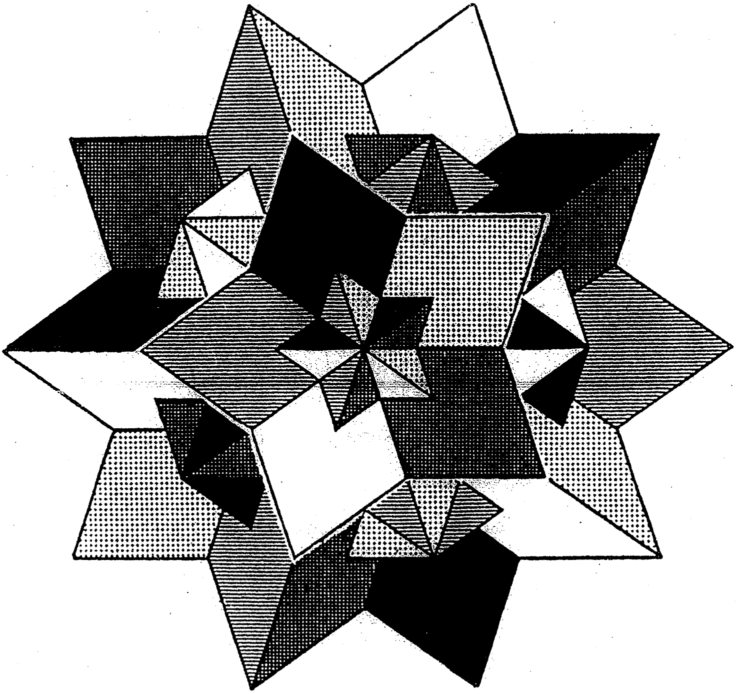
Function

Volume 13 Part 2

April 1989



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FUNCTION is a mathematics magazine addressed principally to students in the upper forms of schools.

It is a 'special interest' journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. FUNCTION is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

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The front cover of this issue depicts what is called a *great stellated triacontahedron*, and is copied from an illustration in H.M. Cundy and A.P. Rollett's book *Mathematical Models*, O.U.P., second edition, 1961. Cundy and Rollett's book tells you, in a detailed and practical way, how to make for yourself models, in cardboard or perspex, sometimes using rods and string, a magnificent variety of mathematical models. They describe the methods they evolved, both to build these models themselves, and to instruct their students how to build them. Read their book and build a great stellated *triacontahedron*!

CONTENTS

Points inside quadrilaterals	R.T. Worley	34
Do you know how to hunt and catch divisors?	Class 2A4 of the Sacré-Coeur Institute of Profondeville	37
Perfect numbers	Translated from *Math-Jeunes*	42
Jesuit and fellow of the royal society: a best fit?	Hans Lausch	44
Solving old problems in new ways – surprising connections between apparently different things	Michael A.B. Deakin	52
Pythagorean triples and complex numbers	Marta Sved	55
Alternative methods of multiplication	Garnet J. Greenbury	60
Approximations to Π		61
Perdix		62
Problems		64

POINTS INSIDE QUADRILATERALS

R.T. Worley

"How can I tell if a point *P* lies inside a quadrilateral *ABCD*?" This was not a problem I had given much thought to until recently, when I was surprised to be asked this question twice within two weeks. My first reaction was that it would be simplest to draw a rough diagram first, but then I learned that my questioner had a large number of points and quadrilaterals stored on a computer, and wanted a procedure that could be programmed.

The first thing to be done when answering such a question is to clarify exactly what is required. Even "the quadrilateral *ABCD*" may not have an obvious meaning. Usually this means the quadrilateral with the four sides being *AB*, *BC*, *CD*, *DA*, as in Fig. 1, but if the points were arranged as in Fig. 2a the quadrilateral my questioner meant may not be the one in Fig. 2b but the one in Fig. 2c which would normally be described as *ABCD*. It turned out that the points *ABCD* were not in a specific order, and the solution would need to order the points in a suitable manner.

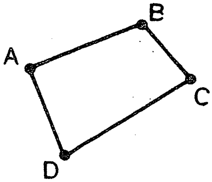


Fig. 1

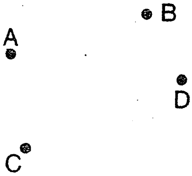


Fig. 2a

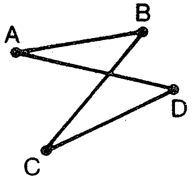


Fig. 2b

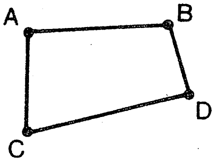


Fig. 2c

When there is no order in the points *ABCD*, it may happen that there is no sensible choice for the quadrilateral. This happens when the points are such that one lies inside (or on the edges of) the triangle formed by the other three points. For example, if the points *ABCD* are as in Fig. 3a then the intended quadrilateral could be any of *ABCD*, *ABDC*, *ADBC*, illustrated in Figs. 3b, 3c, 3d. In this case the quadrilateral is really not properly defined, and so the question of whether or not a point lies inside it cannot be answered.

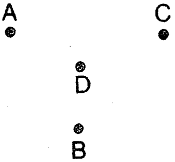


Fig. 3a

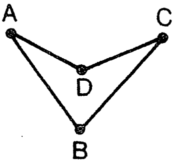


Fig. 3b

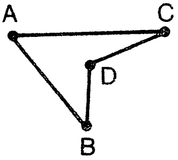


Fig. 3c

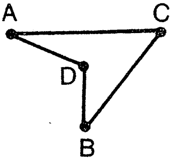


Fig. 3d

Thus the problem to be solved is really two problems:

(i) Given four points A, B, C, D determine if one lies inside or on the edge of the triangle formed by the other three. If not, relabel the points so that they are labelled A, B, C, D in clockwise order around the quadrilateral.

