# Function

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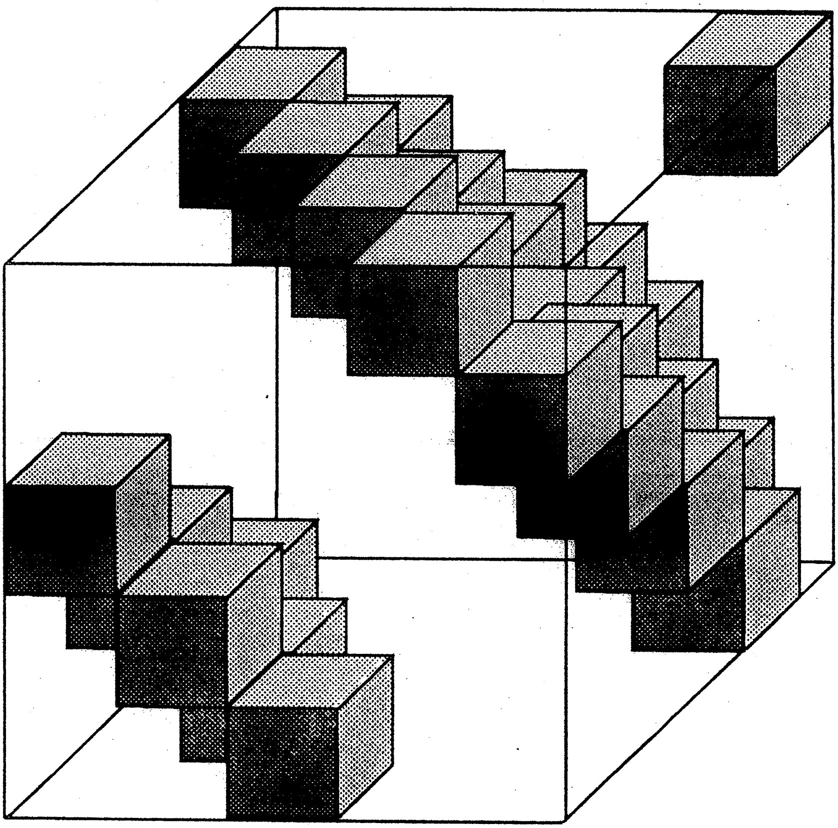

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**FUNCTION**

*Volume 14* *Part 5*

# The Front Cover

The diagram on the front cover is of a $5 \times 5 \times 5$ cube with 25 $1 \times 1 \times 1$ cubes placed inside it. The arrangement of the small cubes is such that if you view the larger cube from a direction perpendicular to any face, your view through the large cube is totally blocked off by the smaller cubes. This arrangement of small cubes is defined as follows. The large cube has its bottom left corner at (0,0,0) and diagonally opposite corner at (5,5,5). Smaller cubes have their top right corner at $(x,y,z)$ if the integers $x$, $y$, $z$ are such that 5 divides $x + y + z$. This 'diagonal' arrangement of cubes occurs in Mark Kisin's article.

# CONTENTS

# THE ABSENT-MINDED SECRETARY AND SIMILAR TALES

Marta Sved, University of Adelaide

The absent-minded secretary was handed ten signed letters. She duly typed ten envelopes, and then placed *each* letter into a wrong envelope. You would think that this was quite an achievement. However, there are exactly 1 334 961 ways to produce such a performance. Of course, you could argue correctly that as soon as you start counting permutations of some kind, you get to astronomical figures before you finish your counting. So look at such an event the probabilistic way. Imagine that our secretary places the letters into the envelopes with closed eyes. Then the probability that each letter goes into the *correct* envelope is

$$1/10! = 1/3{,}628{,}800 \simeq 2.75 \times 10^{-7},$$

whereas the probability that *each* letter goes into a wrong envelope is

$$1{,}334{,}961/3{,}628{,}800 \simeq 0.3678794$$

which is quite a considerable value as probabilities go. Moreover, this number is so close to $1/e$ that your hand-calculator cannot see the difference ($e \simeq 2{\cdot}71828$ is the base of the natural logarithms).

Hopefully, you are by now sufficiently interested in the mathematical analysis of the event. However, before going into it, we will discuss a general logical formula, known as the *Inclusion-Exclusion Principle*, stated in its general form.

We consider a set $S$ of $N$ objects, and a set of properties denoted by $\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Denote the number of objects in $S$ having property $\alpha_1$ (not excluding other properties), by $N_{\alpha_1}$, those having property $\alpha_2$ by $N_{\alpha_2}$, and so on. Similarly denote by $N_{\alpha_1\alpha_2}$ the number of objects in $S$ having both properties $\alpha_1$ and $\alpha_2$, by $N_{\alpha_1\alpha_2\alpha_3}$ the number having all three of $\alpha_1$, $\alpha_2$, $\alpha_3$, continuing in the same manner for all the combinations of the properties. Then the number of those objects in $S$ which have *none* of the listed properties is

$$N_0 = N - N_{\alpha_1} - N_{\alpha_2} - \ldots + N_{\alpha_1\alpha_2} + N_{\alpha_1\alpha_3} + \ldots - N_{\alpha_1\alpha_2\alpha_3} \ldots + (-1)^k N_{\alpha_1\alpha_2\ldots\alpha_k} \qquad (1)$$

As an illustration of this principle, consider the following problem.

*140 people are attending an international conference. Of these, 95 people speak English, 75 speak French, 80 speak German, 50 speak both English and French, 43 speak both English and German, 40 speak both French and German, and 20 speak all three languages. How many people attending the conference speak none of English, French and German?*

We let $\alpha_1$ denote "speaks English", $\alpha_2$ denote "speaks French", and $\alpha_3$ denote "speaks German". Then $N_{\alpha_1}$ denotes the number of people at the conference who speak English, so $N_{\alpha_1} = 95$. Similarly, we are given $N_{\alpha_2} = 75$, $N_{\alpha_3} = 80$, $N_{\alpha_1\alpha_2} = 50$, $N_{\alpha_1\alpha_3} = 43$, $N_{\alpha_2\alpha_3} = 40$, $N_{\alpha_1\alpha_2\alpha_3} = 20$, and $N = 140$. Hence by the formula, the number of people at the conference who speak none of English, French, German is

$$
\begin{aligned}
&N - N_{\alpha_1} - N_{\alpha_2} - N_{\alpha_3} + N_{\alpha_1\alpha_2} + N_{\alpha_1\alpha_3} + N_{\alpha_2\alpha_3} - N_{\alpha_1\alpha_2\alpha_3} \\
&= 140 - 95 - 75 - 80 + 50 + 43 + 40 - 20 \\
&= 3.
\end{aligned}
$$

We can illustrate this using a Venn diagram (Figure 1) in which the rectangular region $C$ represents the people attending the conference, and the circular regions $E$, $F$, $G$ represent the people who speak English, French and German respectively.

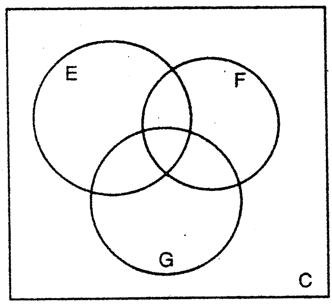



**Figure 1**

We can fill in the number of people in each of the regions as follows. Firstly $N_{a_1\alpha_1\alpha_3} = 20$, so we put 20 in $E \cap F \cap G$ as in Figure 2a. Because $N_{\alpha_1\alpha_2} = 50$, and we know 20 of these lie in $E \cap F \cap G$, we can put the number 30 in the region $(E \cap F)\backslash G$ of Figure 2b.

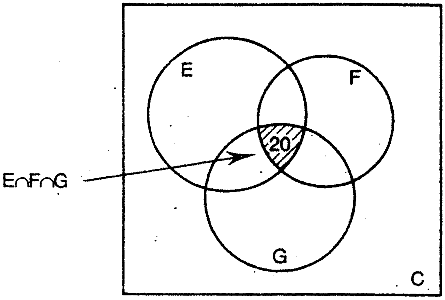



**Figure 2a**

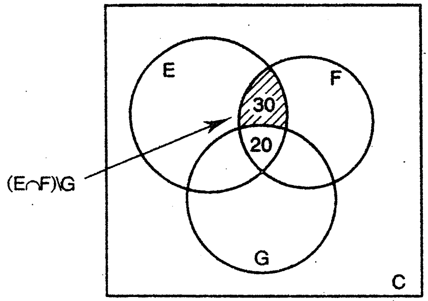



**Figure 2b**

Likewise we can put the numbers 43–20 = 23 in $(E \cap G)\backslash F$ and 40–20 = 20 in $(F \cap G)\backslash E$, as in Figure 2c. Finally, because $E$ contains a total of $N_{\alpha_1} = 95$ people, and we have already counted 30 + 20 + 23 = 73 in the intersection with $F$ and $G$, we can put the number 95 − 73 = 22 in $E\backslash(F \cup G))$, as in Figure 2d, to indicate that 22 people speak only English. Likewise we determine that

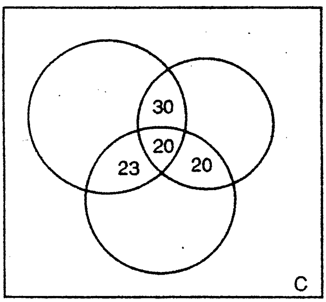



**Figure 2c**

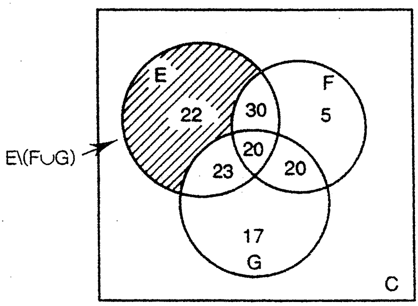


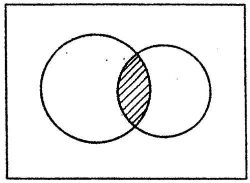

**Figure 2d**

5 speak only French and 17 speak only German. Adding up all the numbers in the circles we find 137 people, leaving only three of the 140 people not speaking at least one of English, French and German.

