Function

Founder Editor G. B. Preston

Volume 16 Part 1 February 1992

A SCHOOL MATHEMATICS MAGAZINE

Aust. Post Publ. No. VBH0171

FUNCTION is a mathematics magazine addressed principally to students in the upper forms of secondary schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide — pure mathematics, statistics, computer science and applications of mathematics are all included. Recent issues have carried articles on advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

* * * * *

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
FUNCTION,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions listed on the inside back cover.

FUNCTION is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $17.00*; single issues $4.00. Payments should be sent to the Business Manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*$8.00 for *bona fide* secondary or tertiary students.

* * * * *

FUNCTION

Volume 16 *Part 1*

(Founder editor: G.B. Preston)

CONTENTS

About Function

(reprinted from Volume 15, Part 1)

Towards the end of 1976, Professor G.B. Preston called a meeting at the Monash University department of Mathematics. He explained to those who attended that Victoria had reached a situation of having no journal of school mathematics, written for school students and giving access to good quality exposition of high-calibre mathematical material. Such journals, he said, were the norm both in the U.K. and in many countries of continental Europe; indeed, other Australian states had them. By contrast, the Victorian journals that once might have played this rôle had instead turned to other things. He saw a need for a specialist journal in this area.

Thus *Function* was born. Its name was one of many supplied to Professor Preston from a variety of people enthused by the idea of such a journal. The "supplier" in this instance was the late Dr. Len Grant, then in the Monash department of Philosophy. Professor Preston singled out this suggestion in preference to all the others because, as he said, "I can think of no idea more central to the whole endeavour of Mathematics". Thus *Function* was christened.

Over the years since then, *Function* has continued to fulfil the aims laid down by that first meeting. The founder editor was Professor Preston himself and he remained active on the editorial board throughout the 14 years that followed – jointly editing *Volume* 14, *Part* 5 with Dr. Rod Worley.

His retirement at the end of 1990 thus sees the end of an era, but *Function* itself continues and remains dedicated to the same ideals as those listed by Professor Preston back in 1976. Essentially these are:

(a) valid and interesting mathematics
(b) high-quality exposition
(c) access to all.

In pursuit of the third of these aims, *Function* has always tried to encourage Mathematics among girls.

The need for an organ like *Function* has recently been significantly increased by the introduction of the new VCE, covering precisely the years (11 and 12) which *Function* is designed to serve.

Late in 1990, the *Mathematical Association of Victoria* printed a compilation of some of its most useful articles under the title *Composite Function* (see p.14). The articles reproduced there were chosen for their especial suitability for student projects. Each year, *Function* publishes more articles, problems, news-items (even cartoons) in the same vein. Recent issues have discussed symmetries in physics, fractals, supercircles, multi-dimensional chess, planetary paths and Mathematics for girls.

We hope teachers and students alike will continue to use *Function*, that it finds its way into more and more school-libraries and that, in particular, students and teachers write for it. *Function* can be a very good place to send that good project for wider circulation and appreciation, or to ask about that question that's always bugged you; and so on.

For details on whom to contact and how to go about it, see the inside front and back covers.

Michael A.B. Deakin
Chief Editor

THE FRONT COVER

Michael A.B. Deakin, Monash University

The front cover is an artist's impression of Hypatia of Alexandria (see pp. 17-22). Hypatia was the first woman currently accepted as a mathematician and she lived in the late 4th and early 5th centuries of our era. The picture has no claim to be a portrait; it was drawn by an artist named Gasparo to illustrate a fictionalised account of Hypatia in a 1908 reader called *Little Journeys to the Homes of Great Teachers*.

It has become the accepted picture of Hypatia (just as, for example, we have an "accepted picture" of Jesus of Nazareth, which is equally inauthentic) but we have no idea of how she really looked. Below is a 19th-century "portrait" of Hypatia by an anonymous artist. We have come to prefer Gasparo's version.

* * * * *

SQUARE ROOTS OF MATRICES

J.B. Miller, Monash University

When x and a are numbers, to say that x is a square root of a means that $x^2 = a$. As we know, every positive number a has two square roots, usually written $\sqrt{a}$ and $-\sqrt{a}$. By convention, $\sqrt{a}$ denotes the positive square root.

The equation

$$X^2 = A \tag{1}$$

can also be considered as an equation between two matrices X and A (the matrices have to be square and of the same dimension). The equation is meaningful because there is a well defined way of multiplying two matrices, and hence a clear meaning for X^2. When (1) holds we say that matrix X is a *square root* of matrix A. It is not so easy to formulate a convention that picks out one of all possible square roots to call $\sqrt{A}$; so we avoid the use of the sign $\sqrt{\ }$ for the time being.

The subject is quite old; one of the early workers on matrix theory, Arthur Cayley, published work on the square roots of 2×2 and 3×3 matrices in 1858, and there has now accumulated a sizeable literature on the general theory, with applications in numerical analysis and statistics. It is by no means so straightforward as the theory of square roots of numbers. Peculiar things happen. Although square roots come in pairs (if X is a square root of A then so is $-X$, since $(-X)^2 = X^2$), the number of square roots which a matrix may have is difficult to predict. I shall try to illustrate this by some examples, using 2×2 matrices.

Suppose that A is given, we write

$$A = \begin{bmatrix} a & b \\ d & c \end{bmatrix}, \quad X = \begin{bmatrix} x & y \\ w & z \end{bmatrix}; \tag{2}$$

and we want to find all matrices X satisfying (1). Since†

$$X^2 = \begin{bmatrix} x & y \\ w & z \end{bmatrix}\begin{bmatrix} x & y \\ w & z \end{bmatrix} = \begin{bmatrix} x^2+yw & y(x+z) \\ w(x+z) & yw+z^2 \end{bmatrix}, \tag{3}$$

we need to solve for x, y, z and w the equations

$$x^2 + yw = a, \tag{4}$$

$$y(x+z) = b, \tag{5}$$

$$z^2 + yw = c, \tag{6}$$

$$w(x+z) = d. \tag{7}$$

† Matrix multiplication is defined by multiplying the rows in the left-hand matrix by the columns of the right-hand one, for example the i-th row by the j-th column gives the element in the i-th row and j-th column of the product matrix.

Here we have four equations which are quadratic in four unknowns. Rather than try to solve them in the general case, consider some particular cases.

1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & c \end{bmatrix}$, c being some non-negative real number.

We put $a = b = 1, d = 0$ in (4) – (7). From (7), either $w = 0$ or $x + z = 0$. By (5), $x + z = 0$ is impossible, so $w = 0$. Then (4) and (6) give $x = \pm 1$, $z = \pm\sqrt{c}$, and (5) gives $y = 1/(x+z)$. This shows that there are four square roots of this matrix, namely

$$\pm \begin{bmatrix} 1 & \dfrac{1}{1+\sqrt{c}} \\ 0 & \sqrt{c} \end{bmatrix}, \quad \pm \begin{bmatrix} 1 & \dfrac{1}{1-\sqrt{c}} \\ 0 & -\sqrt{c} \end{bmatrix}, \tag{8}$$

unless by chance $c = 0$ or 1, in which cases there are only two.

To show dependence upon c, rename A as $A(c)$. As c approaches 0 ($c \to 0$), two of the four roots coalesce; as $c \to 1$, two (a different two) of the roots become meaningless.

Question: What are the roots of $A(0)$ and $A(1)$? When does $A(c)$ have an inverse matrix?

2. The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has no square roots at all. (Try solving (4) – (7) for this A.)

3. The unit matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has infinitely many square roots, namely all matrices of the form

$$\begin{bmatrix} x & y \\ \dfrac{1-x^2}{y} & -x \end{bmatrix},$$

(with $y \neq 0$ but x and y otherwise arbitrary), together with

$$\pm \begin{bmatrix} 1 & 0 \\ w & -1 \end{bmatrix} \text{ and } \pm I \text{ (where } w \text{ is arbitrary).}$$

Q: What square roots has the zero matrix O?

4. $A = \begin{bmatrix} 0 & b \\ d & 0 \end{bmatrix}$, with $bd \neq 0$. Careful examination of all possible solutions of (4) – (7) in this case leads to the following conclusions: no real square root exists unless $bd < 0$; and if $bd < 0$ then there are the four solutions

$$X = \begin{bmatrix} \theta & \dfrac{b}{2\theta} \\ \dfrac{d}{2\theta} & \theta \end{bmatrix}, \text{ where } \theta \text{ is either of the two fourth roots of } -\frac{bd}{4}.$$

So when $bd < 0$ there are 2 square roots of A.

Incidentally, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (the case $b = 1$, $d = -1$) is itself a square root of $-I$. This should immediately put us in mind of the complex imaginary number i.

Q: What are the other real square roots of $-I$?

A square matrix in which all the elements below the main diagonal (the one from the top left corner to the bottom right corner) are zero is called *upper-triangular*. For example, A and X in **1** are upper-triangular.

Upper-triangular matrices are easier to handle in the present context than others. In 3, the unit matrix (which is upper-triangular) has infinitely many upper-triangular roots. Let us pose the question: How many *upper-triangular* roots does a 2×2 upper-triangular matrix have, in general? The form of A in this case is as in (2) with $d = 0$, and we look for solutions X in which $w = 0$. The equations (4) – (7) are now quite simple to handle, though care is needed in enumerating the various cases. If we restrict attention to real A and real roots, the reader will be able to show that the answer is: A has either 4 distinct square roots, or 2, or none, or an infinity of square roots.