(ii) Given a fifth point P, determine if P lies inside the clockwise labelled quadrilateral $ABCD$.

Now that the problem is clearly defined, it must be solved. My original solution involved using a simple test which tells if two points are on the same or opposite sides of the line joining two points. However, on closer investigation it could be rewritten in a more intuitive way based on the formula for the (signed) area of a triangle. Given three points A, B, and C with co-ordinates (x_1,y_1), (x_2,y_2), and (x_3,y_3), respectively, then the expression

$$\mathcal{T} = \mathcal{T}(A,B,C) = (y_1-y_2)(x_1-x_3) - (y_1-y_3)(x_1-x_2)$$

equals twice the signed area of the triangle ABC. It is negative if the triangle is labelled ABC in an anticlockwise direction, and positive if the triangle is labelled clockwise.

There are several ways of obtaining the above formula for $\mathcal{T}$, perhaps the simplest being to take the vector product of vectors representing two of the sides of the triangle, giving a vector whose magnitude is twice the area of the triangle.

Alternatively, one can enclose the triangle in a rectangle $AGHK$, as shown in Fig. 4, with BD and FE parallel to sides of the rectangle. We easily see that twice the area of ABC equals the area of $EGHF$. The area of $EGHF$ is easily calculated in terms of the co-ordinates of A, B, and C.

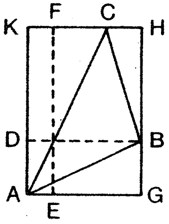


Fig. 4

Using this formula for $\mathcal{T}$ the solution of both parts of the problem is straightforward.

(i) To order the points A, B, C, D to produce a clockwise quadrilateral, one can proceed as follows. Calculate $\mathcal{T}(A,B,C)$ and, if it is negative, exchange the labels B, C so the triangle ABC is labelled clockwise. The point D can lie in any of the seven regions labelled 1,2,...,7 in Fig. 5.

Exactly which region is determined by calculating $\mathcal{T}(A,B,D)$, $\mathcal{T}(B,C,D)$ and $\mathcal{T}(C,A,D)$, and examining Table 1. In the middle columns a "+" denotes that the triangle is clockwise labelled and the signed area should be positive, and a "–" denotes that the signed area should be negative.

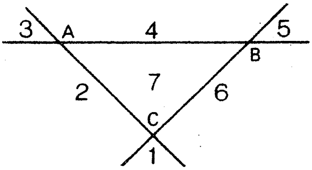


Fig. 5

Region	$\mathcal{T}(A,B,D)$	$\mathcal{T}(B,C,D)$	$\mathcal{T}(C,A,D)$	Comments
1	+	–	–	not a valid quadrilateral
2	+	+	–	clockwise labelling *ABCD*
3	–	+	–	not a valid quadrilateral
4	–	+	+	clockwise labelling *ADBC*
5	–	–	+	not a valid quadrilateral
6	+	–	+	clockwise labelling *ABDC*
7	+	+	+	not a valid quadrilateral

Table 1

Table 1 indicates that valid quadrilaterals occur when exactly one of the three $\mathcal{T}$ values is negative, and gives the relabelling required.

(ii) Having relabelled the points to give a clockwise labelled quadrilateral, it is clear that P lies inside the quadrilateral if and only if all four of $\mathcal{T}(A,B,P)$, $\mathcal{T}(B,C,P)$, $\mathcal{T}(C,D,P)$ and $\mathcal{T}(D,A,P)$ are positive.

Thus the problem can be solved easily using a computer. It requires calculating eight different $\mathcal{T}$ values – four to examine the quadrilateral and then four to decide if P lies inside it.

* * * * *

> A single curve, drawn in the manner of the curve of prices of cotton, describes all that the ear can possibly hear as a result of the most complicated musical performance ... That to my mind is a wonderful proof of the potency of mathematics.
>
> Lord Kelvin

[†]DO YOU KNOW HOW TO HUNT AND CATCH DIVISORS?

Class 2A4 of the
Sacré-Coeur Institute of Profondeville

1. *The number of divisors of a non-zero natural number*

Our teacher asked us to find all the divisors of 90.

Some of us set to work courageously by trying to divide 90 successively by 1, 2, 3, ...

"Each time the division worked, we noted the divisor and the quotient as both being divisors of 90. We stopped as soon as we reached a whole number quotient less than the divisor being used. We then had the following list:

1, 90, 2, 45, 3, 30, 5, 18, 6, 15, 9, 10."

Others found it more interesting to factorize 90 into a product of its prime factors:

90	2
45	3
15	3
5	5
1	

$$90 = 2 \times 3 \times 3 \times 5$$

"We then constructed all divisors by choosing all possible combinations of factors from this factorization of 90."

Working in this way, there are fewer divisions to carry out ... but the risk of omitting some combination of factors, and so missing a factor, is greater!

Could one know in advance how many divisors one should find? If so, this risk would be lessened.

Let us try and find a formula.

[†] A translation of the article "Savez-vour chasser et capturer les diviseurs?", from the magazine *Math-Jeunes*, No. 43, Spring*, 1989. *Math-Jeunes* is published by the Belgian Society of Teachers of Mathematics.

[*Northern hemisphere Spring]

We knew that, except for the order of the factors, there is only one decomposition of a number into a product of prime factors.

Let us use the device of taking powers for collecting equal factors, when for the number 90 we obtain the expression

$$90 = 2^1 \times 3^2 \times 5^1.$$

The construction of a divisor of 90 then reduces itself to calculating a product of the form

$$\circ . \Delta . \square$$

where

$\circ$ represents a divisor of 2^1
Δ represents a divisor of 3^2
$\square$ represents a divisor of 5^1.