The above is just an illustration of the Inclusion-Exclusion Principle. Although we cannot prove it here, the following illustrates how a proof may be constructed.

A simple Venn diagram shows the validity of this principle when two properties $\alpha_1$ and $\alpha_2$ are considered.

The total number of elements in $S$ is represented by the area of the rectangle, while each of the two circles represents the numbers $N_{\alpha_1}$ and $N_{\alpha_2}$ with the shaded area representing $N_{\alpha_1\alpha_2}$. Thus the area not covered by the circles gives $N_0$. It is then clear that



$$N_0 = N - N_{\alpha_1} - N_{\alpha_2} + N_{\alpha_1\alpha_2}.$$

The general formula (1) for $k$ properties can then be derived from here by a mathematical technique known as induction.

Returning to our absent-minded secretary, we state now the general problem, known for some two hundred years as "le problème de rencontres": given the ordered sequence 1, 2 ... $n$, find all permutations where *no* element is at its assigned place. Such a permutation is called a *derangement*. For example, the permutation

3142 is a derangement of 1234.

Denote the number of derangements of 12 ... $n$ by $D_n$. We are going to use the Inclusion-Exclusion principle to prove that

$$D_n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \ldots + (-1)^n. \tag{2}$$

Here the set $S$ considered is the set of all permutations of 12 ... $n$, hence $N = n!$ If the permutation is such that the element $i$ is left at its place, then we say that the permutation has property $\alpha_i$. For example, the permutation

4213 has property $\alpha_2$,

as does the permutation 4231. In addition the latter permutation has property $\alpha_3$, hence it will also be counted in the number $N_{\alpha_2\alpha_3}$ and in the number $N_{\alpha_3}$ when (1) is aplied.

Now $N_{\alpha_1} = (n-1)!$, since there are $(n-1)!$ permutations where the element 1 is fixed. Thus

$$N_{\alpha_1} + N_{\alpha_2} + \ldots + N_{\alpha_n} = \binom{n}{1}(n-1)!.$$

Similarly $(n-2)!$ permutations have properties $\alpha_1$ and $\alpha_2$, and the same applies for any permutation where two elements are fixed, so

$$N_{\alpha_1\alpha_2} + N_{\alpha_1\alpha_3} + \ldots + N_{\alpha_1}N_{\alpha_n} + N_{\alpha_2}N_{\alpha_3} + \ldots + N_{\alpha_{n-1}\alpha_n} = \binom{n}{2}(n-2)!$$

We reason the same way for permutations when $r$ elements are fixed. In particular there is just one permutation, called the identity, having property $\alpha_1\alpha_2\ldots\alpha_n$. Thus the application of (1) yields equation (2) for $D_n$. In particular, $D_{10} = 1{,}334{,}961$, as stated in the beginning.

It is of some interest to write (2) in a slightly different form, by noting that

$$\binom{n}{r}(n-r)! = \frac{n!}{r!}.$$

Thus

$$D_n = n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!}\right).$$

The expression inside the bracket is an approximation of $1/e$.

There are many applications of the Inclusion-Exclusion principle. Combinatorial identities with alternating terms abound, and usually they can be interpreted by this principle. Here we want to touch briefly on two applications. One of them is an important theorem in number theory, the other an amusing problem, which however presented considerable difficulty, and occupied mathematicians of renown for quite a long period of time.

*The Euler totient-function* $\phi(n)$

Let $n$ be a positive integer divisible by the distinct prime numbers $p_1, p_2, \dots, p_r$. Then the number of positive integers less than $n$ and coprime† to $n$ is

$$\phi(n) = n - \frac{n}{p_1} - \frac{n}{p_2} - \dots - \frac{n}{p_r} + \frac{n}{p_1p_2} + \frac{n}{p_1p_3} + \dots + (-1)^r \frac{n}{p_1p_2\cdots p_r}$$

$$= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right).$$

The proof is not difficult and we leave it to you. We suggest that you show that the number of numbers less than 120 and coprime to it is

$$\phi(120) = 32.$$

*The "problème des ménages"*

In how many ways can you seat $n$ married couples around a (round) table so that the wives are seated in alternate seats and no husband sits next to his wife?

The problem appeared in the writings of Cayley (1878) and Lucas (1891), but it was not until 1934 that a closed formula for the solution was published by Touchard. In 1943 Kaplansky proved Touchard's formula by "elementary" combinatorial considerations, using the Inclusion-Exclusion principle. This solution appears now in more recent text-books of combinatorics, but it requires careful exposition, taking up room exceeding the scope of the present article. We satisfy ourselves here by giving the formula for $U_n$, which gives the number of possibilities of placing $n$ husbands, having first seated $n$ wives (which

† Two numbers $m, n$ are called coprime if they have no common divisor $d > 1$. For example, 12, 35 are coprime, while 12, 21 are not because 3 divides both 12 and 21.

can be done in $n!$ ways).

$$U_n = n! - \frac{2n}{2n-1}\binom{2n-1}{1}(n-1)! + \frac{2n}{2n-2}\binom{2n-2}{2}(n-2)! - \ldots + 2(-1)^n.$$

It follows that $U_3 = 1$, but of course, you do not need Touchard's formula for arranging a table for a dinner party for three couples.

With 12 people around your table, there are no less than 80 ways in which you can arrange the husbands after the wives have occupied their places and adhering to the rule of not putting married couples together.

* * * * *

# ROOKS AND MULTI-DIMENSIONAL CHESS BOARDS


## Mark Kisin, first-year student, Monash University


Consider the following problem.

> How many rooks must be placed on a chess-board if every square is to be threatened?*

It is not hard to see that for a normal chess-board the answer is 8, and that for an $n \times n$ chess-board the answer is $n$. Clearly $n$ rooks are sufficient for the job, as they may be placed on the diagonal (Figure 1).

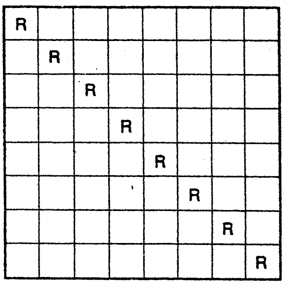



**Figure 1**

On the other hand, if only $n-1$ rooks are placed on the board then by the pigeonhole principle† there must be at least one row not containing a rook, and similarly there must be at least one column containing no rook. It follows that the square contained by this row and this column is not threatened by any rooks (Figure 2).

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* A square is threatened by a rook if it is in the same row or column as the rook.

† If $n-1$ letters are placed in a box containing $n$ pigeonholes, at least one hole has no letters.

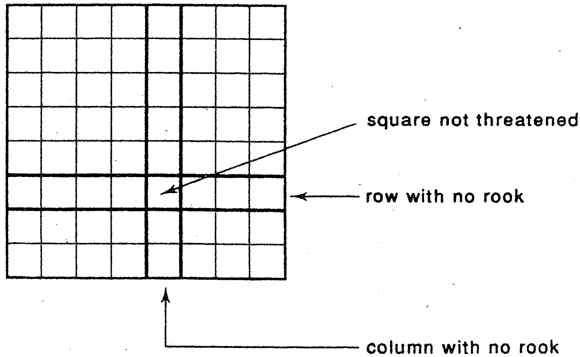



**Figure 2**

Our purpose here is to investigate the situation for the case of an $n \times n \times n$ "chess cube". In other words, to answer the following question.

> Given an $n \times n \times n$ cube, how many rooks must be placed so that every cell is threatened by at least one rook? (A rook may move parallel to the edges of the cube.)

The above problem was one of those proposed and short-listed for the 1988 International Mathematics Olympiad. It was then asked for the case $n = 8$ and was formulated as follows.

> A safe has three dials, each numbered 1 to 8. Due to a defect in the safe mechanism the safe will open if any two dials are in the correct position. How many combinations need to be tried if one is to be able to guarantee opening the safe?

That these two problems are the same may be seen by assigning 'coordinates' to the cells of the cube. Choose a corner as the origin and mark off $1, 2, \ldots, n$ along each edge as in Figure 3.

