratio t_n/n to fall within 0.05 of 1/2 for all values of n

which are greater than N. Dr Conway was so fascinated by this problem, and so sure of the difficulty involved in solving it, that in July 1988, at a lecture he gave at AT&T Bell Laboratories in New Jersey, he offered $10 000 for its solution. Within two weeks, Dr Colin Mallows of Bell Lab sent in the (apparently) correct solution: $N = 3\ 173\ 375\ 556$. Dr Conway duly sent him the $10 000 cheque. Conway had meant to offer $1000 but had made a slip of the tongue, though he didn't really expect anyone to claim anyway. But he and Mallows agreed to the sum of $1000, and the original $10 000 cheque remains framed on Mallows' wall!

Dr Mallows used a Cray supercomputer to obtain his result, so you can see that the Conway sequence is much more complicated than the Fibonacci sequence! In fact, Mallows' original solution was wrong, as he himself later discovered. The correct answer is 6 083 008 742. The story is told by Mallows in the journal *American Mathematical Monthly*, *Vol.* 98, pp. 5-20.

* * * * *

THE CATALYTIC CAMEL

Enzyme chemists are said to use the following example to explain the rôle of a catalyst in a chemical reaction. (A catalyst takes part in a chemical reaction and increases the rate at which it occurs; it is, however, regenerated and is left intact once the reaction is over.)

A sheikh died leaving 17 camels to his three sons. The first was to have half the herd, the second a third, and the third son a ninth. There was some consternation that the terms of his will could not be met until along came an itinerant mathematician riding on *his* camel.

"I will lend you my camel", said he. "You may then give nine camels to the first son, six to the second and two to the third. This will make a total of 17 camels and will leave you able to return my own camel to me!"

* * * * *

A Correction

There were two bad misprints in Peter Kloeden's article "How Big is a Function?" (*Function*, *Vol.* 16, *Part* 3). The formula in the middle of p. 80 should have read

$$\|f\|_2 = \sqrt{\{\int_0^1 |f(t)|^2 dt\}}$$

and seven lines below we should have had

$$\|f - g\|_2 = \sqrt{\{\int_0^1 |f(t) - g(t)|^2 dt} = \sqrt{\{\int_0^1 (t^2 - 2t^3 - t^4)\}}.$$

We apologise for any confusion these errors may have caused.

* * * * *

HISTORY OF MATHEMATICS

EDITOR: MICHAEL A.B. DEAKIN

The Challenge

Diophantus of Alexandria was a late Greek mathematician who lived (probably) about 250 A.D. His major work is the *Arithmetic*, a (now incomplete) set of problems today seen as part of the study of Number Theory.

Problem 8 of Book II of the *Arithmetic* is "To divide a given number into two squares". He gives the illustration

$$16 = \frac{256}{25} + \frac{144}{25}, \qquad (1)$$

it being understood in Diophantus' work that the numbers involved are to be rational.

Nowadays we often study such equations as (1) under the restriction that such numbers are *integers*. Thus we would rewrite Equation (1) as

$$256 + 144 = 400 \qquad (2)$$

or

$$16^2 + 12^2 = 20^2. \qquad (3)$$

This is a special case of the equation

$$a^2 + b^2 = c^2. \qquad (4)$$

Equations such as (4), where we seek *integral solutions*, are now known as Diophantine Equations – in honour of Diophantus.

The emphasis on integral, rather than rational, solutions is due to the influence of a 17th-Century French mathematician, Pierre de Fermat (1601-1665). Fermat worked as a lawyer and politician. Mathematics was purely a hobby for him and of the five volumes of mathematical work published over his name, very little appeared during his lifetime.

He owned an edition of Diophantus' *Arithmetic* and wrote notes in its margin. Opposite Problem 8 of Book II, he had:

> "But it is not possible to split a cube into two cubes, or a squared-square [4th power] into two squared-squares or in general any of the infinitely many powers larger than two into two like terms. I have discovered a truly wonderful proof of this fact. The margin is not large enough to include it." *(My translation.)*

In other words, what Fermat claimed was that the Diophantine Equation

$$a^n + b^n = c^n \qquad (5)$$

has no solutions for $n > 2$. This "fact" is now known as *Fermat's Last Theorem*. It remains unproved and we still do not know therefore if it is indeed a fact.

If we put $n = 1$ into Equation (5) the problem is a trivial one. If we put $n = 2$, we may solve it. Let u, v be integers with $u > v$. Then

$$a = u^2 - v^2, \quad b = 2uv, \quad c = u^2 + v^2 \tag{6}$$

is a solution of Equation (4) and furthermore, all solutions of Equation (4) may be expressed in the form (6). This result, or at least its first half, was indeed known to the Babylonians over 3500 years ago (see *Function, Vol.* 15, *Part* 3, p. 85).