Note that the divisors of

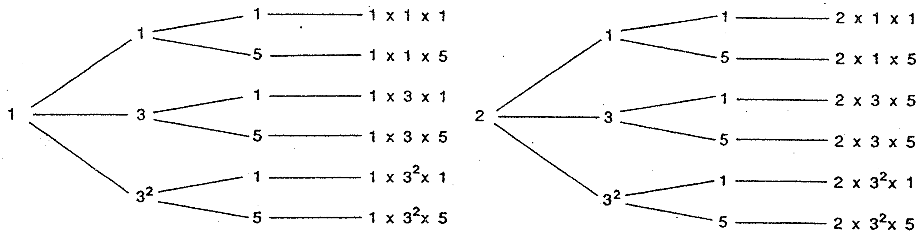
2^1 are 1 and 2 ; there are 2 divisors;
3^2 are 1, 3 and 3^2; there are 3 divisors;
5^1 are 1 and 5 ; there are 2 divisors.

The number of divisors is, in each case, equal to the exponent plus one; this is normal: if a is a prime, then the divisors of

a^α are $a^0 = 1,\ a^1, a^2, \ldots, a^{\alpha-1}, a^\alpha$,
$\alpha+1$ in number.

So as not to miss any divisors, let us construct a tree which will give us all possible choices for products

$$\circ . \Delta . \square$$



Note that the choice of a divisor of 2^1 is followed by the choice of a divisor of 3^2 and then by the choice of a divisor of 5^1.

In proceeding thus, we are certain not to miss any of the divisors, and especially, we can affirm that 90 has 12 divisors:

$$(1 + 1) \times (2 + 1) \times (1 + 1) = 12.$$

Let us generalize the situation.

Let us decompose a non-zero natural number n into a product of distinct prime numbers:

$$n = a^{\alpha} \times b^{\beta} \times c^{\gamma} \times \ldots;$$

then we find the number x of its divisors from the formula

$$\boxed{x = (\alpha+1) \times (\beta+1) \times (\gamma+1) \times \ldots}$$

We can therefore, without calculating the divisors, give the number of divisors of n; this would remove any worries we might have about whether we have found them all.

2. *The product of the divisors of a non-zero natural number*

One of us had the idea of multiplying all the divisors of 90.

Grouping the divisors in pairs, she obtained 90^6. As 6 is half of 12, which is the number of divisors, we immediately deduced the following rule:

$$P = n^{x/2}$$

where

P denotes the product of the divisors of the nonzero natural number n

and

x denotes the number of divisors of n.

Our teacher then suggested that we try this on the number 36. Applying our formula, we get

$$P = 36^{9/2} \quad \ldots \quad \text{bizarre:}$$

we do not know how to calculate this number, since the exponent is not a natural number.

Let us take again the product of the divisors of 36 grouping them together in pairs, as we did for 90:

$$P = (1\times36) \times (2\times18) \times (3\times12) \times (4\times9) \times 6$$

$$= 36^4 \times 6;$$

the cause of the trouble is 6 which remains unpaired because 36 is a perfect square.

Let us try to write everything in terms of 36. We have $6 = \sqrt{36}$, and hence

$$P = 36^4 \times \sqrt{36} = \sqrt{36^8 \times 36} = \sqrt{36^9}.$$

These calculations lead us to the formula

$$\boxed{P = \sqrt{n^x}}$$

Does this formula always work? We have become suspicious, so let us try and prove it.

Let n be a non-zero natural number which has x divisors. Let us denote these by $d_1, d_2, d_3, \ldots, d_{x-1}, d_x$, listed in order of increasing size, so that

$$1 = d_1 < d_2 < d_3 < \ldots < d_{x-1} < d_x = n;$$

carrying out the divisions we obtain

$$\begin{aligned} n &= d_1 \times q_1 \\ n &= d_2 \times q_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ n &= d_x \times q_x \end{aligned} \tag{1}$$

where the quotients $q_1, q_2, \ldots, q_x$ are also the divisors of n, but this time listed in order of decreasing size. We have

$$d_1 \times d_2 \times \ldots \times d_x = q_1 \times q_2 \times \ldots \times q_x. \tag{2}$$

Multiplying together the equations (1) we get

$$n^x = (d_1 \times d_2 \times \ldots \times d_x) \times (q_1 \times q_2 \times \ldots \times q_x),$$

or, in virtue of equation (2)

$$n^x = (d_1 \times d_2 \times \ldots d_x)^2 = P^2$$

and hence

$$P = \sqrt{n^x}.$$

Notice that, if x is even, which always is the case when n is not a perfect square, then one has

$$P = n^{x/2}.$$

Why not generalize and say that this equation always holds? We thus define new powers of n. Our teacher explained that we would meet this notation in our later studies.

3. *The sum of the divisors of a non-zero natural number*

And if one calculates the sum of the divisors of 90 ...

$$1 + 90 + 2 + 45 + 3 + 30 + 5 + 18 + 6 + 15 + 9 + 10 = 234.$$

Let us try once again to find a formula.

Let us begin by looking at what happens when the number is a power of a prime number.

We wish to find the sum of the divisors of a number that can be written as

$$a^{\alpha} \quad (a \text{ a prime number}, \ \alpha \text{ a natural number})$$

Its divisors are

$$1, a, a^2, a^3, \dots, a^{\alpha-1}, a^{\alpha}.$$

Notice that one passes from one divisor to the next by multiplying by a. Let us denote the sum we are seeking by S, so that

$$S = 1 + a + a^2 + a^3 + \dots + a^{\alpha-1} + a^{\alpha}$$

and

$$a \times S = a + a^2 + a^3 + \dots + a^{\alpha-1} + a^{\alpha} + a^{\alpha+1}.$$