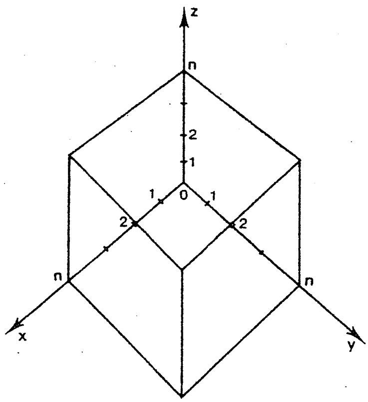



**Figure 3**

If we now label the edges $x, y, z$ then we can describe a cell as $(a,b,c)$ if it is in the position illustrated in Figure 4.

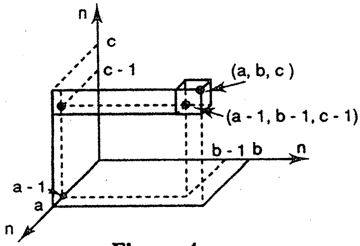



**Figure 4**

It is now clear that a rook's ability to move parallel to the sides of a cube means that a rook placed in cell $(d,e,f)$ threatens cell $(a,b,c)$ if and only if

i) $e = b, f = c$, so that moving parallel to the $x$-edge it changes $d$ into $a$;

ii) $d = a, f = c$, so that moving parallel to the $y$-edge it changes $e$ into $b$; or

iii) $d = a, e = b$ so that moving parallel to the $z$-edge it changes $f$ into $c$.

In other words, if and only if at least two of $d = a$, $e = b$, $f = c$ are true.

This is exactly the same condition for a safe with combination $(a,b,c)$ to be opened with the dials set to $(d,e,f)$. Thus the two problems are indeed the same, with the cells in which the rooks are placed corresponding to the combinations to be tried.

The coordinate notation developed above is very useful, and we shall use it to prove

**Theorem 1.** If $m$ rooks are placed in an $n \times n \times n$ cube such that every cell is threatened, then $m \geq \frac{1}{2}n^2$.

**Proof.** We suppose $m$ rooks are placed so that every cell is threatened. The cube may be considered as consisting of planes of cells. For example, we can let $X(i)$ denote the plane containing the cells $(i,y,z)$ for $1 \leq y \leq n$, $1 \leq z \leq n$ as illustrated in Figure 5a. Likewise we can let $Y(j)$ denote the plane containing the cells $(x,j,z)$ for $1 \leq x \leq n$, $1 \leq z \leq n$, as illustrated in Figure 5b. The plane $Z(k)$ is defined similarly.

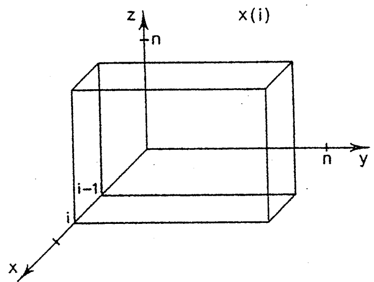



**Figure 5a**
**The cells forming** $X(i)$

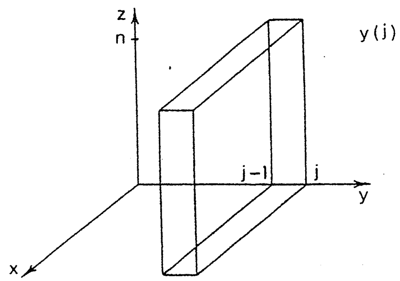



**Figure 5b**
**The cells forming** $Y(j)$

We consider three cases:

**Case i.** Every plane $X(i)$ contains at least $n$ rooks. In this case the $n$ planes $X(1),\ldots,X(n)$ must between them contain at least $n \times n = n^2$ rooks, so $m \geq n^2$ in this case.

**Case ii.** Every plane $Y(j)$ contains at least $n$ rooks. In this case the $n$ planes $Y(1),...,Y(n)$ must between them contain $n \times n = n^2$ rooks, so $m \geq n^2$ in this case.

This leaves the remaining possibility:

**Case iii.** There exists a plane $X(i)$ that contains less than $n$ rooks, and a plane $Y(j)$ that contains less than $n$ rooks. We choose a plane which contains the least number, say $q$ $(q < n)$ of rooks. For simplicity we shall suppose that the plane containing the least number of rooks is $X(a)$ – if it is a $Y$ plane the argument is similar, interchanging the rôles of the $X$ and $Y$ planes.

We regard the plane $X(a)$ as consisting of columns $C(a,j)$, $1 \leq j \leq n$ as illustrated in Figure 6.

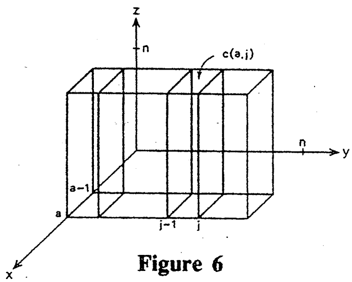



**Figure 6**

Since $X(a)$ is the union of the $C(a,j)$, $1 \leq j \leq n$ we know

$$\text{Rooks}(X(a)) = \sum_{j=1}^{n} \text{Rooks}(C(a,j))$$

where, for a set $S$, Rooks($S$) is the number of rooks in $S$. Since Rooks($X(a)$) = $q$, Rooks($C(a,j)$) $\geq 1$ for at most $q$ values of j. We can therefore select $n-q$ columns $C(a,j)$ which contain no rooks.

Consider one of these selected columns $C(a,j)$. Each of its $n$ cells can be threatened by a rook in either $X(a)$ or $Y(j)$. No rook in $X(a)$ or $Y(j)$ can threaten two cells of $C(a,j)$ – to do that it would have to lie in $C(a,j)$ which we have carefully selected not to contain a rook. As $X(a)$ contains only $q$ rooks which can therefore threaten only $q$ cells of $C(a,j)$ the remaining $n-q$ cells of $C(a,j)$ must be threatened by rooks in $Y(j)$. Thus $Y(j)$ contains at least $n-q$ rooks.

To summarize what we have so far in Case (iii):

a) We can select $n-q$ values of $j$ such that $C(a,j)$ contains no rooks. For each of these values of $j$ the plane $Y(j)$ contains at least $n-q$ rooks. Between them, therefore, these $n-q$ planes contain at least $(n-q) \times (n-q)$ rooks.

b) Every plane $X(i)$, $Y(j)$ contains at least $q$ rooks. In particular the planes $Y(j)$ for the $q$ values of $j$ not selected in (a) each contain at least $q$ rooks, so between them they contain at least $q \times q$ rooks.

Hence the total number of rooks lying in all the planes $Y(j)$, $1 \le j \le n$, must be the sum of those in a) and b), i.e. at least

$$\begin{aligned}(n-q)^2 + q^2 &= n^2 - 2nq + 2q^2 \\ &= \tfrac{1}{2}n^2 + \tfrac{1}{2}(n^2-4nq+4q^2) \\ &= \tfrac{1}{2}n^2 + \tfrac{1}{2}(n-2q)^2 \\ &\ge \tfrac{1}{2}n^2.\end{aligned}$$

Therefore $m \ge \frac{1}{2}n^2$ in this case.

Theorem 1 tells us that we shall need at least $\frac{1}{2}n^2$ rooks, but it does not guarantee that this number will be sufficient. Nor does it help us determine where we should put our rooks to threaten every cell most efficiently. While, in the case of a two-dimensional board, we could place our rooks on a diagonal, this approach fails in general for the three-dimensional case as a cube does not have a 'diagonal' with $\frac{1}{2}n^2$ cells.

*Editor's note. Mark Kisin's article continued analysing how to place the rooks and considered an n-dimensional cube. In its stead we give an argument by Peter Grossman, one of the editors of Function, based on Mark's approach, showing how to place* $\frac{1}{2}n^2$ *(rounded up if* $n$ *is odd) to threaten every cell of an* $n \times n \times n$ *cube.*

An $n \times n \times n$ cube can be split into two three-dimensional 'tee' pieces, as illustrated in Figure 7.

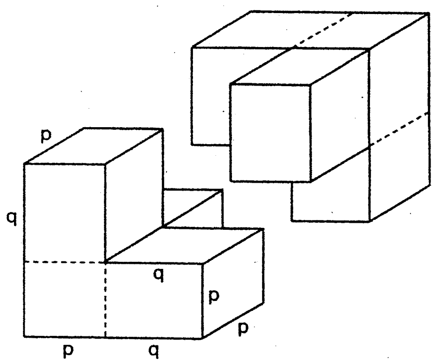



**Figure 7.**

At the corner of one tee is a $p \times p \times p$ cube and at the corner of the other tee is a $q \times q \times q$ cube, where $p + q = n$. We shall place $p^2$ rooks in the $p \times p \times p$ cube to threaten all cells in the first tee. Similarly $q^2$ rooks can be placed in the cube at the corner of the other tee to threaten all its cells.

Consider firstly the two-dimensional equivalent of the tee (Figure 8).

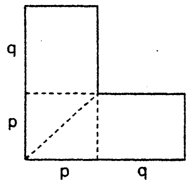



**Figure 8**

Clearly to threaten all cells, we just need to ensure we place $p$ rooks so that we have a rook in each row and in each column. This can be done by placing rooks on the diagonal.

For the three-dimensional case we wish to place $p^2$ rooks in the $p \times p \times p$ cube to have a rook in each column, where columns can be in three directions:

$X(i) \cap Y(j)$ – parallel to $z$-axis

$X(i) \cap Z(k)$ – parallel to $y$-axis

$Y(j) \cap Z(k)$ – parallel to $z$-axis.