Result (6) may be used to prove Fermat's Last Theorem in the case $n = 4$. The proof has an interesting structure and this structure, or method of proof, *is* due to Fermat. It was referred to several times in his correspondence, and, after Fermat's death, his son Samuel found it as a marginal note elsewhere in the much-used copy of Diophantus.

To prove the non-existence of solutions for

$$a^4 + b^4 = c^4, \tag{7}$$

or in fact (more generally)

$$a^4 + b^4 = k^2, \tag{8}$$

Fermat supposed that there *were* such solutions. Of these, he selected that with the lowest value of k. By analysing the properties of this (supposed) solution, he deduced the existence of another solution, with an even smaller k. It thus followed from this contradiction that no solution could exist. See Appendix 1 for details.

This technique of proof, Fermat called the method of "infinite descent". Nowadays it is often referred to as "reverse induction".

Let us now see what else is required if Fermat's Last Theorem is to be proved.

First we note that we have no solutions to Equation (5) for $n = 8, 12, 16, 20, \ldots$. These are all 4th powers, e.g.

$$a^8 + b^8 = c^8$$

could be written

$$(a^2)^4 + (b^2)^4 = (c^2)^4,$$

and, as we have just seen, there are no solutions for a^2, b^2, c^2 in this equation.

Similarly, if we could show that $n = 3$ gave no solutions to Equation (5), then this would rule out $n = 6, 9, 12$ (already ruled out as a 4th power), 15, etc. And if we could rule out $n = 5$, then out would go also $n = 10, 15$ (already ruled out), 20 (also already ruled out), 25, and so on.

Thus to prove Fermat's Last Theorem (given that Fermat himself ruled out the case $n = 4$), we need only prove:

Equation (5) has no solutions if n *is an odd prime.*

After Fermat's death, there was very little progress on the problem for about 100 years. Around 1769, Leonhard Euler, one of the greatest mathematicians of all time, proved the impossibility of solutions in the case $n = 3$. Subsequent work has shown that his proof was incomplete – it needed further work to establish the truth of one of the assertions on which it was based. Nor was it published at the time; it did not see print until 1862.

Long before then, in the early 1800s, the French mathematician Legendre led a systematic attack on the problem. In the course of this work, two types of possible solution to Equation (5) were distinguished:

1. The product *abc* is not divisible by *n* (the so-called "easy case").

2. The product *abc* *is* divisible by *n* (the so-called "hard case").

As a result of these endeavours, a number of partial (and quite technical) results were proved. Active in the research group and closely associated with Legendre was the first woman to contribute a new result to mathematical research. Her name was Sophie Germain and she made a major contribution to the elucidation of the "easy case". As a result of Sophie Germain's Theorem (a technical result whose details I will omit), it was shown by Sophie herself that no Type 1 solutions exist for any $n < 101$. Later Legendre extended this to $n < 197$.

In 1825, Legendre managed to prove that no solution of Type 2 could occur if $n = 5$. The much more difficult case $n = 7$ was not proved until Lamé, another French mathematician, resolved it in 1839.

The major advance of the 19th century, however, was the emergence of a general theory that superseded this case-by-case approach. This consisted of the development of a new branch of Mathematics now called Kummer's Theory of Ideals. Kummer was a German mathematician who, in about 1843, produced what he thought to be a complete proof of Fermat's Last Theorem. However, his "proof" turned out to be flawed and the discovery of the flaw enabled Kummer to point out a similar flaw in a later (1847) alleged proof by Lamé.

Despite the flaws, which meant that the theorem was still not proved in general, the work of Kummer and Lamé did give partial results of very considerable power. In particular, Fermat's Last Theorem was shown to hold for nearly all primes less than 101. (The exceptions were 37, 59 and 67.) The case $n = 37$ was finally proved in 1892, and by 1905 the question had been settled for all n less than 257.

The American mathematician L.E. Dickson extended these results spectacularly in 1908, proving the theorem for all n less than 1700. By 1955, the boundaries had been pushed out even further: the lowest unresolved value of n then stood at $n = 4003$. I don't know what the current "record" is. A 1967 claim that the theorem had been proved for all primes less than 25,000 commanded wide acceptance, and the limit may well have been extended since.