Subtracting the first equation from the second we find that

$$(a-1)S = a^{\alpha+1} - 1,$$

from which

$$S = \frac{a^{\alpha+1}-1}{a-1}. \qquad (3)$$

Let us now turn to the case of a number that decomposes into the product of powers of two distinct primes:

$$n = a^{\alpha} \times b^{\beta}.$$

We know that to find the divisors of n we must take the product of each divisor of a^{α} with each divisor of b^{β}. This gives us the following table of divisors

$$\begin{array}{ccccc}
1 & b & b^2 & \dots & b^{\beta} \\
a & ab & ab^2 & \dots & ab^{\beta} \\
a^2 & a^2b & a^2b^2 & \dots & a^2b^{\beta} \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & & \cdot \\
a^{\alpha} & a^{\alpha}b & a^{\alpha}b^2 & \dots & a^{\alpha}b^{\beta}.
\end{array}$$

Adding the factors line by line, and simultaneously taking out the common powers of a, we have

$$\begin{aligned} S = {} & a^0(1 + b + b^2 + \ldots + b^\beta) \\ & + a(1 + b + b^2 + \ldots + b^\beta) \\ & + a^2(1 + b + b^2 + \ldots + b^\beta) \\ & \quad \vdots \\ & + a^\alpha(1 + b + b^2 + \ldots + b^\beta). \end{aligned}$$

Now taking out the common factor $(1 + b + b^2 + \ldots + b^\beta)$ we get

$$S = (1 + a + a^2 + \ldots + a^\alpha)(1 + b + b^2 + \ldots + b^\beta),$$

so that taking account of formula (3) we get

$$\boxed{S = \frac{a^{\alpha+1}-1}{a-1} \times \frac{b^{\beta+1}-1}{b-1}}$$

The generalization to the case of more than two prime factors is immediate.

Good hunting!

* * * * *

Confusion to Mathematics!

> "And then he [Charles Lamb] and Keats agreed he [Newton] had destroyed all the poetry of the rainbow by reducing it to prismatic colours. It was impossible to resist him, and we all drank 'Newton's health and confusion to mathematics'."

Benjamin Robert Haydon: the above is taken from an account, in the painter Haydon's *Autobiography*, pp. 267-71, of a dinner party, arranged by Haydon, in 1817, so that Keats could meet Wordsworth. Charles Lamb was one of the guests.

* * * * *

PERFECT NUMBERS[†]

One says that a natural number is *perfect* if it is equal to the sum of its proper (natural number) divisors (i.e. its divisors that are less than it). A perfect number n is thus such that

$$S(n) = 2n$$

where $S(n)$ denotes the sum of the divisors of n.

The first two natural numbers that are perfect are 6 and 28. This fact contributed, in antiquity, to the attraction of perfect numbers for mystics, because 6 is the number of days God required for the creation of the world and 28 days is the length of the lunar month. Finding further perfect numbers leads very rapidly to large numbers. The next three are: 496, 8128, 3355036.

Euclid proved that every number of the form

$$2^{p-1}(2^p-1),$$

where $2^p - 1$ is a prime number is perfect. For such a number n, one has

$$S(n) = (2^p-1)(1+2^p-1) = 2^p(2^p-1) = 2n.$$

In the 18th century Euler showed that any *even* perfect number must be of this form. A very simple proof of this was given in 1911 by Léonard Eugène Dickson. Here it is:

Let n be an even number of the form $2^{p-1}\cdot q$, where q is odd and $p > 1$. The sum of its divisors is

$$S(n) = \Sigma(s(1+2+\ldots+2^{p-1})),$$

where s runs over the divisors of q,

$$= (1+2+\ldots+2^{p-1})(S(q))$$

$$= (2^p-1)(q + d),$$

where d is the sum of the proper divisors of q.

Suppose n is perfect. Then

$$S(n) = 2^pq = (2^p-1)(q+d),$$

whence

$$d = q/(2^p-1).$$

[†] Translation of an unsigned article in *Math-Jeunes*, No. 43, 1989 (see the previous article).

Since d and q are natural numbers, we deduce that

(a) q is a multiple of $2^p - 1$

(b) d divides q and is smaller than it;

but then d, which is the sum of the proper divisors of q, is equal to one of its summands, and so must equal the first summand. Consequently q is prime and

$$d = 1 \text{ and } q = 2^p - 1, \text{ a prime number.}$$

Thus n is of the stated form.

As for odd numbers, it seems that none of them are perfect, but I have no knowledge of any proof of this.

* * * * *

JESUIT AND FELLOW OF THE ROYAL SOCIETY: A BEST FIT?

Hans Lausch, Monash University

Once in a while scientists, economists, engineers and other professionals perform measurements consisting of at least three number pairs (x, y) that they believe to be connected by a linear functional relationship, viz. $y = a + bx$ for suitable values a and b. Their beliefs in this relationship may be very strong, and they may not abandon them even if the measurements seem to prove otherwise. It is not the assumption of linear functional relationship that is wrong; the observations must have lacked accuracy, they argue. However, in the same breath they pride themselves on being conscientious observers. Consequently the inaccuracies of their measurements cannot be unduly large; they ought to be as little noticeable as possible. The sole device left for achieving the desired effect in this predicament is a judicious choice of a and b.

Unfortunately, several mathematical interpretations are admitted by the requirement that the inaccuracies be "as little noticeable as possible", or, phrased optimistically: that the line $y = a + bx$ represent the "best of all possible fits" for the data. Of these interpretations the "least-sum-of-squared-residuals" approach is leading in popularity. Suppose that $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are the pairs of observed values. From among all pairs (a,b), one pair is to be determined for which the sum $(a + bx_1 - y_1)^2 + (a + bx_2 - y_2)^2 + \ldots + (a + bx_n - y_n)^2$ is as small as possible. The expressions $a + bx_i - y_i$ are the "errors". Announced by the French mathematician Adrien-Marie Legendre (1752-1833) in the year 1805, this is the well-known Principle of Least Squares. Carl Friedrich Gauss (1777-1855), not without getting himself into a vexatious priority dispute with Legendre, and Pierre Simon Laplace (1749-1827) aided its prevalence by reformulating and developing it within the mathematical theory of probability, which involved the famous normal distribution and its celebrated bell-shaped curve.