To do this we generalize the 'diagonal line' of a square to a 'split diagonal' of a cube, as illustrated, for a cube of side $p = 5$, on the front cover. In Figure 9 we give the vertical slices of this cube, showing the blocked cells, starting from its front face.

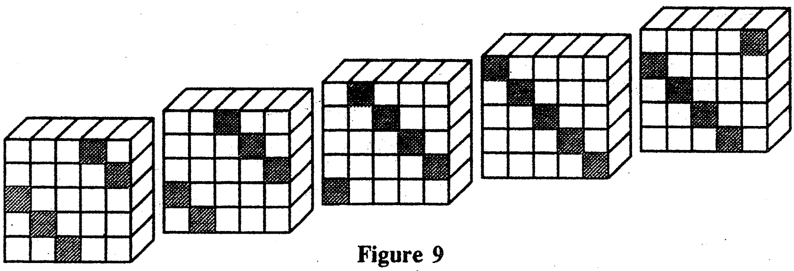


**Figure 9**

The split diagonal consists of all the cells $(x,y,z)$ that satisfy any one of the following three equations:

$$x + y + z = p$$

$$x + y + z = 2p$$

$$x + y + z = 3p.$$

Here $x, y, z$ are each $\geq 1$ and $\leq p$. We shall show that no matter from which axis direction you view the cube, your view along each column is obscured by one of the cells from this split diagonal.

We use this idea to place our $p^2$ rooks in the $p \times p \times p$ cube as follows.

We place a rook in cell $(i,j,k)$ if $i + j + k$ is divisible by $p$. This places precisely $p^2$ rooks, since for any $i$, $j$, $1 \le i$, $j \le p$ there is exactly one $k$, $1 \le k \le p$ for which $i + j + k$ is a multiple of $p$.

Based on this idea we now have

**Theorem:** If $n = p + q$ then $p^2 + q^2$ rooks can be placed in an $n \times n \times n$ cube so that every cell is threatened.

**Proof:** We consider the cube as being split into two three-dimensional tee pieces as described above. One tee has corner the $p \times p \times p$ cube $(i,j,k)$ with $1 \le i \le p$, $1 \le j \le p$, $1 \le k \le p$. We put a rook at each cell of the split diagonal of this cube, i.e. we put a rook in cell $(i,j,k)$ if $1 \le i \le p$, $1 \le j \le p$, $1 \le k \le p$ and $p$ divides $i + j + k$. This places $p^2$ rooks. The other tee has corner the $q \times q \times q$ cube $(i,j,k)$ with $1 \le i-p \le q$, $1 \le j-p \le q$, $1 \le k-p \le q$. We put a rook at each cell of the split diagonal of this cube, i.e., we put a rook in cell $(i,j,k)$ if $1 \le i-p \le q$, $1 \le j-p \le q$, $1 \le k-p \le q$ and $q$ divides $(i-p) + (j-p) + (k-p)$. This places $q^2$ rooks.

We can now check that every cell of the $n \times n \times n$ cube is threatened. For let $(i,j,k)$ be any cell. Suppose $i \le p$ and $j \le p$. Let $r$ be the remainder on dividing $i + j$ by $p$. Then $1 \le p-r \le p$ and $p$ divides $i + j + (p-r)$, so we place a rook at $(i,j,p-r)$ and this threatens cell $(i,j,k)$. A similar argument applies if $i \le p$ and $k \le p$ or if $j \le p$ and $k \le p$. Now suppose $i > p$ and $j > p$. Let $r$ be the remainder on dividing $(i-p) + (j-p)$ by $q$. Then $1 \le i-p \le q$, $1 \le j-p \le q$, $1 \le q-r \le q$ and $q$ divides $(i-p) + (j-p) + (q-r)$, so we place a rook at $(i,j,p-r)$ and this threatens cell $(i,j,k)$. A similar argument applies if $i > p$ and $k > p$ or if $j > p$ and $k > p$. Since either two of $i,j,k$ must be no bigger than $p$, or two must be bigger than $p$, we have shown that every cell $(i,j,k)$ is threatened.

**Corollary:** If $n$ is even, $\frac{n^2}{2}$ rooks can be placed so that every cell is attacked. If $n$ is odd, $\frac{n^2+1}{2}$ rooks can be so placed.

**Proof:** In the first case, put $p = q = \frac{n}{2}$. In the second case, put $p = \frac{n+1}{2}$ and $q = \frac{n-1}{2}$.

* * * * *

# All At Sea - Solutions

In *Function, Vol.* 14, *Part* 3, some problems were posed concerning ships at sea, and solutions were promised. A concise solution to the problems was received from J.C. Barton.

The first problem was:

**The Ship in the Fog**

You are at sea and need to deliver much-needed medical supplies to a cruise-ship. You spot the ship and determine its exact position, both distance and angle. However, a dense fog then comes down; in fact a fog so dense that visibility is for all practical purposes zero. You know that the cruise-ship always travels at the same speed $v$ in a straight line (if we ignore the curvature of the earth), but you know nothing about the direction of that line. Your own speed is $v(1+\varepsilon)$, where $\varepsilon > 0$. Describe a path which will ensure that you meet the cruise-ship sometime.

**Solution.** Let, as in Figure 1, the cruise-ship be sighted at position $O$, a distance $a$ from the supply ship $S$. We observe firstly that to meet the cruise-ship your ship must be at the same place at the same time, and secondly that at time $t$ after being sighted the cruise-ship is at a point $Q$ a distance $vt$ from $O$ on a straight line through $O$.

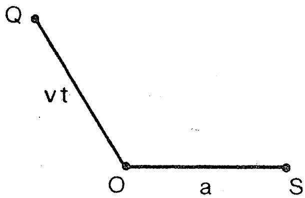



**Figure 1**

Our aim is therefore to describe a path for your ship which has the property that, for every line through $O$, there is a time $t$ when your ship is a distance $vt$ from $O$.

Notice that if the cruise-ship is heading directly towards $S$ it will reach $S$ at time $a/v$, so we can let your ship stay at $S$ until time $a/v$.

Now consider any time $t \geq a/v$.

In Figure 2, your ship is at position $P$ at time $t$ and at position $P'$ at time $t + \delta t$.

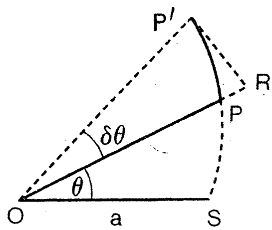



**Figure 2**

We want $P$ and $P'$ to be at the same distance from $O$ as the cruise-ship, so

$$OP = vt,$$

$$OP' = v(t+\delta t) = OR$$

and so $PR = v\delta t$. Because your ship travels at speed $v(1+\varepsilon)$ we have $PP' = v(1+\varepsilon)\delta t$. The triangle $PRP'$ is, to a good approximation, right-angled and so

$$|P'R|^2 = |PP'|^2 - |PR|^2$$

$$P'R = v(\sqrt{(1+\varepsilon)^2-1})\delta t.$$

In other words,

$$\delta\theta = \frac{P'R}{OP} = \frac{v(\sqrt{(1+\varepsilon)^2-1})\delta t}{vt},$$

that is,

$$\frac{\delta\theta}{\delta t} = \frac{(\sqrt{(1+\varepsilon)^2-1})}{t}.$$

In the limit as $\delta t \to 0$ we deduce

$$\frac{d\theta}{dt} = \frac{(\sqrt{(1+\varepsilon)^2-1})}{t}$$

and so, on integrating,

$$\theta = (\sqrt{(1+\varepsilon)^2-1})\log_e t + c.$$

Assuming we measure angles from $OS$, and recalling that $\theta = 0$ at $t = a/v$ (because we let your ship stay at $S$ until time $t = a/v$) we have

$$0 = (\sqrt{(1+\varepsilon)^2-1})\log_e a/v + c$$

$$\text{i.e. } c = -(\sqrt{(1+\varepsilon)^2-1})\log_e a/v.$$

Substituting this we find

$$\theta = (\sqrt{(1+\varepsilon)^2-1})\log_e \frac{tv}{a}. \qquad (1)$$

This, being of the form $\theta = C \log bv$, is a curve known as a logarithmic spiral. The solution is therefore for the supply ship to stay at $S$ until time $a/v$ and then travel in the logarithmic spiral around $O$ given by (1). If $\theta_1$ is the angle of the line $OQ$ along which the cruise-ship is actually sailing, then (1) can be solved to determine the time $t_1$ your ship will meet the cruise-ship. In fact

$$t_1 = \frac{a}{v} e^{\theta_1/\sqrt{(1+\varepsilon)^2-1}}.$$

The solution is illustrated in Figure 3.

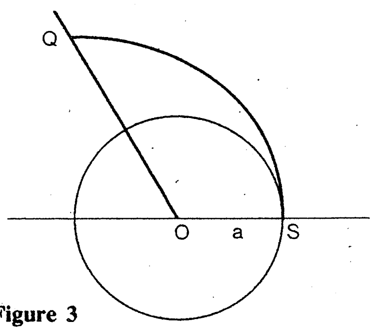



**Figure 3**

You wait at $S$ for time $a/v$ then travel along the spiral
$\theta = (\sqrt{(1+\varepsilon)^2-1})\log_e \frac{tv}{a}$.