Let us turn now to the size of the other numbers a, b, c. Suppose $a < b$, for argument's sake. Then clearly $a < b < c$. [We cannot have $a = b$. Can you see why not?] In 1856, Grunert, a German mathematician, showed that $a > n$. See Appendix 2 for a proof of this result. Thus, even on the 1955 figures, a^n would be a number larger than

$$4003^{4003} = 2.2... \times 10^{14420},$$

i.e. a number with 14421 digits. Later improvements show that the smallest of the three integers a^n, b^n, c^n would have many more digits than this. On the best such result I know, and accepting the claim $n > 25,000$, we would have a being an integer of over 300,000 digits and thus a^n having over 7 500 000 000 digits, with b^n and c^n correspondingly larger!

Another recent result, if one is to play with such large numbers, is a 1990 one which says that no Type 1 (easy case) solution can exist for

$$n < 756\ 800\ 000\ 000\ 000\ 000.$$

The moral is that if Fermat's Last Theorem is false, we will never succeed in writing down the counter-example!

In recent years there has been one spectacular advance in the study of Fermat's Last Theorem. To follow this, rewrite Equation (5) as

$$x^n + y^n = 1, \tag{9}$$

where $x = a/c$, $y = b/c$. The theorem then says that Equation (9) has no *rational* solutions if n is an integer greater than 2. (This formulation goes back to Diophantus' way of looking at things!)

Now Equation (9), under these conditions, is the equation of a curve known as a *super-circle* (see *Function, Vol.* 14, *Part* 4, pp. 101-107, 117-119). The super-circle is classified (along with many other curves) as a "curve of genus greater than 1". In 1922, the British mathematician L.J. Mordell conjectured that no curve of genus greater than 1 could pass through infinitely many points (x, y) where both x, y were rational. This conjecture, the Mordell conjecture, was seen for many years as a very difficult question. It was proved in 1983 by the German mathematician Gerd Faltings (see *Function, Vol.* 7, *Part* 5). Faltings later won a Fields Medal, the mathematical equivalent of a Nobel Prize, for this work.

Thus what we now know is that, for any given n (> 2), there can be at most a finite number of solutions to Equation (5). (Contrast this with the case $n = 2$ where Equations (6) give *infinitely* many solutions.) This is a great advance, but we still have a long way to go before the matter is fully resolved.

The circumstances of its announcement and the ease with which the result may be stated have given a great fascination to the theorem. As one recent author† remarks:

> "[It] has exerted an extraordinary attraction for mathematicians, and especially for amateurs. Rarely a week goes by when editors of mathematics journals do not receive at least one purported proof. Despite the simplicity of the assertion, it has shown remarkable resistance to all attacks."

He goes on to quote Legendre to the effect that:

> "It really seems that a special difficulty attaches to this question and that we still lack the key principle required for its resolution." (*My translation.*)

As he further remarks:

> "This penetrating comment is still true, despite the deep insights gained in the ensuing century and a half."

---

† J.H. Sampson in *Archive for History of Exact Sciences* (1990).

So, did Fermat have a proof as he claimed? The answer must almost certainly be "no". Although Fermat was a considerable mathematician, there were many better among his contemporaries and in the centuries since. Furthermore, there is a much larger body of mathematical knowledge available now than in Fermat's day. Of the many amateur mathematicians who have tried to solve the problem, some at least must have been as able as (and certainly better equipped than) Fermat, himself an amateur. The matter is well summed up by H.M. Edwards in his book *Fermat's Last Theorem.*

> "Any attempt to prove Fermat's Last Theorem by elementary arguments – say by methods which do not use Kummer's theory of ideal prime divisors – must take into account the fact that the single case $n = 7$ resisted the best efforts of the best mathematicians of Europe for many years. Of course it is entirely possible that they were approaching the problem in the wrong way and that there is some simple idea – perhaps discovered by Fermat – which applies to all cases; but, on the other hand, it is more probable that an idea which is valid for *all* n would be found, perhaps in a clumsy form, in an intensive study of *one* n."

Appendix 1

If, in Equation (8), k is to be minimised, then a, b, k will be relatively prime for we can reduce the value of k by dividing out any common factors that may be present. Then, by Equations (6),

$$a^2 = u^2 - v^2, \quad b^2 = 2uv, \quad k = u^2 + v^2 \tag{9}$$

where u, v are relatively prime.

Then b^2 is even and so b must be even. Thus

$$b^2 = 2uv = 2u.2w \text{ (say)} = 4uw,$$

and u, w must be relatively prime. It follows that u, w must each be a perfect square, say,

$$u = r^2, \quad w = s^2. \tag{10}$$

Then

$$a^2 = r^4 - (2s^2)^2$$

i.e.

$$a^2 + (2s^2)^2 = r^4.$$

Once more apply the result (6). For some integers m, n (say),

$$a^2 = m^2 - n^2, \quad 2s^2 = 2mn, \quad r^2 = m^2 + n^2. \tag{11}$$

Just as before, m, n must be perfect squares:

$$m = p^2, \quad n = q^2 \tag{12}$$

(say). But now, by Equation (11)

$$p^4 + q^4 = r^2 \tag{13}$$

and this has the same form as Equation (8). However,

$$r < r^2 = u < u^2 < u^2 + v^2 = k$$

and thus we have the contradiction referred to in the main text.