We all know that for any two real numbers x and y, necessarily $xy = yx$; this fact is so engrained in our use of numbers that it comes as something of a shock to learn that matrix multiplication is not commutative in this way: it is possible, indeed often true, for two square matrices X and Y of the same dimension, that $XY \neq YX$. So we should ask about commutativity properties of roots.

It is easily seen that if A has square roots, they all commute with A. (If $A = X^2$, then $AX = X^2X = (XX)X = X(XX)$ (associative law) $= XX^2 = XA$. But the roots need not commute among themselves. In 3. above it is easy to find a pair of upper-triangular roots of I which do not commute, so even upper-triangular matrices do not escape this compilation.

A last remark: we have assumed almost throughout that all elements of our matrices are real numbers; if we allow complex elements, there are some modifications to the theory. An early theorem, due to G. Frobenius (1896), says that for matrices with complex elements, every matrix which has an inverse also has a square root.

Here are some projects about square roots.

Project 1. Find all possible real solutions of (4) – (7), without restriction on the real numbers a, b, c, d.

Project 2. Show that any real $n \times n$ upper-triangular matrix A, whose diagonal elements are all distinct and positive, has exactly 2^n upper-triangular square roots, all real. (It is also true that these are *all* the matrix square roots of A, it has no non-triangular roots; moreover, the roots all commute among themselves; but these facts are harder to prove.) (You can get a grasp on the case for general n by trying first the cases $n = 2$, $n = 3$.)

Project 3. Show that the 3×3 upper-triangular matrix

$$G = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

has an infinity of square roots, but none of them is upper-triangular.

Project 4. For real positive numbers a and b we always have $\sqrt{(ab)} = \sqrt{a}.\sqrt{b}$. Investigate the validity or failure of the equation

$$\sqrt{(AB)} = \sqrt{A}.\sqrt{B}$$

for pairs of upper-triangular matrices A and B of the same dimension, say 2 or 3. Here it is necessary to interpret the root signs appropriately, in each case.

Project 5. Let A be any matrix of the type described in Project 2. Let $U(A)$ denote the set of all the upper-triangular roots of the unit matrix I which commute with A. Show how all the upper-triangular roots of A can be generated using just one of them, and the set $U(A)$.

Project 6. Investigate the solutions X of the quadratic equation

$$X^2 + BX + XB + C = O,$$

where X, B and C are 2×2 upper-triangular matrices, B and C being assumed given.

* * * * *

HANDLING POLYNOMIALS WITH A COMPUTER ALGEBRA SYSTEM[†]

Cristina Varsavsky, Monash University

As was already explained in a previous edition (October '91), computer algebra systems deal with the manipulation of symbolic and algebraic expressions. These systems are continuously increasing in efficiency and capability.

Since many geometrical and engineering problems are defined by polynomials, polynomial operations enjoy intense research. The aim is to develop efficient algorithms for polynomial greatest common divisors, polynomial factorization, polynomial root isolation etc.

Although many algorithms already existed to perform these calculations, some of them are extremely 'costly' or 'time consuming' and are less suitable for computer algebra than are others. The purpose of this article is to present some of the various algorithms.

Let us start with the formal definition: a *polynomial* is an expression of the form

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 \tag{1}$$

where c_i, $1 \le i \le n$, are the coefficients and n is the degree of $p(x)$. The four basic operations, addition, subtraction, multiplication and division can be performed with polynomials.

[†] Our regular computer section is temporarily suspended as Dr R.T. Worley is on leave.

We will deal with rational coefficients which can be simplified to integer coefficients, because we can always multiply the polynomial by an integer in order to eliminate the denominators.

A first simple example of improving efficiency is evaluating a polynomial $p(x)$ at a given point $x = \alpha$. The most straightforward way to achieve this is to replace x by α in (1) and evaluate each term separately. However, this is not the most efficient way; the Ruffini-Horner algorithm achieves the same result with less calculations using the following nested form (try to prove it!):

$$p(\alpha) = c_0 + \alpha(c_1 + \alpha(c_2 + \dots + \alpha(c_{n-1} + \alpha c_n) \dots)).$$

For example, in $p(x) = x^3 - 3x^2 + 2x + 1$ evaluation at $x = 3$ is given by

$$p(3) = 1 + 3(2 + 3(-3 + 3\times1)) = 7$$

and we performed only three sums and three multiplications while the 'naive' method would have taken seven multiplications and three additions. It can easily be shown that in the general situation of a polynomial of degree n the number of basic operations involved is $2n$ when using the Ruffini-Horner method and $(n^2 + 5n + 2)/2$ for the 'naive' method.

The greatest common divisor

One of the most important operations on polynomials is the greatest common divisor of two polynomials $p(x)$ and $q(x)$, that is, determining a polynomial with the highest degree that divides both $p(x)$ and $q(x)$. We can find a greatest common divisor (*g.c.d.*) of $p(x)$ and $q(x)$ with the help of the *polynomial remainder sequence*. This process (Euclidean algorithm for polynomials) is essentially the same as for the g.c.d. of two integers. Assuming that degree($q(x)$) < degree($p(x)$), we first divide $p(x)$ by $q(x)$ obtaining the first remainder $r_1(x)$:

$$p(x) = q(x)\, h_1(x) + r_1(x) \quad \text{and} \quad \deg(r_1(x)) < \deg(q(x)).$$

The common divisors of $p(x)$ and $q(x)$ are the same as the common divisors of $q(x)$ and $r_1(x)$, because from the above expression we can see that any divisor of $p(x)$ and $q(x)$ is also a divisor of $r_1(x)$, and any divisor of $r_1(x)$ and $q(x)$ is a divisor of $p(x)$. Then

$$\text{g.c.d.}[p(x),\ q(x)] = \text{g.c.d.}[q(x),\ r_1(x)].$$

We can repeat this process with $q(x)$ and $r_1(x)$:

$$q(x) = r_1(x)\, h_2(x) + r_2(x) \quad \text{and} \quad \deg(r_2(x)) < \deg(r_1(x))$$

and consequently

$$\text{g.c.d.}[q(x),\ r_1(x)] = \text{g.c.d.}[r_1(x),\ r_2(x)].$$

We continue until we get a zero remainder. The last non-zero remainder is a g.c.d.$[p(x),\ q(x)]$.

For example, applying the logarithm to

$$p(x) = x^3 + x^2 - x - 1 \quad \text{and} \quad q(x) = 2x^2 + 5x + 3$$

we get the following sequence

$$x^3 + x^2 - x - 1 = (2x^2 + 5x + 3)(\tfrac{1}{2}x - \tfrac{3}{4}) + (\tfrac{5}{4}x + \tfrac{5}{4})$$

$$2x^2 + 5x + 3 = (\tfrac{5}{4}x + \tfrac{5}{4})(\tfrac{8}{5}x + \tfrac{12}{5}) + 0.$$

Therefore a g.c.d.$[p(x), q(x)]$ is $\frac{5}{4}x + \frac{5}{4}$.

Computing a g.c.d. with the Euclidean algorithm can be quite complicated; the coefficients of the remainder sequence can become very large and thus slow down computations. To convince you, just take for example a

$$\text{g.c.d.}[x^{10} - 2x^2 + 1,\ 3x^6 + 5x^4 - 4x^2 - 9x + 21].$$

The 5^{th} remainder is

$$r_5(x) = \frac{262847382927650270734426443}{225822869271742251131603344} - \frac{221347727491527013308134715}{225822869271742251131603344}x.$$

New algorithms had to be developed to avoid or at least simplify these polynomial sequences. One of the outcomes of the research done on this matter is the introduction of so-called *sequences of subresultant polynomials* which reduce the growth of the coefficients in the remainder sequences. It is interesting to mention that this development was based on a paper written in 1853 and buried soon after.

Squarefree factorization

A polynomial $p(x)$ is called *squarefree* if there is no polynomial $q(x)$ such that $q^2(x)$ divides $p(x)$. The simplest way to see whether a polynomial is squarefree is completely factorizing it by finding the roots. For example, the roots of $p(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$ are 2, -3, and -1 (double). Then $p(x)$ can be factorized as follows

$$p(x) = (x - 2)(x + 3)(x + 1)^2$$

and it follows that $p(x)$ is not squarefree because the double root -1 implies the multiple factor $(x + 1)$.

The *squarefree factorization* is obtained by multiplying the linear factors with the same power. In our example, we multiply the two factors with power 1, $(x - 2)$ and $(x + 3)$, and the squarefree factorization is

$$p(x) = (x + 1)^2(x^2 + x - 6)$$

being $x + 1$ and $x^2 + x - 6$ the squarefree factors of $p(x)$.

The decomposition in squarefree factors has many uses in mathematics. Among others, its applications are the decomposition in partial fractions and integration of rational functions.

The 'naive' method of finding the roots to get the squarefree factors will work only for degrees two, three and four, because as you know, it is impossible to solve in general polynomial equations of degree greater than four. Fortunately it is possible to compute the squarefree factors without calculating the roots but using differentiation, g.c.d. calculations and division. Here is a brief description of the algorithm.

Although we do not know how to find the roots, we know that they exist and that any polynomial $p(x)$ can be factorized into a product of linear factors as follows (we assume that the leading coefficient is 1):

$$p(x) = (x - b_1)^{m_1}(x - b_2)^{m_2} \dots (x - b_k)^{m_k}$$

where the b_i are the roots and m_i the corresponding multiplicities.