The next problem was:

**The Abandoned Cruise-ship**

An abandoned cruise-ship whose wheel is locked in position is going round and round in a circle, of which you are at the centre. You know the ship has speed v. Again a fog comes down. This time you know the direction of the ship (from you) and which way it was travelling, but not its distance. Can you determine a path which will enable you to rendezvous with the ship if:

(a) you can choose your speed,
(b) your speed is predetermined?

[Note: If you can solve (b), you can solve (a), but not necessarily *vice versa*.]

Now suppose you don't even know which way round the ship is travelling. Can you still find a path that enables you to rendezvous with the ship? This time assume that you have a top speed, and can travel at any speed less than that.

**Solution.** We let, as in the first problem, the cruise-ship be sighted at position $O$, at distance $a$ from your ship $S$.

As in the previous problem, we must arrange for your ship and the cruise-ship to be in the same place at the same time.

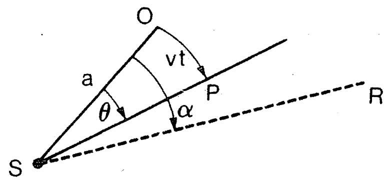



**Figure 4**

At time $t$ the cruise-ship has moved a distance $vt$ from $O$ to $P$, so the angle $\theta$ in Figure 4 is given by

$$\theta = \frac{vt}{a}.$$

If your ship sets out at speed $w$ on the straight line $SR$ making an angle $\alpha$ with $SO$, then at time $t$ it is a distance $wt$ from $S$ on this line. The two ships will therefore be at the same place at the same time if

$$\alpha = \theta \quad \text{and} \quad a = wt,$$

i.e.

$$\alpha = \frac{vt}{a} \quad \text{and} \quad a = wt,$$

so that

$$\alpha = \frac{v}{w}.$$

In other words, for a given speed $w$, the supply ship travels at an angle $v/w$ to the direction $OA$ the cruise-ship was initially seen at.

If the direction (clockwise or anti-clockwise) is not known, then it is not known whether to measure the angle $\alpha$ above in the clockwise or anti-clockwise direction. However, if $\alpha = \pi, 2\pi, 3\pi, \ldots$ it doesn't matter in which direction we measure $\alpha$. We therefore want to choose $w$ so that $v/w = \pi k$ for an integer $k$, i.e. $w = v/\pi k$ for an integer $k$. If our top speed is $w_0$, we choose $k_0$ to be an integer not less than $v/\pi w_0$ and travel at speed $v/\pi k_0$ in a direction making angle $\pi k_0$ with the direction in which the cruise-ship was sighted.

* * * * *

# LETTER TO THE EDITOR

## The Number Pattern

I refer to the pattern which appeared on page 128 of *Function*, *Vol.* 14, *Part* 4:

$$4^2 = 16$$

$$34^2 = 1156$$

$$334^2 = 111556$$

$$\vdots \qquad \vdots$$

To prove the pattern continues indefinitely, it is required to prove that

$$\{3(10^n+10^{n-1}+\ldots+10) + 4\}^2 = 10^{2n+1} + 10^{2n} + \ldots + 10^{n+1} + 5(10^n+10^{n-1}+\ldots+10) + 6$$

$$= 10^{n+1}(10^n+10^{n-1}+\ldots+1) + 5(10^n+10^{n-1}+\ldots+10) + 6$$

for $n = 1,2,3,\ldots$ .

Let

$$K = 10^n + 10^{n-1} + \ldots + 10 = \frac{10^{n+1}-10}{10-1} = \frac{10^{n+1}-10}{9},$$

so that $10^{n+1} = 9K + 10$. It then remains to confirm that

$$(3K+4)^2 = (9K+10)(K+1) + 5K + 6.$$

Similarly constructed patterns arise from numbers such as 65, 66,..., 89 and 35, 36, 37, 38 (among others).

David Shaw,
Newtown, Geelong

# PROBLEMS SECTION

EDITOR: H. LAUSCH

New solutions to *Function* problems continue to reach the editorial office. Many thanks to our contributors from Clayton (Victoria), Duncraig (Western Australia) and Edmonton (Alberta, Canada).

## Solutions

**Problem 12.4.4** Let $k$ be a positive integer and let

$$A = \left\{ 2^i \mid i = 0,1,2,\ldots \right\}.$$

Find all positive integers $n$ such that

numbers $a_1,\ldots,a_k$ $(a_i \neq a_j$ for $i \neq j)$ from $A$,

for which the sum $|n-a_1| + \ldots + |n-a_k|$ is minimal, can be chosen in more than one way.

**Solution** (by Mark Kisin, first-year student at Monash University). Suppose that $2^{m-1} \leq n < 2^m$, that $|n-a_1| \leq |n-a_2| \leq \ldots \leq |n-a_k|$ and that $\sum_{i=1}^{k} |n-a_i|$ is minimal. Now in choosing the sequence $a_1,\ldots,a_k$ ambiguity arises only if $|n-2^i| = |n-2^j|$ $(i \neq j)$ occurs. Without loss of generality, we may assume that $2^i < 2^j$, then $2^i \leq 2^{m-1} \leq n < 2^m \leq 2^j$. Now

$$|n-2^i| = n - 2^i \leq 2^m - 1 - 1 = 2^m - 2.$$

It follows that $|n-2^j| = 2^j - n \leq 2^m - 2$. So,

$$2^j \leq 2^m + n - 2 \leq 2^m + 2^m - 1 - 2 = 2^{m+1} - 3.$$

Thus $j = m$. (Note also that this holds if $|n-2^i| = |n-2^j|$ is replaced by $|n-2^i| \geq |n-2^j|$.) Consequently, if $a_t = 2^q$, then $q = m + 1$ and we have $\{a_1,a_2,\ldots,a_{t-1}\} = \{1,2,\ldots,2^{m-1},2^m\}$. The ambiguity created by $|n-2^i| = |n-2^j|$ will not cause ambiguity in the selection of the numbers $a_i$ unless $m - i = k$, i.e. if and only if

$$a_1 = 2^{m-1},\ a_2 = 2^{m-2},\ldots,\ a_{k-1} = 2^{i+1} = 2^{m-k+1}.$$

Thus $i = m - k$. Hence the desired integers $n$ are of the form

$$\frac{1}{2}(2^m+2^{m-k}) = 2^{m-1} + 2^{m-k-1}, \ (m = k + 1, k + 2, \ldots).$$

Clearly these numbers do yield two choices of the numbers $a_i$ :

$$\{2^{m-1}. \ 2^{m-2}, \ldots, 2^{m-k}\} \ \text{and} \ \{2^{m-1}. \ 2^{m-2}, \ldots, 2^{m-k+2}, \ 2^{m-k+1}, \ 2^m\}.$$

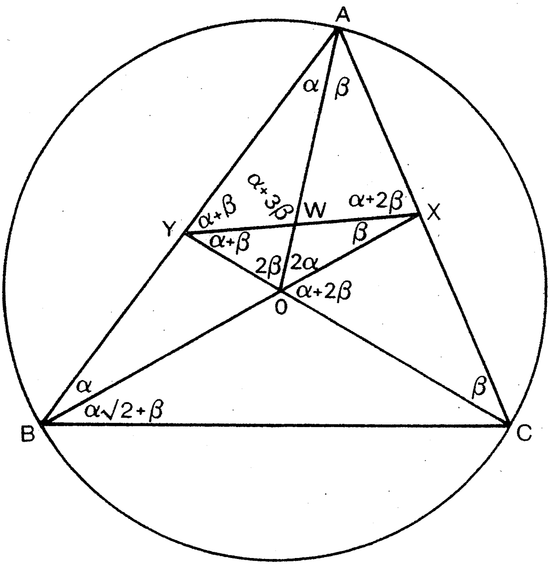
The above also establishes that there are precisely 2 solutions for these values of $n$, for $i$ and $j$ are determined by $m$ (and $k$), and it is impossible that $|n-2^i| = |n-2^j| = |n-2^k|$ when $i,j,k$ are distinct.

*The next problem seems to become our "Problem of the Year"; it is attracting undiminished attention from Function readers:*

**Question 4, Function, Vol. 13, Part 3 (June 1989), p.96.** Let $O$ be the circumcentre of the triangle $ABC$, and let $X$ and $Y$ be the points on $AC$ and $AB$ respectively such that $BX$ intersects $CY$ in $O$. Suppose $\angle BAC = \angle AYX = \angle XYC$; determine the size of this angle.