Appendix 2

In Equation (5), let $c = b + d$. Then

$$a^n = (b+d)^n - b^n$$

and, by the binomial expansion, this is bigger than $nb^{n-1}d$, which is greater than or equal to nb^{n-1}. Thus

$$a^n > nb^{n-1}.$$

But the same argument could be used to show that

$$b^n > na^{n-1}.$$

Now

$$a^{n^2} = (a^n)^n > (nb^{n-1})^n = n^n(b^n)^{n-1}.$$

But

$$n^n(b^n)^{n-1} > n^n(na^{n-1})^{n-1}$$

$$= n^n n^{n-1} a^{(n-1)^2}.$$

Thus

$$a^{n^2} > n^{2n-1} a^{(n-1)^2}$$

and so

$$a^{2n-1} > n^{2n-1}$$

and hence

$$a > n.$$

Similarly, of course, $b > n$ and thus the smaller of a, b (here taken to be a) must exceed n.

I first learned this proof over 20 years ago from my colleague and fellow editor, R.T. Worley.

COMPUTERS AND COMPUTING

EDITOR: R.T. WORLEY

Factoring by Computer

Suppose we are given a large number n to factor. We presume n is odd, for a factor of 2 is easily spotted. One of the simplest ways of factoring is the brute force method of "trial division". For each odd number $d = 3, 5, 7, \ldots$ we see if n/d is an integer. If it is then n has factors d and n/d.

The following is a simple BASIC program segment that could be used in a factoring program.

```
J = SQRT(N)
FOR I = 3 TO J STEP 2
IF INT(N/I) = N/I THEN PRINT "FACTOR";I: STOP
NEXT I
PRINT "NO FACTOR FOUND"
```

This locates the smallest prime factor of N. Do you understand why the limit on I is *SQRT(N)*? Unfortunately, if N has two approximately equal factors this will take a long time to locate the smaller factor.

For numbers that are products of two approximately equal factors, the method known as Fermat factorisation is faster. In this method we attempt to find u, v such that $u^2 - v^2 = n$. Since $u^2 - v^2 = (u+v)(u-v)$ we will manage to factor n. For example, taking $n = 437$, we must have $u^2 \geq 437$, so $u \geq 20.9 \ldots$. We try out the possibility $u = 21$. Trying $u = 21$ we need $v^2 = u^2 - n = 21^2 - 437 = 441 - 437 = 4$, so $v = 2$. Thus we factor 437 as $(21+2)(21-2) = 23 \times 19$.

In this example we were lucky and found a suitable u, v on our first attempt. If we had not been so lucky we would have had to try the possibilities $u = 22, 23, \ldots$ in turn till we succeeded. For example, in factoring $n = 377$ we need $u \geq \sqrt{n} = 19.4 \ldots$. If we try $u = 20$ we find $u^2 - n = 400 - 377 = 23$ is not a perfect square, but trying $u = 21$ we find $441 - 377 = 64 = 8^2$ is a perfect square, so

$$377 = (21+8)(21-8) = 29 \times 13.$$

In some cases many more values of u need to be tried — for example, with $n = 141467$, we need to try $u = 377, 378, \ldots, 414$ until we find the factorisation $(414+173)(414-173)$ of 141467. However, if we factor $3n = 424401$ using Fermat factorisation we only try the values $u = 652, \ldots, 655$ before finding the factorisation $3n = (655+68)(655-68) = 723 \times 587$. Dividing this by 3 gives the factorisation $n = 241 \times 587$. Of course, one could question what inspired us to try Fermat factorisation on $3n$ and not n.

As a general rule, Fermat factorisation is not very good (trial division is much faster on 109637, for example). However, the underlying idea — factoring a difference of squares — is used as the basis of more powerful factoring methods.