Nearly half a century before Legendre's announcement and twenty years before Gauss was born, the Jesuit pater Rogerius Josephus Boscovich – or Rudjer Josip Bošković, as was his Slavonic name – had formulated and put in practice a different principle when trying to find a line that was more or less compatible with observational data he and Christopher Maire (1697-1767) had obtained from measuring the lengths of meridian arcs at various latitudes. Before describing the "Boscovich algorithm" a few biographical remarks might not be inappropriate, as Boscovich ranks among the most versatile minds of the eighteenth century.

Boscovich was born on May 18, 1711, in the Dalmatian Republic of Dubrovnik, an important city-state by the Adriatic Sea; Dubrovnik is today a Yugoslav city. Curiously enough, the future diplomat for the Vatican, Jesuit priest, mathematician and philosopher shares his birthday with Pope John Paul II and Bertrand Russell (but also with Dame Margot Fonteyn and my daughter Monica). He sprang from a well-to-do while not noble family and had five brothers and three sisters. In Dubrovnik he spent his first fourteen years and attended the Collegium Ragusinum, Dubrovnik's well-established Jesuit seminary and secondary school.

Rogerius Josephus Boscovich

In September 1725 he travelled across the Adriatic to the Papal States and continued his education at S. Andrea delle Fratte on the Quirinal, highest among the seven hills of ancient Rome. After two years of noviciate, having studied rhetoric and poetry, he solemnly promised his adherence to perpetual poverty, chastity, and obedience towards his superiors in the Society of Jesus. Then he was transferred to the Collegium Romanum, the finest Jesuit college in Europe. The year 1729 was devoted to philosophy and logic, the years 1730-31 to mathematics. While astounding his teachers with his rapid absorption of arithmetic and algebra, he applied himself feverishly to reading Newton. During the years 1731-32 he concentrated on physics. Each year searching examinations were to be passed in the presence of the college management. Subsequently Boscovich handed on his knowledge in the "*studia inferiora*" to novices of the Collegium.

In 1735 he publicly recited the first 300 lines from his scientific Latin poem, *De Solis ac Lunae Defectibus*; that philosophy and science were subjects for poetry was in the spirit of the age, as also Alexander Pope's creations show. A period followed in which Boscovich wrote treatises on astronomy, on spherical trigonometry, on the nature and use of infinitely great quantities and infinitely small, and on mechanics.

Pope Benedict XIV was concerned about rumours that long-standing cracks in the dome of St Peter's presaged an imminent collapse. In 1742 Boscovich was chosen as one of the Pope's technical advisers. He redeemed the dome by having a number of iron rings inserted to fortify the structure. In 1744 he completed his theological studies and passed his doctoral examination. Next he entered the priesthood and joined the Society of Jesus as a full member. As an archaeologist Boscovich played a rôle in excavating mosaic floors and a sundial of an ancient Roman villa. The Pope commissioned Boscovich and Christopher Maire to measure a meridian arc and to draft a geographical map of the Papal States. In 1752 Boscovich provided his students with a textbook on geometry, trigonometry and algebra.

As an adept hydrographer he went to Lucca in 1756 on his first assignment outside the Papal States. He was to settle a quarrel between the Republic of Lucca and the Austrian Grand Duchy of Toscana that had arisen over adjacent waters. This business necessitated a journey to Vienna in 1757. In Vienna his social life was very active: the Austrian aristocracy entertained him; he conferred with many scholars; "Kaiserin" Maria Theresia, Archduchess of Austria and Queen of Bohemia and Hungary, impressed him by her charm, insight, and breadth of outlook; and her husband Emperor Franz I saw him in audience. Having been the Pope's counsel in architectonic matters, he was now asked to pronounce on the building that housed the Imperial Library and on the roof covering the belfry of Milan cathedral.

Amidst all these activities, Boscovich was preparing his masterpiece, a monumental Theory of Natural Philosophy. In it he postulated a world that is composed exclusively of finitely many mathematical points which attract or repel one another depending on their distance. Boscovich's radical atomism attracted the attention of the scientists Priestley, Faraday, Maxwell and Lord Kelvin, as well as the philosophers Friedrich Nietzsche ("Beyond Good and Evil") and Ernst Cassirer. A Boscivichian hypothesis served the physicist J.J. Thomson in introducing the earliest concepts of the new atomic physics. Shortly after the book had appeared, the philosopher Moses Mendelssohn analysed the opus in an extensive and brilliantly written review, which concluded with a praise of Boscovich's extraordinary imaginative capacities, notwithstanding his apparent errors: "After all, it is worthier and benefits Truth much more to deviate from Her with genius than to repeat insipidly what others have already expressed in a better way".

After a sojourn in Rome, Boscovich moved abroad, first to Paris on a diplomatic mission for his native Dubrovnik. In London he talked to the best British opticians of the day, John Dollond and James Short. Again he had social and political contacts with fashionable society, such as Lord Marlborough's daughter Lady Pembroke and the Lords Shrewsbury and Stanhope, whereas a clear majority of his friends and acquaintances were scientists. He toured both Oxford and Cambridge, and he met Benjamin Franklin at a demonstration of Franklin's experiments with electricity. Boscovich had arrived in London as a corresponding member of the French Academy, a circumstance that facilitated being introduced to Fellows of the Royal Society and eventually becoming one himself. On January 15, 1761, Roger Joseph Boscovich was received into this august assembly.

After England he visited, in his capacities as a scientist and as a diplomat, Belgium, Holland, Lorraine, Germany, Venice, Turkey – where he inspected Gallipoli – and Poland. On his arrival back in Rome he was confronted with a centuries-old problem: the drainage of the Pontine marshes. The French mathematician Clairaut sent a note of warning: "Take great care, I beg you, not to overtax yourself and don't go and ruin your health which is precious to Mathematicians". Still, Boscovich drew up a report, which served as a basis for succeeding attacks on the marshes.