Here are two more solutions:

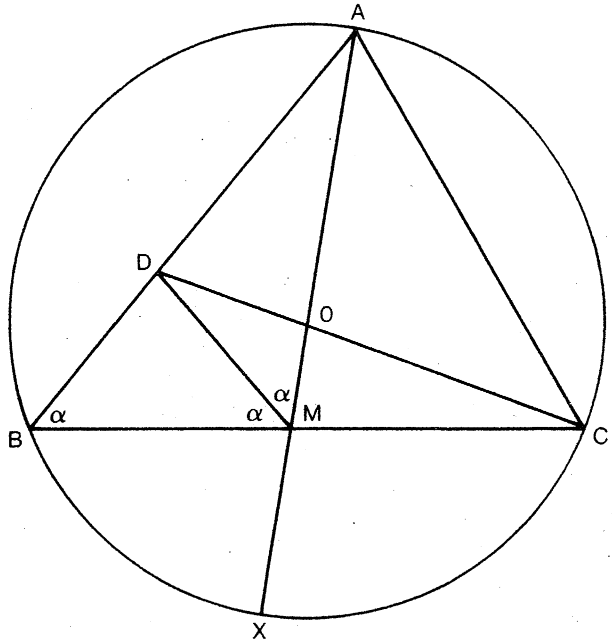
**Solution** (by Andy Liu, University of Alberta, Edmonton, Alberta, Canada).



Join $AO$, cutting $XY$ at $W$. Let $\angle BAO = \alpha$ and $\angle CAO = \beta$. The angles marked in the diagram are easily computed, and we have $3\alpha + 4\beta = 180°$. Now $\Delta OXYE \cong \Delta YCA$. Hence $OY.AC = AY.XY$. Also $\Delta AYW \cong \Delta BAX$. Hence $AB.WY = AY.AX + AY.XY$. It follows that $OY.AC = AB.WY$, so that $\Delta ABC \cong \Delta YOW$. From $2\beta = \angle WOY = \angle ABC = \frac{3\alpha}{2} + \beta$, we have $3\alpha = 2\beta$. It follows easily that $\alpha = 20°$ and $\beta = 30°$, so that the desired angle is $50°$.

*The second solution has been sent to Function by Kin Yan Chung, Duncraig, Western Australia, member of the Australian team at the 29th International Mathematical Olympiad, Canberra, 1988. As Kin Yan Chung labelled the points that occur in the problem with different letters, we rephrase the problem accordingly:*

Let $O$ be the circumcentre of the acute-angled triangle $ABC$. Let $M$ be the point where $BC$ meets $AO$ produced, and let $D$ be the point where $AB$ meets $CO$ produced. Suppose that $\angle AMD = \angle DMB = \angle CBA$. Find the size of this common angle.




**Solution.** Without loss of generality, we may suppose the circumradius is 1. We shall use directed angles. Let $\alpha = \angle AMD = \angle DMB = \angle CBA$. Extend $AO$ to meet the circumcircle at $X$. Now, $\angle COA = 2\alpha = \angle AMB$, so $\angle CMO = \angle MOC = \pi - 2\alpha$, and hence $\angle OCM = 4\alpha - \pi$. Since $\Delta ABC$ is acute angled, $O$ lies inside $\Delta ABC$, so we require $\angle OCM \geq 0$, i.e.

$$\alpha < \frac{\pi}{3}. \qquad (1)$$

Further, $\angle DOM = \angle COA = 2\alpha$, so $\angle MDO = \pi - 3\alpha$ and since $O$ lies inside $\Delta ABC$, we require $\angle MDO > 0$, i.e.

$$\alpha < \frac{\pi}{3}. \qquad (2)$$

Applying the sine rule to $\Delta OCM$ gives

$$OM = OC\,\frac{\sin(4\alpha-\pi)}{\sin(\pi-2\alpha)} = \frac{-\sin 4\alpha}{\sin 2\alpha} = -2\cos 2\alpha.$$

By a well-known result, $BM.MC = XM.MA$. $\Delta OMC$ is isosceles, since $\angle CMO = \angle MOC = \pi - 2\alpha$., so $MC = OC = 1$. Hence $BM = BM.MC = AM.MX = (1+OM)(1-OM) = 1 - 4\cos^2 2\alpha$. Applying the sine rule to $\Delta BMD$ and

$$DM = BM\,\frac{\sin\alpha}{\sin(\pi-2\alpha)} \quad \text{and} \quad DM = OM\,\frac{\sin 2\alpha}{\sin(\pi-3\alpha)},$$

whence $(1 - 4\cos^2 2\alpha)\,\frac{\sin\alpha}{\sin(\pi-2\alpha)} = -2\cos 2\alpha\,\frac{\sin 2\alpha}{\sin(\pi-3\alpha)}$, or

$(1 - 4\cos^2 2\alpha)\frac{\sin\alpha}{\sin 2\alpha} = -2\cos 2\alpha\,\frac{\sin 2\alpha}{\sin(\pi-3\alpha)}$. We transform this equation step-by-step into equivalent equations:

$(1 - 4\cos^2 2\alpha).\sin\alpha.\sin 3\alpha = -2\cos 2\alpha.\sin^2 2\alpha$ $(\sin 2\alpha \neq 0,\ \sin 3\alpha \neq 0$ by (1), (2)).

$(1 - 4\cos^2 2\alpha).\frac{1}{2}[\cos 2\alpha - \cos 4\alpha] = -2\cos 2\alpha.(1\ \cos^2 2\alpha)$

$(1 - 4\cos^2 2\alpha)(\cos 2\alpha - 2\cos^2 2\alpha + 1) = -4\cos 2\alpha.(1 - \cos^2 2\alpha)$

$-(1 - 4\cos^2 2\alpha)(\cos 2\alpha - 1)(2\cos 2\alpha + 1) = 4\cos 2\alpha.(\cos 2\alpha + 1)(\cos 2\alpha - 1)$

$-((1 - 4\cos^2 2\alpha)(2\cos 2\alpha + 1) = 4\cos 2\alpha.(\cos 2\alpha + 1)$ [since $\cos 2\alpha \neq 1$ by (1) and (2)]

$$8\cos^3 2\alpha - 6\cos 2\alpha = 1$$

$$4\cos^3 2\alpha - 3\cos 2\alpha = \frac{1}{2}$$

$$\cos 6\alpha = \frac{1}{2}.$$

Hence $6\alpha = 2k\pi \pm \frac{\pi}{3}$, $k \in Z$, or $a = \frac{\pi}{3}k \pm \frac{\pi}{18}$, $k \in Z$. The only $\alpha$ that satisfies both (1) and (2) is $\frac{\pi}{3} - \frac{\pi}{18} = \frac{5\pi}{18} = 50^\circ$. It is a simple matter to now check that $50^\circ$ does result in the given figure.

# Problems

## a. Remember?

*In the last Function issue, you were asked to keep the following problem ready for this issue:*

**Problem 14.4.11** It is known that all natural numbers can be written in the binary system, using only 0 and 1 as digits. But if $(-2)$ is used as a *base* instead of 2, can *all* integers (negative or positive) be expressed as a sum of different powers of $(-2)$?

[Examples: $73 = (-2)^6 = (-2)^4 + (-2)^3 + (-2)^0$;
$-55 = (-2)^7 + (-2)^6 + (-2)^4 + (-2)^3 + (-2)^0$.]

*The problem continues:*

**Problem 14.5.1** (a) What can you say about an integer in this system if its number of digits is even?

(b) If a number can be represented as a sum of different powers of (–2), is the representation unique?

## b. An edible problem

*Function editor P.A. Grossman communicated this most appetizing algorithmic problem, known as the Pancake Problem. Get your spatulas ready!*

**Problem 14.5.2** Given a stack of pancakes of varying diameters, rearrange them into a stack with decreasing diameter (as you move up the stack) using only "spatula flips". With a spatula flip you insert the spatula and invert (i.e., turn upside down) the (sub)stack of pancakes above the spatula. Design an algorithm that correctly solves the pancake problem for a stack of $n$ pancakes with at most $2n$ flips. Count exactly how many flips your algorithm uses in the worst case.

## c. Historical problems from Cambridge, England

*After last issue's medieval problem we turn to the more recent past. Here are three Cambridge Senate-House Problems. The first one was posed in 1860, the other two were given in 1878:*

**Problem 14.5.3** Solve the simultaneous equations in the unknowns $x, y, z$ :

$$x^2 - yz = a^2$$

$$y^2 - zx = b^2$$

$$z^2 - xy = c^2.$$

**Problem 14.5.4** $ABC$ is a triangle, $O$ its incentre. Show that $AO$ passes through the circumcentre of $BOC$.

**Problem 14.5.5** Between three towns, $A$, $B$, $C$ there is a continual migration of families, so that the number of families in each town is unaltered, while the whole number of families migrating at any specified time is always even. Show that, if by the end of any time an even number of families left $A$ for $B$, then by the end of the same time the number of families that have left $B$ for $A$ is also even.

*The next problem is of the same provenance (1878), only that here it has been modified to make it a trifle harder:*

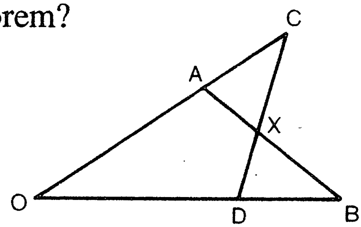
**Problem 14.5.6** Let $x, y, z$ be the three non-zero real numbers which are distinct from each other. If $x + y + z = 0$, what is the product

$$\left\{ \frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z} \right\} \cdot \left\{ \frac{x}{y-z} + \frac{y}{z-x} + \frac{z}{x-y} \right\} ?$$

* * * * *

# HISTORY OF MATHEMATICS SECTION

## A Tasmanian Theorem?




The diagram at right is the emblem of the Mathematical Association of Tasmania and it illustrates a little-known theorem of Euclidean geometry. Stated briefly, the theorem says that:

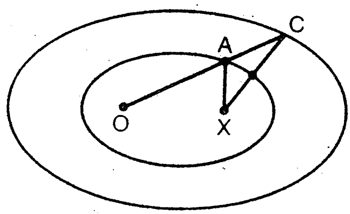
*If* $OA + AX = OD + DX$, *then* $OC + CX = OB + BX$.