PROBLEMS AND SOLUTIONS

EDITOR: H. LAUSCH

SOLUTIONS

Problem 16.1.1 (Juan Bosco Romero Márquez, Valladolid, Spain). Solve the following equation for integers x and y with $y \geq x > 0$:

$$y - x = y^x - x^y.$$

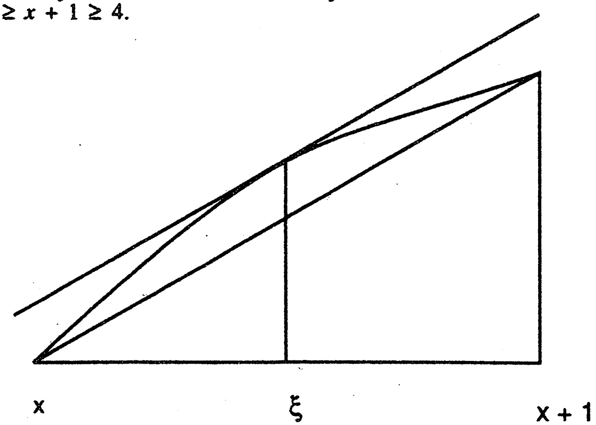
Solution (by Seung-Jin Bang, Seoul, Republic of Korea). Note that $(x, y) = (x, x)$ and $(x, y) = (1, y)$ are solutions of the equation. Suppose $x = 2$. Then $y = 3$ is a solution of the equation $y - 2 = y^2 - 2^y$. Since $y^2 - 2^y \leq 0$ whenever $y \geq 4$, there is no solution $(2, y)$ with $y \geq 4$.

Suppose $y \geq x + 1 \geq 4$, and let $f(y) = x \ln y - y \ln x$. Since $\frac{x}{\ln x} \leq x + 1 \leq y$, we have for the derivative f' of the function f that $f'(y) = \frac{x}{y} - \ln x \leq 0$. It follows that $f(y) \leq f(x+1) = x \ln (x+1) - (x+1) \ln x$, hence

$$f(y) \leq x[\ln (x+1) - \ln x] - \ln x. \tag{1}$$

Consider the graph of the function g defined by $g(t) = \ln t$ over the abscissa between x and $x + 1$. At some point $(\xi, g(\xi))$ with $x < \xi < x+1$, this graph has a tangent that is parallel to the chord joining the points $(x, g(x))$ and $(x+1, g(x+1))$.†

Chord and tangent having the same slope means that $g(x+1) - g(x) = g'(\xi)$, i.e. $\ln (x+1) - \ln x = \frac{1}{\xi}$. Now, by (1), $f(y) \leq \frac{x}{\xi} - \ln x < 1 - \ln x < 0$. Hence $y^x < x^y$ whenever $y \geq x + 1 \geq 4$.



† We refer to the so-called Mean Value Theorem of Differential Calculus.

The answer is therefore: the solutions (x, y) are $(2, 3)$, (t, t) and $(1, t)$, where t is an arbitrary positive integer.

PROBLEMS

a) In-laws and Tasmanian tigers

Problem 16.4.1 (another one by Lewis Carroll and proposed by G.J. Greenbury, Up. Mt. Gravatt, Queensland). The Governor wished to give a small dinner party and invited his father's brother-in-law, his brother's father-in-law, his father's in-law's brother, and his brother-in-law's father.

How many guests attended?

Problem 16.4.2 (adapted to Australian conditions from Alexander Yakovlevič Halameiser's entertaining reader *Mathematics ? – Entertaining !*, Moscow 1990). "Some years ago", remembered Hansel, "I encountered a Tasmanian tiger in the Dandenongs† when I had a barbecue with Red Riding Hood. It was on April 1." "Really", laughed Gretel mockingly, "and you would, of course, remember the day of the week on which you had your very strange encounter with the big bad wolf, ... pardon me, I mean ... with your Tasmanian tiger and that girl whatshername." "I am afraid I don't remember the day of the week", replied Hansel, who felt quite embarrassed, "but wait, ... I do remember that there were three Sundays in this month falling on even-numbered days. Is that of any help to you?" "It is indeed", said Gretel, making herself sound very important. "I don't believe a word of your Tasmanian tiger story, but at least I now know the day of the week on which you and that girl whatshername had a barbecue in the Dandenongs! It was on a ... ." She could not finish her sentence because a ferocious looking animal was sitting down beside her and ... .

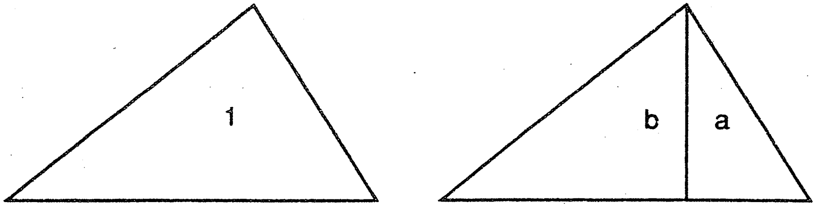
What day of the week was Gretel just about to mention?

b) Triangles and ratios

Problem 16.4.3 (P.A. Grossman, Caulfield). Prove that every triangle is "approximately isosceles", which is to say that in every triangle there are two sides whose lengths are in a ratio that is less than $(1 + \sqrt{5}) : 2$.