In 1764 Boscovich took up the chair of mathematics at the University of Pavia after an invitation by the Austrian government. Besides lecturing and building an astronomical observatory he busied himself with university reform. He suggested that the syllabus needed improving, the methods of instruction called for changing. Since the dictation of notes by the professor was time-eating and the taking of notes by the students often inaccurate, he wanted proper textbooks to be used instead, and he recommended a long list. The elements of a subject, he submitted, could be taught at a lesser cost. Accordingly, senior staff could be spared to engage in further research and not in grounding beginners. These were sound ideas, yet none too popular.

His old age was overshadowed by a painful ulceration of his leg, which an unqualified Belgian sawbones with an enviable reputation for cures was able to arrest only temporarily. Furthermore, the Jesuit Order had come increasingly under pressure: in 1772 the Jesuit Seminary was closed by the Pope, and in 1773 Maria Theresia agreed to suppress the Order in Austria. Boscovich's career had its continuation in Paris, where he was to stay for nine years. As Director of Naval Optics of the French Navy he occupied himself intensely with the theory of the achromatic telescope.

The French Royal Press's costly engrossment in the American War of Independence prevented the publication of Boscovich's collected works on optics and astronomy and so he returned to Italy. During November 1786 doctors noted signs of mental disturbance in the ailing Boscovich. He was tormented by ideas of persecution, shame, a futile fear of poverty, and by worry of losing his reputation, of being ridiculed owing to overlooked misprints or to errors being discovered in his works. When his condition was growing worse, he attempted suicide. His leg trouble recurred, a catarrh flared up. Finally an abscess on the lung manifested itself. On February 13, 1787, he passed away. He was buried in Milan.

When the Kingdom of Serbs, Croats and Slovenes had been created after the First World War, the young state passed in review the Southern Slavs' great sons and daughters of the days gone by. Interest in Boscovich was soon rekindled and has persisted ever since.

The Boscovich Algorithm

Given n points ("observations")

$$(x_1,y_1),\ (x_2,y_2),\ldots,(x_n,y_n)$$

in the plane, Boscovich defined a line $y = a + bx$ as to be most nearly in accord with all of the observations if it satisfies the two conditions:

(1) The sum of the numbers $a + bx_i - y_i$ (the positive and negative corrections to the y-values or "errors") is zero.

(2) The sum of the numbers $|a + bx_i - y_i|$ (the absolute values of the corrections or "errors") is as small as possible.

Note that if in condition (2) "absolute values" were replaced by "squares", we would obtain precisely the Principle of Least Squares.

Condition (1) states that a and b, the intercept and slope of the best-fitting line $y = a + bx$, must satisfy the equation

$$(3)\quad (a+bx_1-y_1) + (a+bx_2-y_2) + \dots + (a+bx_n-y_n) = 0$$

which can also be written in the form

$$(4)\quad \bar{y} - a - b\bar{x} = 0$$

where

$$\bar{x} = (x_1+x_2+\dots+x_n)/n \quad \text{and} \quad \bar{y} = (y_1+y_2+\dots+y_n)/n$$

are the arithmetic means of the n observed values of x_i and y_i, respectively.

Condition (2) states that a and b must satisfy the condition

$$(5)\quad |a+bx_1-y_1| + |a+bx_2-y_2| + \dots + |a+bx_n-y_n| = \text{minimum.}$$

Replacing a in condition (5) by its value

$$(6)\qquad a = \bar{y} - b\bar{x}$$

implied by equation (4), it is seen that condition (2) in conjunction with condition (1) requires that the slope b shall satisfy

$$(7)\quad |(y_1-\bar{y}) - b(x_1-\bar{x})| + |(y_2-\bar{y}) - b(x_2-\bar{x})| + \dots$$
$$\dots + |(y_n-\bar{y}) - b(x_n-\bar{x})| = \text{minimum.}$$

No serious harm is done by assuming that $\bar{x} = 0$ and $\bar{y} = 0$ as we are free to make the substitutions $Y_i = y_i - \bar{y}$ and $X_i = x_i - \bar{x}$. This implies $a = 0$ (the actual a can be recovered from equation (6), once b is evaluated), so that after this simplication a line as required, being of the form $y = bx$, will pass through the origin, while condition (7) now reads

$$(8)\quad |y_1-bx_1| + |y_2-bx_2| + \dots + |y_n-bx_n| = \text{minimum.}$$

Consequently, determination of a "Boscovich line" corresponding to a given set of observational points reduces to determining its slope b from condition (8).

Using beautiful dynamic imagery in order to find b, Boscovich reasoned thus:

In Figure 1, let A, B, C, D and E denote the five observational points (x_1,y_1), (x_2,y_2), (x_3,y_3), (x_4,y_4), and (x_5,y_5), respectively. Consider a line LL' through the origin and rotated clockwise through a very small angle from the y-axis. At this "moment" the vertical distances of the observational points from the line LL' (i.e. the absolute values of the differences $y_i - bx_i$ will be "enormous", and so will their sum S, i.e. the left-hand side of condition (8)).