In Australia, this result is widely known as "Urquhart's Theorem" after the late Mac Urquhart who taught at the University of Melbourne from 1932 to 1944 and at the University of Tasmania from 1947 to 1966. During this latter period, he founded and became the first president of the Mathematical Association of Tasmania.

Urquhart did indeed discover the theorem for himself (in 1964), though, as we shall see, he was not the first to do so. He succeeded in proving it, but did not publish his proof. (Indeed, Urquhart never published any of his mathematical results.) He did, however, tell the proof to various people including Dr Fred Syer, formerly of the University of Melbourne. He, in his turn, told the proof to John Barton, who sent it to *Function*, and we published it in *Volume* 2, *Part* 3.

John Barton is a regular correspondent to *Function*. On this occasion, he wrote us a quite long article of which we published only a part – that part I have just summarised and the actual proof of Urquhart's Theorem. We left out a lot, with regret, I might say, but it would have been rather heavy going for the average reader of *Function*.

However, I will summarise it here omitting the details.




The statement $OA + AX = OD + DX$ means that $A$ and $D$ both lie on an ellipse with foci at $O$, $X$. Thus if we put pins at $O, X$ and connect them by means of a string of length greater than $OX$, we can draw an ellipse by stretching this string taut by means of a pencil whose possible path is the ellipse shown in the diagram at right. If we used a slightly longer string we would get another ellipse with the same foci $O, X$. The statement $OC + CX = OB + BX$ means that $C, B$ must lie on this other "confocal" ellipse.

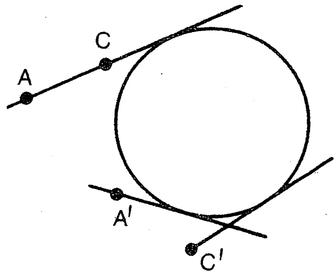
What John Barton did was to cast Urquhart's theorem into the form of a statement about confocal ellipses, a statement he then proceeded to prove by means of trigonometry and coordinate geometry. He speculated that the theorem might actually be already known in this form and a number of us looked quite hard in the literature of confocal ellipses but with only marginal success.

In order to understand the extent of what we found, we need some more background.

One of the mathematicians who learned of Urquhart's result was Basil Rennie, who was for many years Professor of Mathematics at James Cook University in Townsville. He re-organised Urquhart's original proof and greatly improved its presentation. In this form it was first published, some ten years before the *Function* article, as an appendix to an article by George Szekeres in the *Journal of the Australian Mathematical Society.*

Rennie's version of the proof begins with a lemma (i.e. a preliminary result) that says:

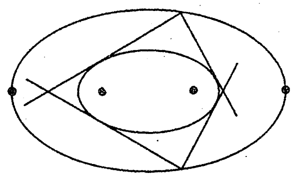
*Take a circle and two points $A$, $C$ outside it; then draw tangents from $A$, $C$ to the circle; then if the sum or difference of the lengths of these tangents equals the length $AC$, $A$, $C$ lie on a common tangent to the circle.*




See the diagram: $A$, $C$ satisfy this condition; $A'$, $C'$ do not.

Rennie next showed that, under the hypothesis of Urquhart's theorem, all four lines $OC$, $CD$, $AB$, $OB$ must be tangent to a common circle. From this he readily deduced the final result.

In looking through the literature, the best we could find was a theorem due to the French geometer Chasles (1860). This said that if we took an ellipse outside a smaller confocal ellipse and drew tangents to the inner from two points on the outer (as shown), then the four tangents all touched a common circle.

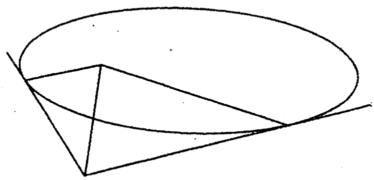


Now suppose the inner ellipse to shrink until in the limit it becomes the line-segment connecting the two foci. We then have Rennie's second result (that $OC$, $CD$, $AB$, $OB$ all touched a common circle) as a consequence of Chasles's more general result. This was the best we could find at that time (1981) and it seemed to say that, while Urquhart's theorem could be seen as a ready consequence of Chasles's result, nonetheless the priority of statement and full proof still lay with Urquhart.

However, I later learned that this is not correct. I was looking through the 1841 volume of the *Cambridge Mathematical Journal* (for a completely different purpose) when my eye fell on a two-page note by one ADM.

It was the fashion for authors in this journal at that time to publish their work under rather cryptic pseudonyms. ADM, however, was Augustus De Morgan, a leading mathematician of the day, now perhaps better remembered for his friendship with and encouragement of George Boole.

What De Morgan wanted to show was that if, from a point outside an ellipse, one draws two tangents to that ellipse, the segments cut off between the point itself and the points of tangency subtend equal angles at either focus. See the diagram at right.



This was in fact a known result when De Morgan wrote. What he claimed to have was a new proof and the proof he gave used Urquhart's Theorem as a lemma. He proved this in the course of proving his main result. Thus Urquhart's Theorem should really be called De Morgan's Theorem, unless of course an even earlier account turns up.

Finally, notice one interesting point: Urquhart's proof was published by Barton, Rennie's by Szekeres, the Chasles version by me; Barton's proof seems never to have been published and De Morgan's appeared under a semi-pseudonym. It must be very rare indeed for mathematicians to be so modest about their accomplishments.

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# COMPUTER SECTION

EDITOR: R.T. WORLEY

## Computing and the Fibonacci Numbers

The sequence of Fibonacci numbers 1,1,2,3,5,..., which is defined by the rule

$$F_1 = 1, F_2 = 1$$

$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 3$$

occurs in a number of places in computer algorithms. They occur, for example, in a sorting algorithm and in a numerical integration algorithm. We shall look at these, and then discuss how $F_n$ may be computed efficiently.

A sorting algorithm may be based on the idea of merging increasing "runs". For example, if we are sorting integers, we can take two increasing runs 3, 7, 11 and 4, 6, 8, 12, 15 and merge them to produce the single increasing run 3, 4, 6, 7, 8, 12, 15. Based on this idea we can sort a list of integers as follows.

1. Split the list into two pieces, and regard each piece as consisting of a number of increasing runs. For example, split the list 1 3 2 5 15 4 16 8 13 7 9 25 10 into two lists

   1 3 2 5 15 4 16          8 13 7 9 25 10

which we regard as consisting of increasing runs which are underlined

1 3 2 5 15 4 16          8 13 7 9 25 10.

2. Merge pairs of increasing runs from the lists. In our example this would produce

1 3 8 13  2 5 7 9 15 25  4 10 16

We repeat the splitting and merging ideas until the list is sorted. This idea is used when we have very large lists to be sorted, so the lists are held in, for example, disk files. A file is normally available for reading or for writing, so our above idea requires two readable files, containing the runs to be merged, and a writable file, where the merged runs will be placed.

In one clever way of implementing this idea, the list is initially split unevenly, with $N_1$ runs in the first list (stored in file **A**) and $N_2$ $(N_2 < N_1)$ in the second list, stored in file **B**. During the merging step we merge pairs of runs, so we can merge the first $N_1$ runs from file **A** with the $N_2$ runs of file **B**, putting the resulting $N_2$ merged runs on file **C**. We have 0 runs left in file **B** and $N_1-N_2$ runs left in file **A**. The following table illustrates this:

| | file **A** runs | file **B** runs | file **C** runs | operation |
|---|---|---|---|---|
| before | $N_1$ | $N_2$ | 0 | $N_2$ runs merged **A**,**B** → **C** |
| after | $N_1-N_2$ | 0 | $N_2$ | |

We can now merge runs from file **A** and file **C**, putting the result in file **B**, and so on.

We have the following two example tables giving what happens with different values of $N_1$, $N_2$.

| **A** | **B** | **C** | operation to be performed | |
|---|---|---|---|---|
| 13 | 8 | 0 | merge 8 runs from **A**,**B** → **C** | |
| 5 | 0 | 8 | merge 5 runs from **A**,**C** → **B** | |
| 0 | 5 | 3 | merge 3 runs from **B**,**C** → **A** | |
| 3 | 2 | 0 | merge 2 runs from **A**,**B** → **C** | |
| 1 | 0 | 2 | merge 1 run from **A**,**C** → **B** | * |
| 0 | 1 | 1 | merge 1 run from **B**,**C** → **A** | |
| 1 | 0 | 0 | | |

**Example 1.**

| A | B | C | operation to be performed |
|---|---|---|---|
| 12 | 9 | 0 | merge 9 runs from A,B → C |
| 3 | 0 | 9 | merge 3 runs from A,C → B |
| 0 | 3 | 6 | merge 3 runs from B,C → A |
| 3 | 0 | 3 | merge 3 runs from A,C → B |
| 0 | 3 | 0 | ?? |

**Example 2.**

In our second example we have reached a situation where we need to split again. We split the 3 runs on **B**, writing the first run to file **A** and the next two runs to file **C**, at which time we have reached a situation identical to that marked with an asterisk in the first example, and we continue as from there.