Problem 16.4.4. The figure opposite shows two copies of the same right-angled triangle, whose area in one copy is divided by the altitude in the ratio $a : b$ $(a < b)$, the whole being held to have unit area. The three portions thus have areas 1, a, b.

† Hill country east of Melbourne.



Further copies of the triangle are to be divided into constituent right-angled triangles, by repeated insertions of altitudes in such a way that no two subdivisions have the same area, anywhere over the full set of copies. How many copies in all will this allow?

c. Draga's choice

Departmental artist Draga Gelt of Monash University's Department of Earth Sciences communicated the following problem to Function *which she had selected from the Slovenian mathematics and science magazine* Presek, *Volume 18 (1990/91), issue #2, and translated for our readers:*

Problem 16.4.5. Let f be the function which is defined by $f(x) = \frac{x-1}{x+1}$. For each positive integer n, let

$$f^{n}(x) = \underbrace{f(f(\ldots f(x)}_{n \text{ times}} \ldots)).$$

Draw the graph of $f^{1992}(x)$.

d. One Day of Mathematics

For some years Tag der Mathematik *("Day of Mathematics") has been a distinctive school-mathematics competition in Southwest Germany. Some of its features are reminiscent of the well-established one-day competition* Canberra Mathematics Day. *Many thanks to competition organizer Professor Gudrun Kalmbach, Ulm, for sending her report, which we summarize:*

Tag der Mathematik *took place on 21 March, 1992, at five different locations simultaneously. Prizes were given for the best team effort and for six individual performances (each full score). Moreover the Emmy-Noether-Prize*† *was awarded to the best girl student.*

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† Named after Emmy Noether (d. 1935), one of the most brilliant algebraists this century has seen. Emmy Noether left Germany in 1933. Her father was Max Noether (1844-1921), whose contributions to modern algebraic geometry are fundamental.

I. GROUP CONTEST

Fractions should be given in their simplest form, decimals being rounded to two places (i.e. with an accuracy of ± 0.005). Surds need not be transformed into decimals.

I.1 (15 points)

For each real number x, let $f(x)$ be defined by

$$f(x) = \frac{1}{3}x^2 - \frac{1}{3}x^2 + \frac{1}{24}.$$

(a) Determine the relative maxima, minima and inflexion points of f.

(b) Is $|f(x)| \geq 1$ whenever $|x| < 1$?

(c) For which values of x does $f(x) = 0$ hold?

I.2 (20 points)

(a) Someone is standing at the foot of a hill that has perfect conic shape and plans its circumvention along a path on the surface of the cone. The hill is 120 m high, and the radius of its bottom circle is 50 m.
Find the shortest path.

(b) To what altitude (measured from the base of the cone) does this path lead?

I.3 (15 points)

(a) Let n be a positive integer. Compute values a and b such that $\frac{1}{n(n+3)} = \frac{a}{n} + \frac{b}{n+3}$.

(b) If $a_n = \frac{1}{n(n+3)}$, determine $a_1 + a_2 + ... + a_k$ for every positive integer k.

I.4 (8 points)

Let a_i, b_i be integers $(i = 1, 2, 3)$ such that $b_i \neq 0$ $(i = 1, 2, 3)$ and $|b_1 b_2 b_3| \neq 1$. Each pair (a_i, b_i) and (b_i, b_j), for $i \neq j$, has highest common factor 1. Prove that

$$\frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3}$$

is not an integer.

II. INDIVIDUAL CONTEST

II.1 (20 points)

In a regular octagon $ABCDEFGH$ of side length $2a$ join the midpoints P, Q, R, S of the sides AB, CD, EF, GH, respectively, and the midpoints T, U, V, W of the sides BC, DE, FG, AH, respectively, to obtain two quadrilaterals. Determine the area of the eight-cornered star covered by the rectangles $PQRS$ and $TUVW$.
(Hint: exploit symmetries and the presence of parallel lines.)

II.2 (20 points)

First names and family names of four students are known to be: Arnold, Bernhard, Conrad and Dietrich.

We also know that:

(a) the family name of each student is always different from that student's first name;

(b) Conrad's family name is not Arnold;

(c) Bernhard's family name coincides with the first name of that student whose family name coincides with the first name of the student with the family name Dietrich.

What are the first and the family names of all four students?

II.3 (20 points)

Let u and f be the real functions defined by

$$u(x) = 4 - \sqrt[3]{x^2} \quad \text{and} \quad f(x) = \sqrt{u(x)^3},$$

$f(x)$ being non-negative for all real values x.

(a) Determine the set of all real numbers x for which $f(x)$ makes sense.