The derivative of p is obtained by applying the product rule

$$\begin{aligned} p'(x) &= m_1(x - b_1)^{m_1 - 1}(x - b_2)^{m_2} \dots (x - b_k)^{m_k} \\ &+ m_2(x - b_1)^{m_1}(x - b_2)^{m_2 - 1} \dots (x - b_k)^{m_k} \\ &+ \dots \dots \\ &+ m_k(x - b_1)^{m_1}(x - b_2)^{m_2} \dots (x - b_k)^{m_k - 1}. \end{aligned}$$

You can verify this for two or three factors. What this expression tells us is that the derivative of $p(x)$ consists of as many terms as there are roots of $p(x)$ and in each of those terms we have the same factors as in $p(x)$ all to the same power except one whose power is one less. Then we derive

$$r(x) = \text{g.c.d.}[p(x), p'(x)] = (x - \mathrm{b}_1)^{m_1 - 1}(x - b_2)^{m_2 - 1} \dots (x - b_k)^{m_k - 1}.$$

That is, the g.c.d. of $p(x)$ and $p'(x)$ has the same linear factors of $p(x)$ but their powers are reduced by one. Consequently, by dividing $p(x)$ and $r(x)$ we get rid of the multiplicity of the linear factors

$$t(x) = \frac{p(x)}{r(x)} = (x - \mathrm{b}_1)^{m_1 - 1}(x - b_2)^{m_2 - 1} \dots (x - b_k)^{m_k - 1}.$$

Then

$$v(x) = \text{g.c.d.}[t(x), r(x)]$$

will have the same factors of $p(x)$ with power equal to or greater than two. Hence $u(x) = \frac{t(x)}{v(x)}$ will give us the product of factors of $p(x)$ with power one. Repeating this process with $r(x)$ instead of $p(x)$ we can get the product of all linear factors of $p(x)$ of multiplicity two. Similarly for factors of multiplicity three, and so on.

Let us apply this algorithm to our example $p(x) = x^4 + 3x^3 - 3x^2 - 11x - 6$.

$$p'(x) = 4x^3 + 9x^2 - 6x - 11$$
$$r(x) = \text{g.c.d.}[p(x), p'(x)] = x + 1$$
$$t(x) = x^3 + 2x^2 - 5x - 6$$
$$v(x) = \text{g.c.d.}[t(x), r(x)] = x + 1$$
$$u(x) = x^2 + x - 6$$

and $x^2 + x - 6$ is the squarefree factor of $p(x)$ with exponent one. To compute the product of linear factors of $p(x)$ with power two we follow the same process but with $r(x)$ instead of $p(x)$:

$$r(x) = \text{g.c.d.}[x + 1, 1] = 1$$
$$t(x) = x + 1$$
$$v(x) = 1$$
$$u(x) = x + 1$$

which indicates that $(x + 1)^2$ is also a factor of $p(x)$. Since $r(x)$ is already a constant the process concludes here giving the squarefree decomposition

$$p(x) = (x^2 + x - 6)(x + 1)^2.$$

Real roots

Solving polynomial equations is a problem that has moved many mathematicians for centuries. They succeeded in solving by radicals the general equations of the second, third and fourth degree. They demonstrated the impossibility of solving "algebraically" equations of degree five or more.

After that discovery the problem of finding real roots of a polynomial was split into two simpler ones: first to isolate the real roots and then to approximate to any desired degree of accuracy.

Isolation is the process of finding disjoint intervals such that each contains exactly one root and every root is contained in some interval. This problem has been studied by many famous mathematicians like Descartes and Fourier and intense research is being done by computer algebraists since the birth of the subject.

In order to isolate the roots we first need to know their number. We can suppose that the polynomial $p(x)$ does not have multiple roots, because as we proved in the previous section, the multiplicity can be removed when dividing $p(x)$ by g.c.d.$[p(x), p'(x)]$. Cardano and Descartes observed that by just looking at the number of sign variations in the sequence of coefficients of $p(x)$, we can get an upper bound for the number of positive roots. More precisely, if V is the number of sign variations in the sequence $c_n, c_{n-1}, \ldots, c_1, c_0$, and p is the number of positive roots of $p(x)$ then

$$V = p + 2m \quad \text{where } m \text{ is a positive integer.}$$

For example, consider the polynomial $p(x) = x^4 + x^3 - x^2 + x - 2$. The sequence of coefficients is $[1, 1, -1, 1, -2]$. Since the sign changes three times we conclude that $p(x)$ either has three positive roots or it has one. We can use the same result for negative roots by just changing x by $-x$. In this case

$$p(-x) = (-x)^4 + (-x)^3 - (-x)^2 + (-x) - 2 = x^4 - x^3 - x^2 - x - 2.$$

Then the sequence is $[1, -1, -1, -1, -2]$ with only one sign variation. That means that there is exactly one negative root. Consequently $p(x)$ has either one positive, one negative, and two complex roots or one negative and three positive roots. This must be determined by further investigation.

We can obtain the exact number of roots within a given interval by using the *Sturm sequence* associated with the polynomial $p(x)$ and defined as follows:

$$p_0(x) = p(x)$$

$$p_1(x) = p'(x)$$

and for $i = 2, 3, \ldots$:

$$p_i(x) = -\text{ remainder in the division of } p_{i-2}(x) \text{ by } p_{i-1}(x)$$

which is very similar to the one we use to calculate a g.c.d.. How do we use this sequence to get the number of roots in the interval (a, b)? Very simple: substitute a and b into the Sturm sequence and count the sign variations in each of the resulting sequences. If we call those numbers $V(a)$ and $V(b)$, then the exact number of roots of $p(x)$ in (a, b) is given by $V(a) - V(b)$.

Let us apply this algorithm to the polynomial we used before. The corresponding Sturm sequence is

$$p_0(x) = x^4 + x^3 - x^2 + x - 2$$

$$p_1(x) = 4x^3 + 3x^2 - 2x + 1$$

$$p_2(x) = \frac{11}{16}x^2 - \frac{7}{8}x + \frac{33}{16}$$

$$p_3(x) = \frac{448}{121}x + \frac{256}{11}$$

$$p_4(x) = -\frac{27225}{784}.$$

If we want to know the number of roots in the interval $(-3, 3)$, we evaluate the Sturm sequence at $x = -3$ and at $x = 3$ and we obtain

$$[40, -74, \frac{87}{8}, \frac{1472}{121}, -\frac{27225}{784}] \text{ and } [100, 130, \frac{45}{8}, \frac{4160}{121}, -\frac{27225}{784}]$$

where the number of sign variations are three and one respectively and we conclude that $p(x)$ has exactly 2 roots in the interval $(-3, 3)$. But we cannot yet say that $p(x)$ has exactly 2 real roots, because we do not know whether all the roots are within that interval. Fortunately, there are some results to bound the roots of a polynomial. One of them says that any root α is bounded by

$$|\alpha| \le 2 \max\left(\left|\frac{c_{n-1}}{c_n}\right|, \left|\frac{c_{n-2}}{c_n}\right|^{1/2}, \left|\frac{c_{n-3}}{c_n}\right|^{1/3}, \ldots, \left|\frac{c_0}{c_n}\right|^{1/n}\right).$$

In our example any root α is bounded by

$$2 \max\left(\left|\frac{1}{1}\right|, \left|\frac{-1}{1}\right|^{1/2}, \left|\frac{1}{1}\right|^{1/3}, \left|\frac{-2}{1}\right|^{1/4}\right) = 2.$$

Therefore $-2 \leq \alpha \leq 2$, and since we already know that $p(x)$ has two roots in $(-3, 3)$, we conclude that $p(x)$ has exactly two real roots; the other two must be complex.

To isolate the two roots we split the interval into two sub-intervals. The Sturm sequence evaluated at zero is $[-2, 1, \frac{33}{16}, \frac{256}{11}, \frac{-27225}{784}]$ and consequently $V(0) = 2$. Given that $V(-3) = 3$ and $V(3) = 1$ we derive that $p(x)$ has one root in $(-3, 0)$ and another in $(0, 3)$, and we have a complete isolation of the real roots.

Once the roots are isolated we start the process of *approximation* to any desired accuracy. This is quite simple: if (a, b) is one of the isolating intervals, we calculate the middle point $c = \frac{a+b}{2}$ and determine the signs of $p(a)$ and $p(b)$. Those of $p(a)$ and $p(b)$ are certainly different because there is exactly one root in (a, b) and the graph of $p(x)$ crosses the x-axis only once. If $p(c) = 0$ then c is the root, but if $p(c) \neq 0$ its sign is either equal to that of $p(a)$ or $p(b)$. We reject the interval $[a, c)$ or $(c, b]$ on which the sign does not change and keep the other in which the root is to be found. In this way the size of the isolating interval is divided by two. We repeat this process until the interval is small enough.

In our example, if we want to approximate the root in $(0, 3)$, since we have $p(0) = -2$, $p(3) = 100$, $p(1.5) \cong 5.7$ we discard the interval $(1.5, 3)$ and split $(0, 1.5)$ in two. Now $p(0.75) \cong -1.07$, then the root is bound to be in the interval $(0.75, 1.5)$. Splitting again we get $p(1.125) \cong 0.88$. Then the root is within $(0.75, 1.125)$. We keep halving to get as close as we want to the root of $p(x)$.

This method will give us a root to any desired accuracy but it is also well known for its slowness. There are some other algorithms to speed up the process of approximation but as they need new definitions, they will not be mentioned here.