The question that arises is:

What is the best way of initially splitting the runs into file A and file **B**?

It turns out that the best way is, for some $n$, to put $F_n$ runs in file **A** and $F_{n-1}$ in file **B**. (It may be necessary, if $N$ is not of the form $F_n + F_{n-1}$, to consider there to be some empty runs added after the $N$ runs.) Thus the Fibonacci numbers arise in this sorting algorithm.

We now look at how the Fibonacci numbers arise in numerical integration. The definite integral

$$\int_a^b f(x)dx$$

can be estimated by an average

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

taken over $N$ points $x_1,\ldots,x_N$, in the interval $(a,b)$. Likewise an integral

$$\int_A f(x,y)dx\,dy$$

over a region $A$ of the plane can be estimated by an average

$$\frac{1}{N}\sum_{i=1}^{N} f(x_i,y_i)$$

taken over $N$ points $(x_i,y_i)$, $i = 1,\ldots,N$ in the region A. Of course, the average is just an estimate of the integral, and depends on the points $(x_i,y_i)$ chosen. There is one way of choosing the points $(x_i,y_i)$, known as a "lattice point rule" which can be used for the case where the region A is the square $0 \le x \le 1$, $0 \le y \le 1$. The lattice point rule $R_{b,N}$, for an integer $b$, involves points $(x_i,y_i)$, $1 \le i \le N$, where $x_i = i/N$ and

$y_i = \frac{k_i}{N}$ where $k_i$ is the remainder on division of $ib$ by $N$. Rather surprisingly, the best lattice point rules are those where $N, b$ are Fibonacci numbers $F_n, F_{n-1}$ for some $n$.

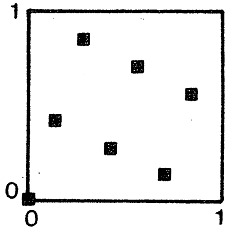


**Figure 1.**

**The points used by $R_{3,7}$**

Finally we consider calculation of the Fibonacci numbers by computer. Suppose we wish to write a function *Fib(n)* which given a parameter $n$ calculates the Fibonacci number $F_n$. Perhaps the simplest way, in Pascal†, is to use the definition

```
function Fib1(n:integer):integer
begin
    if  n < 1  then begin
        writeln("Error:Fib(n) can be calculated
               only for  n > 1");               | A
        Fib1 := -1                              |
        end                                     |
    else if  n <= 2  then  Fib1 := n            | B
    else  Fib1 = Fib1(n-1) + Fib1(n-2)          | C
end;
```

It is interesting to observe how long this takes to calculate the Fibonacci numbers. The following are the times taken on a PC.

| $n$ | 0–21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 |
|---|---|---|---|---|---|---|---|---|---|
| time for $F_n$ | 0 | 1 | 1 | 3 | 4 | 6 | 9 | 16 | 25 |

**Table 1. Time on a PC to compute $F_n$ using Fib1, to nearest second**

† This uses a method known as recursion where a function calls itself. This is normally not possible in BASIC.

The times seem quite long as $n$ gets larger, and seem to be the sum of the previous two times. One would think that the time taken to calculate $F_{n+1}$ should be longer than the time taken to calculate $F_n$ simply by the time it takes to add $F_n$ to $F_{n-1}$, but this does not seem to be the case.

A close look at the function above will explain why. If we ignore the possibility that $n < 1$, then either statement **B** is executed or statement **C**. Suppose we let $T_n$ denote the time this takes to calculate $F_n$. Then if $n \le 2$ we have $T_2 = 1$ (supposing executing one statement takes one unit of time). If, on the other hand, $n > 2$ then we execute statement C – this involves calculating both Fib1($n$–1) (which takes time $T_{n-1}$) and Fib1($n$–2) (which takes time $T_{n-2}$). We therefore have the "equations"*

$$T_1 = T_2 = 1$$

$$T_n = T_{n-1} + T_{n-2} \qquad n \ge 3$$

for the time. These are identical with the equations for the Fibonacci numbers, so calculating $F_n$ by Fib1($n$) will take about $F_n$ units of time.

Intuitively this seems to be wrong. The major cause of the problem turns out to be that in line C, not only does it calculate Fib1($n$–1) (which requires Fib1($n$–2)), but it then actually recalculates Fib($n$–2) to add it to Fib1($n$–1).

Consider the following alternative method, which instead of calculating just $F_n$ calculates both $F_n$ and $F_{n-1}$. It returns the value $F_n$, and in its second parameter fnminus1, it returns $F_{n-1}$.

```
function  Fib2(n : integer, var fnminus1 : integer):integer
begin
      var temp1, temp2 : integer
            if  n < 1  then begin
                  writeln("Fib2(n) should only be called if  n ≥ 1");
                  fnminus1 := –1;
                  Fib2 := –1
                  end
            else if  n = 1  then begin
                  Fib2 := 1;
                  fnminus1 := 0
                  end
```

---

* One could perhaps argue that $T_n = T_{n-1} + T_{n-1} + 1$, but we are making quite an approximation in assuming one statement takes one unit of time that the difference is not significant.

```
else begin
      temp1 := Fib2(n–1,temp2);
      Fib2 := temp1 + temp2;
      fnminus1 := temp1
      end
end;
```

In this case, if $S_n$ denotes the time taken to calculate Fib2($n$), we have the "equations"

$$S_1 = 2$$

$$S_n = S_{n-1} + 2 \quad \text{for } n > 1$$

which imply $S_2 = S_1 + 2 = 4$, $S_3 = S_2 + 2 = 6$, ... and clearly $S_n = 2n$. The times taken on a PC, given in Table 2, verify this.

| $n$ | 0–29 |
|---|---|
| time for $F_n$ | 0 |

**Table 2. Time on a PC to calculate $F_n$ using Fib2; to nearest second.**

* * * * *

# Necessity for proof; emergence of mathematics

"Between the workable empiricism of the early land measurers who parceled out the fields of ancient Egypt and the geometry of the Greeks in the sixth century before Christ there is a great chasm. On the remoter side lies what preceded mathematics, on the nearer, mathematics; and the chasm is bridged by deductive reasoning applied consciously and deliberately to the practical inductions of daily life. Without the strictest deductive proof from admitted assumptions, explicitly stated as such, mathematics does not exist. This does not deny that intuition, experiment, induction, and plain guessing are important elements in mathematical invention. It merely states the criterion by which the final product of all the guessing, by whatever name it be dignified, is judged to be or not to be mathematics."

E.T. Bell,
*The Development of Mathematics*

# AN IDEA

K.R.S. Sastry, Addis Ababa, Ethiopia

When the need arises, such as preparation of test questions, then one can quickly construct the equation of a line and of a circle or equations of two circles so that they intersect at lattice points – the points having both coordinates integers. Here is an illustration:

(I) Write down the equation of a circle in the following form:

$$(x-3)^2 + (y-1)^2 = 4^2 \text{ (say).}$$

(II) By inspection find two lattice points on (I). This is easily done. For example, $|x-3| = 4$, $y - 1 = 0$ or $x - 3 = 0$, $|y-1| = 4$. For definiteness, let the points chosen be $(-1,1)$ and $(3,5)$.

(III) Use the chosen points (II) to determine the equation of the line $y = x + 2$ that contains them.

Then $y = x + 2$ and $(x-3)^2 + (y-1)^2 = 4^2$, that is,
$y = x + 2$, $x^2 + y^2 - 6x - 2y - 6 = 0$, with common points $(-1,1)$ and $(3,5)$. (5)

Using (5) and a translation of the plane one can construct "different" problems. For instance, $x \to x-3$, $y \to y-4$ transforms (5) into

$$y = x + 3,\ x^2 + y^2 - 12x - 10y + 45 = 0;\ (2,5),\ (6,9).$$

In general, if we begin with the circle equation $(x-p)^2 + (y-q)^2 = r^2$ the above procedure leads to the line equation $y = \pm x + b$. If a less simple line equation is desired then one can begin with the circle equation as

$$(x-p)^2 + (y-q)^2 = r^2 + s^2.$$

For example, with the circle equation $(x-1)^2 + (y+2)^2 = 13$ and the lattice points given by $x - 1 = -3$, $y + 2 = 2$; $x - 1 = 2$, $y + 2 = 3$, that is $(-2,0)$ and $(3,1)$, we obtain the line equation $x - 5y + 2 = 0$. (5′)

We can use (5) or (5′) to obtain the equation of a family of circles with the same intersection points simply by using the line as the common line of intersection of that circle family. For example, the circles

$$x^2 + y^2 - 6x - 2y - 6 = 0$$

and

$$x^2 + y^2 - 6x - 2y - 6 + \lambda(x-y+2) = 0$$

intersect at $(-1,1)$ and $(3,5)$ for any value of $\lambda$.

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