(b) For which values of x does the derivative f' of f exist?

(Hint: $\sqrt[3]{x} = \sqrt[3]{|x|}$ for $x \geq 0$ and $\sqrt[3]{x} = -\sqrt[3]{|x|}$ for $x < 0$.)

Tag der Mathematik *also had a "speed-test"*.

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"What delighted me most about mathematics was that things could be proved".

Bertrand Russell

THE CAR AND THE GOATS

This puzzle appeared in a US magazine *Parade* (a syndicated supplement to a number of Sunday newspapers) in its "Ask Marilyn" column.

> A TV host shows you three numbered doors, one hiding a car (all three equally likely) and the other two hiding goats. You get to pick a door, winning whatever is behind it. You choose Door No. 1, say. The host, who knows where the car is, then opens one of the other two doors to reveal a goat, and invites you to switch your choice if you so wish. Assume he opens Door No. 3. Should you switch to Door No. 2?

The problem was the subject of a recent article by Leonard Gillman (*American Mathematical Monthly*, Jan. 1992) and was also set as a problem in *Mathematical Digest*, a South African counterpart of *Function*.

The solution published by Marilyn was that you should switch and choose Door No. 2. Marilyn argued thus:

> The chance is 1/3 that the car is actually at Door No. 1, and in that case you lose when you switch. The chance is 2/3 that the car is either at Door No. 2 (in which case the host perforce opens Door No. 3) or at Door No. 3 (in which case he perforce opens Door No. 2) and in these cases, the host's revelation of a goat shows you how to switch and win.

Marilyn thus claimed that there is a 2/3 chance that the car is behind Door No. 2. (This was also the answer published (though with a disclaimer) by *Mathematical Digest*.) But as Professor Gillman reports:

> This led to an uproar featuring "thousands" of letters, nine-tenths of them insisting that with Door No. 3 now eliminated, Door No. 1 and Door No. 2 were equally likely; even the responses from [university academics] voted her down two to one.

Gillman, however, thought both answers wrong and produced a third answer. He set q equal to the probability that the host would open Door No. 3 in the event that the car was actually behind Door No. 1. (Thus $1 - q$ would be the probability that he would open Door No. 2; it is assumed that in such an event he would not open Door No. 1.)

Gillman then calculated that the probability that the car is behind Door No. 2, given that the host has opened Door No. 3 to reveal a goat, is

$$P = \frac{1}{1+q} .$$

Now we don't know the value of q, but as it is a probability, it lies between 0 and 1. If $q = 0$, the host only opens Door No. 3 when the car is definitely *not* behind Door No. 1 – i.e. it is behind Door No. 2. Then $P = 1$. On the other hand, if $q = 1$, the host opens Door No. 3 no matter whether the car is behind Door No. 1 or Door No. 2 – he has a fixed attraction to Door No. 3. In such a case, $P = 1/2$ and there is no advantage to be gained by switching. (This is the case considered by Marilyn's critics.)

For all other values of q, we have

$$1/2 < P < 1.$$

In particular, if $q = 1/2$, and the host opens Door No. 3 (as opposed to Door No. 2) quite at random, we have $P = 2/3$, which is Marilyn's answer.

But now note that Gillman has come down clearly on the side of those who say that you should switch. The minimum value of P is 1/2, so at best you're preserving the odds, and as long as $q < 1$, you're improving them. So, at very worst, on Gillman's analysis, you can't lose by switching.

Gillman goes on to claim that the Car and Goats problem is the same as another. It goes by various titles and has turned up in various places. We ran it as Problem 1.3.7 in the first volume of *Function*; it can also be found in Martin Gardner's *Mathematical Games* column in *Scientific American* (Oct., 1959). This is a problem about three prisoners awaiting execution. The details will be omitted here, but it ends up with a suggestion of 1/2, 2/3 as possible answers to (ostensibly) the same problem.

The question, however, remains as to whether Gillman's analysis goes far enough. If not, the car and goats problem is more complicated. Suppose you are on the show and the host then opens Door No. 3 to reveal the goat. He then goes on to invite you to choose Door No. 2 – changing your initial choice which is Door No. 1. This is all a complete surprise, of course. Before you chose Door No. 1, you had no inkling that the host would introduce this new element into the game. One very common reaction would surely be that the host was "up to something" and perhaps trying to diddle you out of a car.

Now, on Gillman's analysis, given that the host has opened Door No. 3 to reveal a goat, the probability that the car is behind Door No. 1 is $1 - P$ and that it is behind Door No. 2 is P; that is to say

$$\frac{q}{1+q} \quad \text{and} \quad \frac{1}{1+q}$$

respectively.