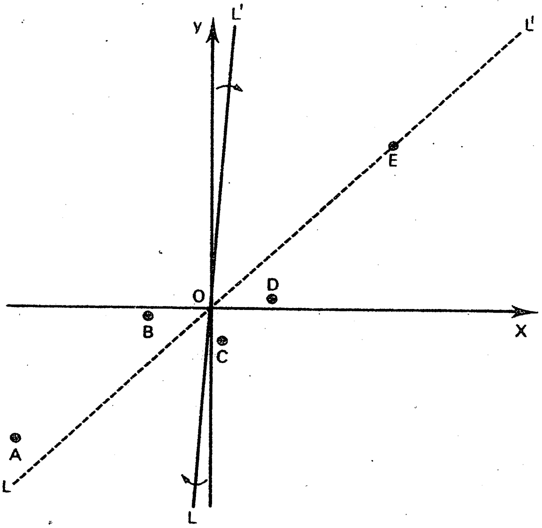


Fig. 1

If now the line LL' be rotated slowly in a clockwise direction about the origin, all of the vertical distances will decrease until the line intersects the "first" of the points, E in Fig. 1, after which the distance from the first point (E) to the line will steadily increase, but the distances from the other points (A, B, C, and D) will continue to decrease until the line "reaches" the "second" point, A in Fig. 1, after which the distance from the second point to the line will steadily increase, and so forth.

By virtue of the similarity of the triangles formed at any "instant" by the x-axis, the vertical lines through the respective observational points, and the rotating line LL', it is evident that the change in the distance of the line from any observational point (x_i,y_i) produced by a given rotation of the line LL' in the clockwise direction is proportional to the distance of x_i from the centre of rotation, i.e. from the origin, which is $|x_i|$; and the "change" will be positive for points "already passed" and negative for points "not yet reached". Hence the corresponding change in the sum S will be proportional to the difference of the sums of the "horizontal distances" $|x_i|$ of (a) the points "already passed" and (b) the points "not yet reached" ; and the change in S will be negative until the first of these sums is not less than the second. Consequently, the sum S, i.e. the left-hand side of condition (8), achieves its minimum value at the "moment" when the line LL', rotating clockwise about the origin, "reaches" the first observational point such that the sum of the "horizontal distances" $|x_i|$ of all points "reached" or "passed" is not less than one half of the sum of the "horizontal distances" of all the points, i.e. $|x_1| + |x_2| + \ldots + |x_n|$; and the desired value of b in equation (8) is the slope of the line at this "moment".

Summarizing Boscovich's reasoning, we arrive at the following method, the Boscovich algorithm, for determining b in condition (8):

Consider only those terms of the sum

$$(9) \qquad |y_1-bx_1| + |y_2-bx_2| + \ldots + |y_n-bx_2|$$

for which $x_i \neq 0$ (since the lines $y = bx$ pass through the origin for every choice of b, observations (x_i,y_i) with $x_i = 0$ can be discarded. These observations have, of course, influenced the values of x_i and y_i in equation (7)).

For each i with $x_i \neq 0$, calculate the corresponding "slope"

$$(10) \qquad b_i = y_i/x_i .$$

Arrange these b_i in descending order of magnitude, thus

$$(11) \qquad b_{(1)} \geq b_{(2)} \geq b_{(3)} \geq \ldots \geq b_{(m)} \; (m \leq n).$$

(The line LL' passes through $(x_{(1)},y_{(1)})$ first, then through $(x_{(2)},y_{(2)})$, and so forth.)

Arrange the absolute values of the corresponding numbers x_i in the same order, thus

$$(12) \qquad |x_{(1)}|, \; |x_{(2)}|, \ldots, |x_{(m)}|$$

and compute their sum

$$D = |x_{(1)}| + |x_{(2)}| + \dots |x_{(m)}|. \tag{13}$$

Then the sum S of (9) will be a minimum for $b = b_{(r)}$, the r-th term of the sequence (11), where r is the smallest integer for which the partial sum of the first r terms of the sequence (12) equals or exceeds one half of the sum of the entire sequence, i.e. the smallest r for which

$$|x_{(1)}| + |x_{(2)}| + \dots + |x_{(r)}| \geq D/2. \tag{14}$$

We conclude the description of the Boscovich algorithm with a numerical example.

Suppose that

$$\begin{aligned}
(x_1, y_2) &= (\ 6,\ \ 4),\\
(x_2, y_2) &= (-4,\ \ 4),\\
(x_3, y_3) &= (\ 2,\ -7),\\
(x_4, y_4) &= (\ 0,\ -1),\\
(x_5, y_5) &= (-3,\ \ 0),\\
(x_6, y_6) &= (-1,\ \ 0).
\end{aligned}$$

A Boscovich line for this set of observations is to be determined.

Note that both the x-values and the y-values sum to 0, hence $\bar{x} = 0$ and $\bar{y} = 0$, and the problem appears in the "simplified" form (8).

Since $x_4 = 0$, we discard (x_4, y_4). We calculate the b-values as in (10):

$$\begin{aligned}
b_1 &= 2/3,\\
b_2 &= -1,\\
b_3 &= -7/2,\\
b_5 &= 0,\\
b_6 &= 0.
\end{aligned}$$

Then we arrange them as in (11):

$$b_{(1)} = 2/3 \geq b_{(2)} = 0 \geq b_{(3)} = 0 \geq b_{(4)} = -1 \geq b_{(5)} = -7/2.$$

Writing down, in accordance with (12), the absolute values of the corresponding x-values in the same order, we obtain the sequence

6, 1, 3, 4, 2 (or, which makes little difference, 6, 3, 1, 4, 2).

The sum of its terms is $D = 16$.

We form the sums 6, $6 + 1 = 7$, $6 + 1 + 3 = 10$ and note that $6 < D/2$, $7 < D/2$, whereas $10 > D/2$. Hence we have found a suitable value for b, namely $b = b_{(3)} = 0$, and $y = 0$, the *x*-axis, is therefore a Boscovich line for the given set of observations.

Problem: Are there sets of observations that admit more than one Boscovich line?

Food for thought: what other lines besides the Legendre ("principle of least squares") and the Boscovich line may be reasonably deemed "best fits"?

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SOLVING OLD PROBLEMS IN NEW WAYS – SURPRISING CONNECTIONS BETWEEN APPARENTLY DIFFERENT THINGS

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Let us begin our story where many good mathematical stories begin: with Archimedes. Archimedes was interested in (among many other things) the calculation of areas and volumes. And let me, to make things concrete for us, concentrate on one such problem – that of determining the area under a parabola.