I hope this article gives you an insight into how polynomials are being handled by computer algebra systems. Summarizing, we can say that it is not just a matter of using any existing technique. Computer algebraists are working hard on improving algorithms introduced by earlier mathematicians and also introducing new ones.

* * * * *

Even in the best families ...

"Money, mechanization, algebra. The three monsters of contemporary civilization."

(Simone Weil, sister of the
algebraist André Weil)

* * * * *

ONE GOOD TURN

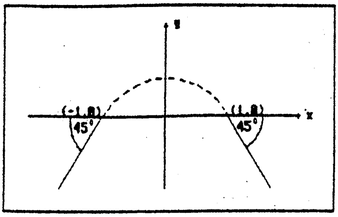
Kent Hoi, Student, University High School[†]

Introduction

In the construction of roads and railways particular care has to be taken in the construction of bends. The case of joining two straight sections of road or rail by a curve is to be considered.

Two roads which meet the x-axis at 45°.

The ends to be joined are 2 miles apart.



First a suitable bend is required so that there is no sudden change in direction. This will be so if the gradient of the tangent to the curve at the join is equal to the gradient of the straight section. It is also necessary to ensure that the rate of change of the tangential direction is the same for the straight section and the curve at the join, so there will not be any sudden jerk to the steering wheel.

In Summary

In order to ensure a smooth transition from the straight section to the curved section we require:

(a) gradient of the straight section at joint = gradient of the curve at join;

(b) curvature of the curve at join = curvature for the straight section at the join = 0, i.e. point of inflection.

Find functions which give a curve with all the requirements listed above. Then decide which one is the most suitable for the bend.

Finding

Investigate arcs of non-polynomial functions.

The most obvious non-polynomial function would be a trigonometric function, since logarithmic functions or exponential functions do not have any turning points, while the circle, ellipse and hyperbola do not have points of inflection. The arc could not possibly be a tangent function, and the cosine function is only a translation of a sine function, so we can treat them the same. Since the maximum y value is when $x = 0$, therefore it is most likely to be a cosine graph.

[†] An excerpt from a VCE project.

The first x-intercept of a standard cosine graph is of $x = \pm\pi/2$. This means we are required to dilate the standard cosine graph by a factor of $2/\pi$ from the y-axis, so the distance between the two x-intercepts is 2 units.

$$y = \cos\frac{\pi x}{2}$$

$$\frac{dy}{dx} = -\frac{\pi}{2}\sin\frac{\pi x}{2}.$$

When $x = -1$ or 1, $\frac{dy}{dx} = \pm\pi/2$, and this does not satisfy the conditions listed previously. So we need to dilate the function from the x-axis as well by the same factor.

$$y = \frac{2}{\pi}\cos\frac{\pi x}{2}$$

$$\frac{dy}{dx} = -\sin\frac{\pi x}{2}.$$

Now this satisfies the first condition with the gradient.

$$\frac{dy}{dx} = -\sin\frac{\pi x}{2}$$

$$\frac{d^2y}{dx^2} = -\frac{\pi}{2}\cos\frac{\pi x}{2}.$$

When $x = -1$ or 1, the curvature is 0. This function satisfies all the conditions.

Investigate an arc of a quartic of the form $y = a(x^2-1)(x^2-b)$.

Expand the above equation, since it is easier to differentiate.

$$y = ax^4 - (b+1)ax^2 + ab$$

$$\frac{dy}{dx} = 4ax^3 - 2a(b+1)x$$

$$\frac{d^2y}{dx^2} = 12ax^2 - 2ab - 2a.$$

The concavity for $(-1, 0)$, $(1, 0)$ must be 0.

Therefore

$$12a - 2ab - 2a = 0$$
$$10a - 2ab = 0$$
$$2a(5 - b) = 0$$
$$a = 0 \text{ or } 5 - b = 0$$

since $a \neq 0$, $\therefore b = 5$.

When $x = -1$, $\frac{dy}{dx} = 1$, and when $x = 1$, $\frac{dy}{dx} = -1$.

Therefore

$$4a - 2ab - 2a = -1$$
$$2a - 10a = -1$$
$$-8a = -1$$
$$a = 1/8$$

$\therefore\ y = (x^2 - 1)(x^2 - 5)/8$ satisfies all the conditions.

I decided to check if there were other functions which also satisfy all the conditions. Since there were no other non-polynomial functions covered in class, I continued to investigate with polynomial functions. I used the form $a(x^2 - 1)(x^{2n} - \mathrm{b})$. Since the function must be raised by an even power to be symmetrical and this will ensure a function with an axis of symmetry at $x = 0$, I first tried $a(x^2 - 1)(x^4 - \mathrm{b})$ and found that $(x^2 - 1)(x^4 - 9)/16$ also satisfied all the conditions. Then I continued to investigate with different functions such as $a(x^2 - 1)(x^4 - b)$ and found that $1/24(x^2 - 1)(x^6 - 13)$ satisfied the requirements.

After a further two functions I worked out the following functions which satisfied the conditions and plotted a graph with all of them on the same axis.

(1) $y = \frac{1}{8}(x^2 - 1)(x^2 - 5)$

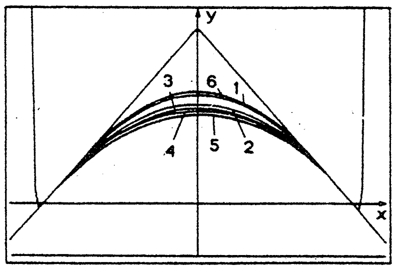
(2) $y = \frac{1}{16}(x^2 - 1)(x^4 - 9)$

(3) $y = \frac{1}{24}(x^2 - 1)(x^6 - 13)$

(4) $y = \frac{1}{32}(x^2 - 1)(x^8 - 17)$

(5) $y = \frac{1}{400}(x^2 - 1)(x^{100} - 201)$

(6) $y = \frac{2}{\pi} \cos\frac{\pi x}{2}$.



There is an infinite number of different polynomial functions which can satisfy all the conditions, so long as they are of the form $a(x^2-1)(x^{2n} - \mathrm{b})$.

* * * * *

Editors' Note: Kent has very ably looked at the problem set by the examiners. Indeed, there are more matters discussed in this project than we can include here. However, in practice the problem is actually more complicated. The curvature, strictly speaking, is not

$$\frac{d^2y}{dx^2} \quad \text{but} \quad \left\{1 + \left[\frac{dy}{dx}\right]^2\right\}^{-3/2} \frac{d^2y}{dx^2}$$

as the more complicated expression measures the curvature *experienced by the driver as the car travels along the curve.* This and other considerations greatly complicate matters.

* * * * *

HISTORY OF MATHEMATICS SECTION

EDITOR: M.A.B. DEAKIN

Hypatia of Alexandria

The first† woman in the history of mathematics is usually taken to be Hypatia†† of Alexandria who lived from about 370 A.D. to (probably) 415. Ever since this column began, I have had requests to write up her story. It is certainly a fascinating and a colourful one, but much more difficult of writing than I had imagined it to be; this is because so much of the good historical material is hard to come by (and not in English), while so much of what is readily to hand is unreliable, rhetorical or plain fiction.

I will come back to these points but before I do let me fill in the background to our story.

Alexander the Great conquered northern Egypt a little before 330 B.C. and installed one of his generals, Ptolemy I Soter, as governor. In the course of his conquest, he founded a city in the Nile delta and modestly named it Alexandria. It was here that Ptolemy I Soter founded the famous Alexandrian Museum, seen by many as an ancient counterpart to today's universities. (Euclid seems to have been its first "professor" of mathematics; certainly he was attached to the Museum in its early days.) The Museum was for centuries a centre of scholarship and learning.

Alexandria fell into the hands of the Romans in 30 B.C. with the suicide of Cleopatra. Nevertheless, the influence of Greek culture and learning continued. Two very great mathematicians are associated with this second period. *Diophantus* (who lived around 250 A.D.) wrote a number of books but most particularly an algebra text that will come into the story later. A little less than a hundred years after Diophantus came the great geometer *Pappus*. Shortly after Pappus, however, the Museum fell into a decline. Alexandria became a prey to sectarian violence between various factions of Christians, different groups of Greek "pagans" (including a number of so-called Neoplatonic groups), Jews and others.

Riots occurred and did much to damage the Museum, in particular destroying its great libraries – the last going up in smoke in 392 when the temple of Serapis was put to the torch by a riotous throng.

The last known member of the Museum (very likely its last president) was *Theon of Alexandria*, a minor mathematician and astronomer. He made few if any original contributions to mathematics but his work as an editor has been very useful to later generations. His daughter, *Hypatia*, is the heroine of our story. She was not associated with the Museum, but headed the Neoplatonic school, another institution. She thus belonged to one of the "pagan" groups, and met her death on this account, brutally hacked to pieces by a Christian lynch-mob in a year that is usually put at 415.

---

† This assessment may, however, need to be revised. Winifred Frost of the University of Newcastle believes she has found an earlier claimant. We hope to bring this development to you in a future issue.

†† The strictly correct pronunciation probably approximates *heew–pah–TEE–ah*, but it is usual and acceptable to pronounce the name as *high–PAY–sha*. (Much as we say "Paris" in the usual way and don't attempt the French pronunciation, which is more like *par–HEE*.)

After this, and perhaps in part because of it, the focus of Neoplatonist thought moved to Athens. In the years that followed, *Proclus* (to whom we owe the preservation of much of Euclid's *Elements*) and other mathematicians frequented the Neoplatonic school in Athens. The last two names associated with the school are those of philosophers rather than mathematicians: *Isidorus* and his pupil *Damascius*† . In 529, the emperor Justinian, enforcing Christianity, closed the school and Damascius went into exile in Persia.