But you, as a player, might well assign weights to these, depending on your degree of belief in the host's motives. Let these weights be p, $1 - p$ respectively. The new probability that you would assign to the car's being behind Door No. 1 then becomes

$$\frac{pq}{1+pq-p} = Q \quad \text{(say).}$$

If $p = 1/2$, we get Gillman's result. If you were *certain* the host was playing you for a sucker – inducing you to forgo the car behind Door No. 1 and choose the goat behind Door No. 2 – then $p = 1$ and $Q = 1$. Naturally you'ld not switch. At the other end of the scale $p = 0$, which would correspond to conviction that the host was trying to do you a great favour. In this case, $Q = 0$ and of course you would switch.

But now remember that you have no way of knowing the value of q, whereas you might be brave enough to estimate a value for p. In this case you might well say that your best estimate for the unknown q was $q = 1/2$. In this case,

$$Q = \frac{p}{2-p}\,.$$

The matter now depends entirely on your subjective assessment. In order to consider a switch, you'ld need to rate $Q \leq 1/2$, i.e. $p \leq 2/3$. In other words, you'ld need to be 2/3 convinced that the host was trying to do you a favour, before deciding to switch.

So the problem is quite a complicated one – indeed, there is a lot more that can be said about it, though it is left out here. Perhaps the moral is that in translating "real life" situations into mathematics, even quite stylised ones such as this, we need to be very careful not to discard essential features of the story.

ARMSTRONG NUMBERS

Problem 3.5.1 concerned the so-called *Armstrong Numbers*. An Armstrong number of order m is an m-digit number, the mth powers of whose digits add up to the number itself. The best-known is 153 as

$$1^3 + 5^3 + 3^3 = 153.$$

Clearly the numbers 0, 1, 2, ..., 9 are all Armstrong numbers of order 1. There are none of order 2 (the proof of this was part of the problem we set). Indeed, it is not hard to show that 153 is the smallest non-trivial Armstrong number.

In January 1990, *Pythagoras*, a Dutch counterpart of *Function*, published a list of Armstrong numbers up to order ten. There are 26 entries in their list. 153, 370, 371, and 407 are the four of order 3. The largest they list is 4 679 307 774, which is of course of order ten. Larger Armstrong numbers may have been found but are not listed. The solution to Problem 3.5.1 proved that no Armstrong numbers could exist with orders greater than 60 and very probably this bound can be lowered. There are thus only finitely many Armstrong numbers.

Armstrong numbers were discussed by Tim Hartnell in his column in the Computer Section of *The Australian*. On April 19, 1988 he quoted a letter from Michael F. Armstrong:

> "In the late 1960s, I wrote a short paper intended to spoof serious mathematical papers, entitled A Brief Introduction to Armstrong Numbers."

That paper included the passage:

> "The concept of the Armstrong number is not a hard one to grasp, but due to the utter uselessness of same, some thought must be devoted to the subject if complete comprehension is desired."

H.C. Bolton, Emeritus Professor of Mathematical Physics at Monash, brought this material to our attention. He writes:

> "I still think that the 'Armstrong numbers' are a good computing exercise!"

Armstrong may well have hoaxed himself by producing a rather more useful "uselessness" than he knew!

* * * * *

REMEMBERING PI

Back in 1980, *Function* (*Vol.* 4, *Part* 1) published an article on Pi. This included the mnemonic:

> "How I want a drink, alcoholic of course, after the
> heavy lectures involving quantum mechanics. All of
> thy geometry, Herr Planck, is fairly hard ...".

If you count the letters in each word you find successively 3 1 4 1 5 9 2 6 5 3 5 8 9 7 9 3 2 3 8 4 6 2 6 4 ... , the first 24 digits in the decimal expansion of π.

Others have attempted the construction of such mnemonics, even bursting into verse for the purpose.

See, I have a rhyme assisting
My feeble brain its tasks ofttimes resisting.

or

Sir, I send a rhyme excelling
In sacred truth and rigid spelling
Numerical sprites elucidate
For me the lexicon's dull weight.

These two have been sitting in our files so long that we now take no record of where they came from or how they got there, but we did put them into *Function*, *Vol.* 14, *No.* 4, p.127 (without a full explanation).

This one:

Now I know a charm unveiling
An artful charm for tasks availing
Intricate results entailing

comes from *Mathematical Digest*, a South African counterpart of *Function*. They got it from a radio quiz show. From the same sources comes yet another:

PIE

I wish I could determine pi
Eureka! cried the great inventor.
Christmas pudding, Christmas pie
Is the problem's very centre.

While none of these go as far as the 24 digits of the first, perhaps they are a little more memorable. On the other hand, perhaps not.

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