Let us now look more closely at those turbulent times around the year 400. A good place to begin finding out about a mathematician from the past is a 16-volume work called the *Dictionary of Scientific Biography* (DSB). The article on Theon in that work is by G.J. Toomer; it is authoritative and well-researched. It tells us what happened and also *how we know* that it happened. There are notes with clear and detailed references to where Toomer got the information. All this is how a scholarly article should be.

[These articles in *Function*, by contrast, are popular articles and are not intended as scholarly. So in most cases I don't give all my sources; usually they are readily accessible, and in any case readers seeking further information can always write to me, as some have. In the present case, I will give rather more detail than is my usual custom, but not to the point of excess. I hope to prepare a scholarly article on the subject for publication elsewhere.]

Toomer, to get back to *Theon*, tells us that Theon was the author of a number of "Commentaries". These were editions, with extra notes, of the works of famous authors. In many cases, the original works get lost and modern editors have to work from such Commentaries. Theon wrote Commentaries on Euclid's *Elements* (and in places these provide the basis for the modern text), two other books by Euclid, the *Data* and the *Optics*, and two works by the astronomer *Ptolemy* (about 100-170 A.D.)†† , the *Almagest* and the *Handy Tables*. Over and above these he wrote a book on the astrolabe, an astronomical instrument with navigational applications. This book may or may not have been a Commentary on an earlier (now lost) book by Ptolemy. It too is lost, but perhaps not entirely.

So Toomer tells us a lot about Theon. Regrettably the DSB article on Hypatia is not up to that work's usually high standard. The sources are only perfunctorily indicated, and in many cases not given, credence is given to a work of avowed fiction, and some statements are plain wrong. So I had rather more work to do than I anticipated when I set out to write this article; however, the extra work has led me to some very interesting reading.

Historians distinguish between *primary* sources (the original documents on which all subsequent work depends) and *secondary* sources (those which retell, explain, comment on and judge the material in the primary sources). Unless one is *oneself* expert in the period, the language (in this case patristic Greek) and the questions of textual authenticity and interpretation, secondary sources are vital. In this instance, the sources are rather hard to come by. I have succeeded in laying my hands on all the primary sources and most but not all of the best secondary ones.

The primary sources on Hypatia come under two headings: (a) the *Suda Lexicon*, (b) the *Patrologiae Graecae*. The *Suda Lexicon* is a 5-volume work from the 10th Century A.D. It is an alphabetical compilation for all the world like an encyclopedia of today. Until recently it was called the *Suidae Lexicon* and its supposed author was called *Suidas*

---

† Damascius may have some minor claim on mathematical history as an editor of Euclid, but the case is disputed.

†† Note that this is not Ptolemy I Soter, but a different chap.

(rather as if some 30th Century writer were to talk of Britannicus and his wonderful Encyclopedia!). Anyhow, the *Suda Lexicon* (the name is now thought to be related to the Greek for "fortress" – the stronghold of knowledge) is a compilation from earlier sources.

There is quite a long entry on Hypatia in the *Suda* and this derives from an earlier such encyclopedia (the *Onomatologus* of *Hesychius Milesius*) and also from Damascius' *Life of Isidorus.*

Hesychius Milesius was also known as Hesychius the Illustrious. The name Hesychius was quite common and so one needed to say which Hesychius was being discussed. (In particular, and confusingly, Hesychius of Alexandria has nothing to do with the story.) Hesychius Milesius' *Onomatologus* now survives only through one very imperfect copy and what has found its way into later works like the *Suda*. *Damascius* we met briefly earlier. His life of Isidorus is now lost, but fragments survive as quotations in other writings.

The *Patrologiae Graecae* are by and large in better shape. They form a work of over 150 volumes collecting the writings (in Greek) of persons important in the early Christian Church. The most scholarly edition comes with a parallel translation into Latin. Of the texts in this collection that concern Hypatia, most are letters from *Synesius*, one of her pupils, but who either was or became a Christian, indeed a bishop. There are also letters from Synesius not to Hypatia but making mention of her. The other major source in the *Patrologiae Graecae* is a passage in the *Ecclesiastical History* by *Socrates Scholasticus*, who lived only shortly after Hypatia. (This is not, of course, *the* Socrates; *he* was some 850 years dead by this time.) The remaining references are meagre. There is a sentence in the 6th Century *Chronicle of John Malalas* and a short paragraph in an early 5th Century chronicle by the ecclesiastical historian *Philostorgius*. It is to Philostorgius that we owe the opinion that Hypatia was a better mathematician than her father Theon, and it's possible that he had many more interesting things to say – but we don't know. The version of Philostorgius that has come down to us is an abridgement by the 9th Century scribe *Photius*.

Photius himself wrote a sentence on Hypatia. It will endear him neither to women nor to mathematicians. It went:

> "Isidorus greatly outshone Hypatia, not just because he was a man and she a woman, but in the way a genuine philosopher will over a mere geometer."

It is believed that this sentence is in fact copied from Damascius' *Life*, the lost work that in part informed the *Suda*.

The last and least of the Christian fathers with anything to say is *Nicephorus Callistus* who lived in the 14th Century and whose account merely paraphrases Socrates Scholasticus.

So – there are our sources. What do they tell us?

As always, much less than we'd like to know. But a good deal is agreed. Hypatia was a public figure who taught philosophy and mathematics. She attracted a large following and probably held some kind of official post. She was unmarried – in fact determinedly celibate. She was a Neoplatonist, born probably sometime around 370 and murdered in (almost certainly) 415 by a mob of Christian fanatics.

There are arguments over details: which of the many brands of Neoplatonism did she profess? Which Christian faction killed her and why? Was Cyril, the bishop of

Alexandria, implicated in her death? And so on.†

I will not dwell on these matters, but turn rather to what we can learn of Hypatia's mathematics and what the sources tell us of *that*.

Precious little really, I'm afraid. *That* she was a mathematician is widely agreed. She is variously described as a philosopher, a mathematician, a geometer and an astronomer. What we would like to know is what as a mathematician, geometer or astronomer it was that she did.

The most explicit statement is a 12-word passage in the *Suda*. Yes, just 12 words (and almost half of these subject to disputed readings or various interpretations). However, there is a general consensus as to what they say:

> "She wrote a Commentary on Diophantus, [one on] the astronomical Canon, and a Commentary on Apollonius's Conics."

Take these in reverse order. *Apollonius*, who lived around 200 B.C., was a very great geometer. He codified much of what we know about the conic sections (ellipse, parabola, hyperbola). Regrettably, Hypatia's Commentary on this work is totally lost.

When it comes to the "astronomical canon", we are on slightly firmer ground. Most scholars agree that what she wrote was a Commentary on one of Ptolemy's works: either the *Almagest* or the *Handy Tables*. It will be remembered that Theon, Hypatia's father, wrote Commentaries on both these works. Various authors have suggested that Hypatia collaborated with him in one or other or both of these enterprises.

Theon's Commentary on the *Almagest* has twice been edited in modern times: once last century and once this. The 20th Century edition is a work of great scholarship. Its editor, a Professor *Rome*, suggests that what Hypatia did was to revise her father's Commentary on Book 3 of the *Almagest*. An inscription by Theon is preserved in the best manuscripts saying that he is using the text as revised by 'my philosopher-daughter, Hypatia'.

Neugebauer (a historian of Mathematics whom we met in *Function, Vol.* 15, *Part* 3), however, thinks that this is not the work referred to in the *Suda*, which he thinks is a now lost Commentary on Ptolemy's *Handy Tables*.

The remaining work attributed to Hypatia is her Commentary on Diophantus. Most writers assume that Hypatia's Commentary was an edition of his major work, the *Arithmetic*.

By "arithmetic" we should understand "number theory", which used to be called "higher arithmetic". Diophantus has given his name to several branches of modern mathematics. A diophantine equation, for example, is one to be solved in integers. Thus, for example, the diophantine equation

$$x^2 + y^2 = 25$$

has solutions (0, ±5), (±3, ±4), (±4, ±3), (±5, 0).

But I digress. Diophantus's *Arithmetic*, like Euclid's *Elements*, was a collection of 13 "books". We know this from the introduction. Of these 13, however, we have only six (presumed to be the first six).

---

† We may dismiss a further ground of dispute. She was *not* Isidorus's wife, although the *Suda* says at one point that she was. The passage is almost certainly spurious (some blame Photius). Besides, Isidorus was either unborn or in nappies when Hypatia was killed.

Diophantus's writings were collected and edited by the 19th Century French scholar, *Paul Tannery*. Tannery put forward the suggestion that it was these six books on which Hypatia "commented". This could have been because Books 7-13 were already lost by Hypatia's time, but Tannery preferred the alternative view that Books 1-6 were preserved *because* Hypatia commented on them. (Recent research based on Arab translations has complicated this theory however; it is the subject of great controversy.) Tannery and Heath, however, suggest that what has come down to us is in fact not Diophantus's original but Hypatia's Commentary. If this is so then we are much indebted to Hypatia, for without her we would miss most of the surviving work of Diophantus.

But now, if this is right, then what survives of Diophantus's work would incorporate whatever comment Hypatia wrote, and so we would have a small legacy of her mathematics hidden in the work of Diophantus. In 1885, Sir Thomas Heath brought out the first English edition of Diophantus. This suggests that the most obvious place to look for such interpolated material is at the start of Book 2. Problems 1-5 of Book 2 are mere repetitions of problems that already appeared in Book 1. Problems 6, 7 look "different" from Problems 8, 9, etc.

It seems very much as if what we see is an edition designed as a student text. Problems 1-5 could be seen as "revision". Then Problems 6, 7 are "exercises" before we move on to the "new theory" of Problems 8, 9, etc. Thus, if we do see Hypatia's hand in Diophantus's *Arithmetic*, she poses (in essence) the problem of solving for x, y the simultaneous equations

$$x - y = a, \quad (x^2 - y^2) - (x - y) = b,$$

where a, b are known. This is Problem 6. Problem 7 is essentially the same.

Hardly, I'm afraid, stuff to raise one's voice about.

The other source of specific information on Hypatia's mathematics is the writings of Synesius. Of Synesius' letters to Hypatia, six and a little bit survive. She is mentioned in a number of others – the precise number depending on which editor one believes. Two of these letters are relevant to an evaluation of Hypatia's mathematics.

One is Letter 15. It is puzzling. He writes that he is "in such a bad way" that he has to have a "hydroscope". He asks her to make him one and sends quite detailed instructions and specifications. Clearly he greatly respects her abilities – indeed relies on them.

But what is he talking about? What is a "hydroscope"? And why should he be in such urgent need of one? The question has attracted attention for over 300 years at least. Normally a "hydroscope" implies a water-clock, but why should he be so desperate for a water-clock? *Fermat*, the 17th Century mathematician, suggested that what Synesius needed (being very ill) was a *hydrometer* to measure the density of drinking water or medicine of some sort. Now, certainly, the letter has a text which is compatible with this story. But does one really judge drinking water or measure medicine by finding its *density*? Was he perhaps *making* his own medicine? The whole matter leaves me perplexed.

Finally we return to the astrolabe. The term "astrolabe" is applied to a wide variety of astronomical or navigational instruments. (For an accessible article on the astrolabe see *Scientific American*, Jan. 1974.) Essentially all the various instruments that went by the name were models of the heavens. Some were "armillary spheres" – large, and necessarily clumsy, 3-D structures. Other, later, varieties were portable 2-D instruments in which geometric projections made for what were handy dedicated analogue computers.

This account comes from Neugebauer who suggests that Theon's "lost" work on the astrolabe is alive and well – the common core to various suspiciously similar works which he sees as Commentaries on an earlier work: Theon's. It may well be that it was Ptolemy who showed how to construct the handier 2-D instrument and that Theon's book in its turn derived from Ptolemy. This seems to Neugebauer the most likely course of events.

In any case, Synesius wrote a covering letter (it isn't listed as a letter and is to be found elsewhere in his writings – however, it is a letter) to one *Paionos* to accompany the gift of an astrolabe. In it, he states that he designed the astrolabe himself but with help from Hypatia and had it crafted by the very best of silversmiths. The implication is that the knowledge derived by (probably) Ptolemy was passed on through Theon to Hypatia and thus to her pupil Synesius.

This then exhausts all we know of Hypatia's mathematics. It is commonly said that Theon was a transmitter of mathematics rather than a creator of it. He edited the works of others rather than developing theories of his own. The same would seem to be true of his daughter. She was widely respected as a teacher – eminent, influential, even charismatic in her day. But we have no evidence that she was anything more than this.

There has been an often stated view that she was a better mathematician than Theon. This derives from the passage in Philostorgius, which may however mean merely that she was the more widely acclaimed in her day. Indeed, we could argue that Theon was in fact the greater. In 640 or 642, the Arabs conquered Alexandria. What texts we have of Greek mathematics often come to us through Arab translations and Commentaries. This is true of Diophantus's *Arithmetic* and also of some of Theon's work. It is not unreasonable that there is a principle of selection here – the best work is what has survived; the Arabs saved what they thought worth saving. One would not like to push this notion too far, nonetheless a good proportion of Theon's work survives and almost none of Hypatia's.

Whatever judgement we make of her contribution to mathematics, she was certainly a remarkable woman. She certainly was a mathematician, a philosopher and a charismatic teacher. It would be nice to know more of her.

* * * * *

A True Equality

The German university town of Göttingen is famous for its mathematicians and theoretical physicists, among them the very great mathematician David Hilbert (1862-1943). Overlooking the town are two hills known as *die Gleichen* (the equals). Hilbert was fond of saying that this was not because they were the same height, nor because they presented the same aspect to the viewer.

"Why the name then?", people would ask.

Hilbert is said to have attributed the name to the incontrovertible fact that they were the same distance from one another!

* * * * *

LETTERS TO THE EDITOR

A letter from Peter Grossman and Keith Anker appeared in *Function, Vol.* 15, *Part* 5. It concerned the geometric mean and showed that it could be viewed as a special case of the more general "power mean" described by K.McR. Evans in *Function, Vol.* 15, *Part* 4. Regrettably the letter was so badly misprinted as to be virtually incomprehensible. We decided the only recourse was to print it again – this time as it should have been. Our sincerest apologies to both the authors, and also to our readers!

The Geometric Mean

In his concluding remarks, K. Evans (*Function, Vol.* 15, *Part* 4) pointed out that three of the four means he had discussed (the arithmetic and harmonic means and the root mean square) are special cases of the *power mean* (of two positive numbers x_1, x_2) defined by

$$M(n, x_1, x_2) = \left\{ \frac{x_1^n + x_2^n}{2} \right\}^{1/n} \tag{1}$$

Specifically, the arithmetic and harmonic means and the root mean square correspond to the cases $n = 1$, $n = -1$ and $n = 2$ respectively.

We would like to point out that there is a sense in which the fourth mean considered by K. Evans, namely the geometric mean, can also be regarded as a special case of the power mean, corresponding to $n = 0$. Of course, we cannot just put $n = 0$ in Equation (1), since the expression would be undefined; however, it is the case that

$$\lim_{n \to 0} M(n, x_1, x_2) = \sqrt{x_1 x_2} \tag{2}$$

where the right-hand side is the geometric mean of x_1 and x_2.

If we are prepared to take for granted that this limit exists, then Equation (2) can be established as follows. First observe that

$$\left\{ \frac{x_1^n + x_2^n}{2} \right\}^{1/n} = \left[\frac{x_1^{n/2} x_2^{n/2} (x_1^{n/2} x_2^{-n/2} + x_1^{-n/2} x_2^{n/2})}{2} \right]^{1/n}$$

$$= \sqrt{x_1 x_2} \left[\frac{(x_1/x_2)^{n/2} + (x_1/x_2)^{-n/2}}{2} \right]^{1/n} .$$

Therefore it is sufficient to show that

$$\lim \left(\frac{\nu^{n/2} + \nu^{-n/2}}{2} \right)^{1/n} = 1,$$

where $\nu > 0$.

Let $L = \lim \left[\frac{v^{n/2} + v^{-n/2}}{2} \right]^{1/n}$. Now put $m = -n$ to find

$$L = \lim_{m \to 0} \left[\frac{v^{-m/2} + v^{m/2}}{2} \right]^{-1/m} = \left[\lim_{m \to 0} \left[\frac{v^{-m/2} + v^{m/2}}{2} \right]^{1/m} \right]^{-1} = L^{-1}.$$

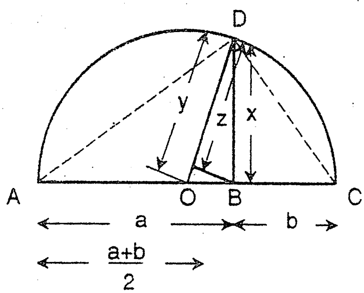
Hence $L^2 = 1$. Since the limit clearly cannot be negative, we must have $L = 1$.

Peter Grossman, Keith Anker
Monash University, Caulfield Campus

* * * * *

More on Averages

In *Function, Volume* 15, *Part* 4, p. 99, a diagram illustrates the construction of the geometric mean, $G(a, b) = \sqrt{(ab)}$, of two positive numbers a, b. The diagram can be extended to show the arithmetic mean, $A(a, b) = \frac{a+b}{2}$, and the harmonic mean, $H(a, b) = \left[\frac{a^{-1} + b^{-1}}{2} \right]^{-1}$. This extension appears in Note 75.18 in *The Mathematical Gazette* (U.K.), *Volume* 75, *No*. 472, June 1991, but was known to Pappus of Alexandria, a mathematician of the 4th century A.D.



Exercise: Use similar triangles to prove that

$$x = G(a, b)$$
$$y = A(a, b)$$
$$z = H(a, b).$$

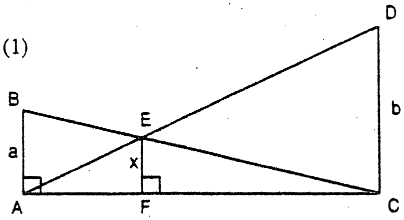
Notice also that $z \le x \le y$, i.e.
$H(a, b) \le G(a, b) \le A(a, b)$.

Geometry abounds with illustrations of the geometric mean and various power means

$$M(n, a, b) = \left[\frac{a^n + b^n}{2} \right]^{1/n}, \; n \in R \setminus \{0\},$$

of two positive numbers a, b. The following are exercises for the reader.

(1)

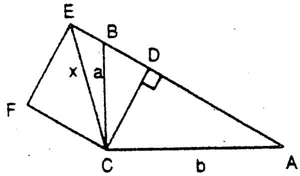


In the figure $\overline{AD} \cap \overline{BC} = \{E\}$.
If $AB = a$, $CD = b$, $EF = x$, prove that

$$x = \tfrac{1}{2}H(a, b)$$
$$= \tfrac{1}{2}M(-1, a, b).$$

What is the locus of E as the length AC changes?

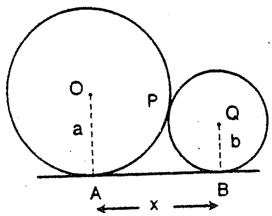
(2)



ABC is a triangle right-angled at C. $\overline{CD} \perp \overline{AB}$. A square $CDEF$ is shown on side $\overline{CD}$. If $BC = a$, $CA = b$, $CE = x$, prove that $x = M(-2, a, b)$.

The next three exercises have been adapted from "Japanese Temple Geometry Problems", H. Fukagawa and D. Pedoe, 1989, Charles Babbage Research Centre, Winnipeg, Canada.

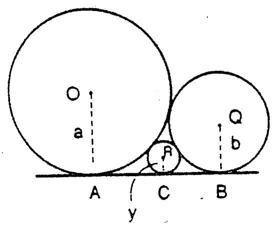
(3)



Two circles with centres O, Q and radii lengths a, b touch at P and have a common tangent $\overleftrightarrow{AB}$ as shown. If $AB = x$, prove that

$$x = 2G(a, b).$$

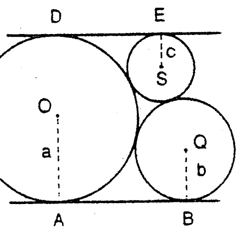
(4)



In the diagram of Question (3) a third circle, centre R, radius length y, is drawn touching the other two circles and $\overleftrightarrow{AB}$. Prove that

$$y = \tfrac{1}{4}M(-\tfrac{1}{2}, a, b).$$

(5)



a > b

In the diagram of Question 3 a tangent to the larger circle is drawn parallel to $\overleftrightarrow{AB}$. A third circle, centre S, radius length c, is drawn touching this tangent and the other two circles. Prove that

$$a = 2G(b, c).$$

K. McR. Evans

[Problem (4) relates very closely to "The Kiss Precise", Function, Vol. *15*, Part *4*. *Eds.]*

PROBLEMS SECTION

EDITOR: H. LAUSCH

We cordially welcome our subscribers in 1992 and hope they will enjoy this section of Function *by trying their hands and minds on the problems or by puzzling fellow subscribers with their own problems. At the moment there is no shortage of imaginative solutions and new problems that readers sent in (mainly during the later parts of 1991). Thank you all for your contributions!*

Solutions

One solution, by John Barton (North Carlton), to the following problem has already been published in Function, *Volume 15, Part 4. Here is a different solution by Andy Liu (Edmonton, Alberta, Canada):*

Problem 15.1.3. A person sits for an examination in which there are four papers with a maximum of m marks for each paper; show that the number of ways in which a total of $2m$ marks may be obtained is $\frac{1}{3}(m+1)(2m^2+4m+3)$.

Solution. Represent the $2m$ marks by $2m$ circles, and use 3 strokes to divide them into 4 parts. The number of ways of arranging $2m$ circles and 3 strokes in a row is $\binom{2m+3}{3}$. However, we must rule out those cases where we have at least $m+1$ circles between two consecutive strokes. We first remove $m+1$ circles. Now there are $\binom{m+3}{3}$ ways of arranging $m-1$ circles and 3 strokes in a row. We can add the $m+1$ circles back into any of the 4 parts. It follows that the desired answer is $\binom{2m+3}{3} - 4\binom{m+2}{3} = \frac{(m+1)(2m^2+4m+3)}{3}$.

And Seung-Jin Bang (Seoul, Republic of Korea) counted the marks quite differently – with the same result, of course:

Solution. Let x_i $(i = 1, 2, 3, 4)$ be the points obtained from the i-th paper. Then $x_1 + x_2 + x_3 + x_4 = m$, $0 \le x \le m$, $i = 1, 2, 3, 4$. Let $x_1 + x_2 = l$ and $x_3 + x_4 = 2m - l$. Suppose $0 \le l \le m - 1$.

The number of solutions (x_1, x_2) of the equation $x_1 + x_2 = l$ is $\binom{l+1}{l} = l + 1$ and the number of solutions $(x_3 + x_4) = 2m - l$ is $l + 1$. It follows that the number of solutions (x_1, x_2, x_3, x_4) of the equations $x_1 + x_2 = l$, $x_3 + x_4 = 2m - l$, $0 \le x_i \le m$, $i = 1, 2, 3, 4$ is $(l+1)^2$.

We conclude that the number of required ways is

$$2\sum_{l=0}^{m-1}(l+1)^2 + (m+1)^2 = \frac{m(m+1)(2m+1)}{3} + (m+1)^2$$

$$= \frac{(m+1)(2m^2+4m+3)}{3}.$$

Known by the name "Steiner-Lehmus Theorem", the next problem is a "classic". Garnet J. Greenbury (Up. Mt. Gravatt, Queensland) proposed it to Function. *Quite overwhelming were readers' responses, being often not just "simple" solutions but little treatises, containing interesting historical facts.*

Problem 15.1.5. If the bisectors of two angles of a triangle are equal, the triangle is isosceles. – We want a *Euclidean proof.* [Trigonometric proofs are acceptable.]

We begin with a solution submitted by Andy Liu, who comments: "This is essentially Exercises 13, 15 and 16 in Section 2.1 of Howard Eves' A Survey of Geometry. *Although heavily algebraic, it does not use trigonometry, co-ordinates or anything beyond Euclidean geometry."*

Solution.

a. STEWART'S THEOREM. Let D be a point between B and C. Then $AB^2CD + AC^2BD = AD^2BC + BC.BD.CD$ for any point A.

Proof. First note that if A coincides with any of B, C or D, equality holds by virtue of $BC = BD + CD$. Suppose A coincides with a point H between C and D. Then

$$HB^2CD + HC^2BD = (BD+HD)^2(HD+HC) + HC^2BD$$

and

$$HD^2BC + BC.BD.CD = HD^2(BD+HD+HC) + (BD+HD+HC)BD(HD+HC).$$

It is routine to verify that these two expressions are equal. Equality can be established in the same way if A coincides with a point H such that C is between D and H. Finally, suppose A is any point not on the line BC. Let H be the foot of the perpendicular from A to BC. Then

$$AB^2 + AC^2BD = HB^2CD + HC^2BD + AH^2(CD+BD)$$

$$= HD^2BC + BC.BD.CD + AH^2BC$$

$$= AD^2BC + BC.BD.CD.$$

b. ANGLE-BISECTOR FORMULA. Let the bisector of $\angle BAC$ intersect BC at D. Then

$$AD^2 = AB.AC\left[1 - \frac{BC^2}{(AB+AC)^2}\right].$$

Proof. We have \F(BD,AB) = \F(CD,AC) and $BD + CD = BC$. Hence $BD = \frac{AB\,.\,BC}{AB+AC}$ and $CD = \frac{AC\,.\,BC}{AB+AC}$. The desired result follows from Stewart's Theorem.

c. STEINER-LEHMUS THEOREM. Let the bisector of $\angle ABC$ and $\angle ACB$ intersect AC and AB at E and F resepectively and let $BE = CF$. Then $AB = AC$.

Proof. By the Angle-Bisector Formula,

$$BE^2 = BA.BC\left[1 - \frac{AC^2}{(BA+BC)^2}\right] \text{ and } CF^2 = CA.CB\left[1 - \frac{AB^2}{(CA+CB)^2}\right].$$

From $BE^2 = CF^2$ we have

$$(AB-AC)[AB(AC+BC)^2 + AC(AB+BC)^2 + AB.BC.CA + BC^3].BC(AB+BC+CA) = 0.$$

It follows that $AB = AC$.

K.R.S. Sastry (Addis Ababa, Ethiopia) introduces his proof with remarks that might be useful to Function *readers:*

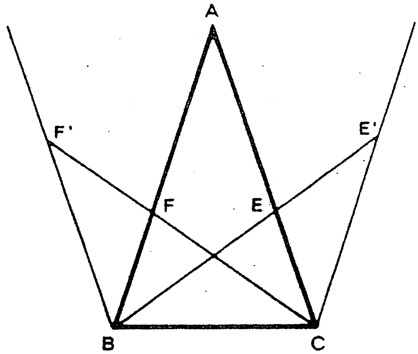
"I am writing about 15.1.5, the Steiner-Lehmus Theorem. It deserves a discussion in the History section of Function [any volunteer authors? Ed.]. *Here is some background material that I think readers will find of interest.*

In 1840 Professor Lehmus asked the Swiss geometer Steiner for a proof of 15.1.5. Steiner soon found one but did not publish immediately. In 1850 Lehmus found his own. The first published proof was in 1842 by the Frenchman Rougevain. See Mathematics Magazine, *a publication of Mathematical Association of America (MAA), volume 47 (March 1974), p. 87, 'On the Steiner-Lehmus Theorem' by Mordechai Lewin for other details.*

In an interview for the Two-Year College Mathematics Journal of MAA *(now* College Mathematics Journal, CMJ*) published in Vol. II, Number 1, January 1980, page 10, the celebrated geometer H.S.M. Coxeter says: "There has been an enormous number of proofs of that, I should think over a hundred."*

In fact, in response to a request from Tony Trono of Burlington High School, U.S.A., in the Reader Reflections Section of the December 1980 issue of the Mathematics Teacher *(published by the National Council of Teachers of Mathematics of the U.S.A.) he received 80 different proofs from around the world. There have been a number of incorrect proofs of the Steiner-Lehmus Theorem. For one that is in print, see "Fallacies, flaws and flimflam, FFFZ, The Steiner-Lehmus Theorem" on p. 50, vol. 20, No. 1, in the January 1989 issue of* CMJ.

Solution.



Suppose BE and CF are the internal angle bisectors such that $BE = CF$.

We wish to show that $AB = AC$. Let BE extended meet the parallel through C to BA at E'. Similarly the point F' is defined. If (the angle at) B is different from (the angle at) C, then suppose that $B > C$.

Then $AC > AB$. Also $A + B > A + C$ and hence $CF' > BE'$ or $FF' > EE'$, whence

$$CF = BE. \tag{1}$$

Now the triangles ABE and $CE'E$ are similar. Therefore $\frac{AE}{EC} = \frac{AB}{CE'} = \frac{BE}{EE'}$. Likewise $\frac{AF}{FB} = \frac{AC}{BF'} = \frac{CF}{FF'}$. Hence $\frac{AB}{CE'} \cdot \frac{BF}{AC} = \frac{BE}{EE'} \cdot \frac{FF'}{CF}$, i.e. $\frac{AB}{AC} = \frac{FF'}{EE'}$ because $BF' = BC = CE'$ and $BE = CF$.

But from (1) $\frac{AB}{AC} < 1$ while $\frac{FF'}{EE'} > 1$, so $B \leq C$. Similarly $B \geq C$, and the theorem follows.

Our contributor adds: "In fact the analogous result for the external angle bisectors is not necessarily true. See Mathematics Magazine, *vol. 47, p. 52f. (January 1974). Here is the example by (the late) Prof. Charles Trigg.*

Consider triangle ABC in which $\angle BAC = 132°, \angle ABC = 36°, \angle ACB = 12°$*. Let AE, CD be the bisectors of the exterior angles at A and C, respectively, terminated by the sides BC, BA (extended) at E and D. Then AE = CD, but ABC is not isosceles."*

Editor's historical comments: Jakob Steiner (b. 1796 in Utzenstorf near Solothurn, d. 1863 in Bern) was the son of a Swiss farmer. "Among the mathematicians of Berlin University there have been several 'originals', among whom Jakob Steiner was the oddest", writes Kurt-R. Biermann in his history of Berlin University (Die Mathematik und ihre Dozenten an der Berliner Universität 1810-1933, *Berlin 1988). The German mathematician Daniel Christian Ludolf Lehmus (b. 1780 in Soest, d. 1863 in Berlin), taught at Berlin University in 1814/15; from 1826 onwards he was instructor at the Berlin Artillery and Engineering School.*

We hope to present more contributions on the Steiner-Lehmus Theorem in subsequent issues.

A solution to another of last year's problems has been provided by Dieter Bennewitz (Koblenz, Germany):

Problem 15.4.1 (from *Mathematical Spectrum*). A man has 3 sons. The age of the youngest times the sum of the ages of the other two is 1495; the age of the second son times the sum of the ages of the other two is 1767. How old are the sons?

Solution. Let $x < y < z$ be the respective (integer) ages of the sons. Then

$$x(y + z) = 1495$$

$$y(x + z) = 1767.$$

It follows that

$$z(y - x) = 272.$$

The number 272 can be written as a product of two positive integers in the following says:

$$\begin{aligned} 272 &= 272 \;.\; 1 \\ &= 136 \;.\; 2 \\ &= 68 \;.\; 4 \\ &= 34 \;.\; 8 \\ &= 17 \;.\; 16. \end{aligned}$$

Hence $z = 272, 136, 68, 34$ or 17.

If $z = 34$, then $y - x = 8$, so that $1495 = x(y + z) = x(x + 8 + 34)$, which is a quadratic equation; $x = 23$ is the only positive solution; we obtain immediately $y = 31$, $z = 34$. All the other cases do not result in integer solutions.

Problems

Problem 16.1.1 (Juan Bosco Romero Márquez, Valladolid, Spain). Solve the following equation for integers x and y with $y \geq x > 0$: $y - x = y^x - x^y$.

Garnett J. Greenbury selected the following Lewis Carroll puzzle for our readers young and old:

Problem 16.1.2. A bag contains one counter known to be either white or black. A white counter is added, the bag shaken and a counter drawn out which proves to be white. What is now the chance (probability) of drawing a white counter?

* * * * *

The 32nd International Mathematical Olympiad

The 1991 International Mathematical Olympiad (IMO) took place in July at Sigtuna, about halfway between the capital Stockholm and the old city of Uppsala. Teams of up to six students from 55 countries sat the contest. It consisted of two four-and-a-half-hour examinations held on subsequent days. Altogether 20 gold medals were awarded, 51 silver and 84 bronze. Each student, who missed out on a medal but obtained a perfect score for the solution to at least one problem, received an honourable mention. Nine students headed the list with perfect scores (42 marks, i.e. 7 marks for each of their solutions to *the six problems); four of them were from the former USSR, one being Yevgeniya Mallinikova, who won her third Gold. In the unofficial ranking of countries, the USSR came first (241 out of 252 possible points), second was the People's Republic of China (231), and Romania ended up third (225). The team from united Germany took fourth position (222). The list continues with: 5. USA (212), 6. Hungary (209), 7. Bulgaria (192), 8. Iran and Vietnam (191), 10. India (1987).*

Given an unprecedentedly strong competition, the Australian team did very well, indeed, in achieving 20th place (127), only one behind Austria and the United Kingdom (142 points each). Function *congratulates*

bronze medal winners: Anthony Henderson (year 10, 30 points), NSW, Sydney Grammar School;
Angelo di Pasquale (year 12, 25 points), Vic, Eltham College;
Joanna Masel (year 12, 22 points), Vic, Methodist Ladies' College;

and

winners of an honourable mention:

Luke Kameron (year 12, 18 points), NSW, Knox Grammar School;
Justin Sawon (year 12, 18 points), SA, Heathfield High School.

Here are the problems of the 1991 IMO:

FIRST DAY

1. Given a triangle ABC, let I be the centre of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI.BI.CI}{AA'.BB'.CC'} \leq \frac{8}{27} .$$

2. Let $n > 6$ be an integer and $a_1, a_2, ..., a_k$ be all the natural numbers less than n and relative prime to n. If

$$a_2 - a_1 = a_3 - a_2 = ... = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

3. Let $S = \{1, 2, 3, ..., 280\}$. Find the smallest integer n such that each n-element subset of S contains five numbers which are pairwise relatively prime.

SECOND DAY

4. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, 3, ..., k$ in such a way that at each vertex which belongs to two or more edges the greatest common divisor of the integers labelling those edges is equal to 1.

5. Let ABC be a triangle and P an interior point in ABC. Show that at least one of the angles $\angle PAB$, $\angle PBC$, $\angle PCA$ is less than or equal to $30°$.

6. An infinite sequence $x_0, x_1, x_2, ...$ of real numbers is said to be bounded if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.

Given any real number $a > 1$, construct a bounded infinite sequence $x_0, x_1, x_2, \ldots$ such that

$$|x_i - x_j|\,|i - j|^a \geq 1$$

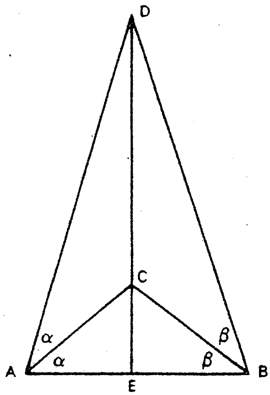
for every pair of distinct non-negative integers i, j.

* * * * *

SOFIA SOPHISM

According to the *Concise Oxford Dictionary*, a *sophism* is "a false argument intended to deceive". Here is a nice little sophism from Sofia, the capital of Bulgaria. It is a "proof" that every right-angled triangle is isosceles. Devised by Alexander Kyuchukov, it appeared in the Bulgarian magazine *Matematika, no.* 6 (1990). We got it from the South-African publication *Mathematical Digest No.* 83 (1991).

Starting with triangle ABC, we construct $\angle DAC = \alpha$ and $\angle DBC = \beta$ as shown. Join DC and extend it to meet AB in E.



Since the angle bisectors of a triangle are concurrent, DC bisets $\angle D$.

So $\angle ADC = \frac{1}{2}(180° - 2\alpha - 2\beta) = 90° - \alpha - \beta$.

Now $\angle ACE = \alpha + \angle ADC$ (exterior angles), and it follows that $\angle ACE = 90° - \beta$.

We now use the fact that $\angle C = 90°$, so that $90° - \beta = \alpha$. Therefore $\angle ACE = \alpha$, and $AE = EC$. A similar argument shows that $BE = EC$. So E is the midpoint of AB.

So the angle bisector DE is also a median, and it follows that triangle ADB is isosceles. It may now be shown that $\angle DAB = \angle DBA$, and hence $\alpha = \beta$. So triangle ABC is isosceles.

The epithet "sophisticated" is generally regarded today as complimentary, but it was not always so. Coming from the same root as "sophism" it used to mean deceitful or misleading. Can you find the sophisticated step in Kyuchukov's "proof"?

* * * * *

Registered for posting as a periodical – "Category B"
ISSN 0313 – 6825

* * * * *

Published by Monash University Mathematics Department