The parabola is one of the simplest geometric curves we can draw – indeed it's the very simplest of them, except for the straight line (which is so simple it doesn't "curve" at all) and possibly the circle.

Represent the parabola as $y = p^2$ and concentrate on the values of p lying between nought and one. If I plot enough of these, I can get a very accurate parabolic curve. Now on my graph paper, I can also draw two lines: the horizontal base line, and a vertical line at $p = 1$. These two lines together with the parabolic curve enclose a sort of triangular area. It isn't exactly a triangle, of course, because one of its sides isn't straight. Perhaps we can call it a "quasi-triangle".

Archimedes set out to determine the area of this quasi-triangle.

Now it's not at all clear how to do this, but you have probably learned, as I did, to *approximate* such areas by counting the little squares of graph paper that go to make them up.

If we do this on reasonably standard graph paper, where one square of unit area is made up of a hundred smaller squares each of area 0.01, the process is relatively straightforward. The straight base and the vertical end of our quasi-triangle each run neatly along the sides of these little squarelets. The third, the parabolic, side is a little more complicated, however, as it ploughs *through* the little squarelets instead of going along their edges.

Now when we come to do a count, we find there are 24 squarelets definitely inside the quasi-triangle and 19 "maybe's" that are part in and part out. So we really have two estimates of our area: an *under*estimate of 0.24, found by counting only the definites and ignoring the maybe's, and an *over*estimate of 0.43 found by treating the maybe's on the same basis as the definites and lumping them all together.

If we take the average of these two estimates, we get 0.335, very close to one third, and we can speculate that the correct answer is exactly one third. How could we check this? Well, we could use finer and finer graph paper with, say, 10,000 or perhaps a million little squarelets to the square. And if we did, we would find, in each such case, an under- and an overestimate with an average very close to, but not exactly, one third.

How do I know? Well, this is where Archimedes comes in. Not that he actually did such experiments - that is not mathematics. Mathematics comes in when we *imagine* them being done. And such imaginary experiments are not shackled by practical considerations such as "where shall we get such fine graph paper?" or "will we need a microscope to count the little squarelets?". We can *imagine* the squarelets to be as small as we please - and when we do this, we find that the two estimates both get closer and closer to one third, so that if we could go on forever, the discrepancies by which the overestimate exceeded one third and the underestimate fell short both would become vanishingly small.

So our quasi-triangle has an area of 1/3 and we know this for sure as a result of Archimedes' calculation. *A consequence is that the average value of the height* p^2 *is also 1/3.*

For the average height multiplied by the length of the base (which here is one) must give the area. *So the average value of* p^2 *taken over all the values of* p *between nought and one is one third.*

Now I want to tell you another story. This one begins with a quite different imaginary experiment. I take a coin and toss it twice. What is the probability of getting two heads? This problem was considered by the French encyclopaedist d'Alembert, who argued like this. Three things can happen: I can get two heads, one head or no heads. So the chance of getting two heads, according to d'Alembert, is one third.

But Diderot, another of the French encyclopaedists, didn't see things this way. According to Diderot, d'Alembert had got the answer wrong by overlooking a subtle point. In two tosses of the coin there are actually *four* things that can happen: we could get head-head, head-tail, tail-head or tail-tail. The "one-head" case can happen in two separate ways. So according to Diderot, the probability of two heads in one quarter, which corresponds to there being a probability of 1/2 of getting a head on each toss.

This is in fact the answer you'll find today in any basic test on probability today, but did you notice that it contains an assumption? The coin is supposed to be fair - that is, equally likely to give heads or tails. Now when I spelled out the problem I didn't say anything about that. For all I told you, the coin might have been a double-headed penny. Or, at the opposite end of the scale, try spinning an American 1-cent coin on a table.

So let the probability of getting heads on a single toss be p. The probability of getting heads twice in two tosses is then p^2.

So far, so good. There's only one trouble: I don't know the value of p. Now a possible response to this, in fact quite a reasonable one, is to say: "Well, I've gone as far as I can go – I can't be expected to give complete solutions to problems if you don't supply complete data". But actually in real life we very often do have to make decisions on the basis of incomplete data. We calculate outcomes as best we can and use these "best estimates" as the basis for our decision. We can't expect always to get it right, of course, but in the day-to-day business of living there's no back of the book where we can sneak a look at the answers.

So the question boils down to this: How can we best estimate the probability of two heads if we have no data on the coin? I will give two possible approaches to this. Both begin by agreeing that the experiment shall consist of two tosses of a coin, the number of heads resulting to be recorded, and the question being to estimate the probability that this number is two.

One mathematician, let us call him *M1*, would argue like this. I have no information whatever about the possible bias of this coin, but I know that the state of having no information may itself be analysed. It corresponds to maximising a quantity called entropy, and furthermore it is known that entropy is maximised when the outcomes are all equally likely. The description of the experiment says only that we *count* heads; it makes no mention of the order of their arrival, and so there are 3 outcomes: 2 heads, 1 head or no heads. So until I get some more information I must assume these outcomes to be equally likely and assign to each a value of 1/3. I therefore estimate the probability of two heads as 1/3. Thus argues *M1*. Notice that the actual calculation performed by *M1* is trivially simple. But it is by no means simple*minded*, for it finds its justification in some rather deep theory.

Our other mathematician, *M2*, argues differently. He says: Look, I know the value is p^2 – for some value of p. The only trouble is that I have no idea what that value is. What I'd better do is consider all the possible values of p all the way from 0 to 1, calculate the value of p^2 for each and find the average of all these results.

Now there isn't much deep theory here. The approach is pretty straightforward. But it does leave *M3* with a rather difficult calculation. In fact it's exactly the calculation I talked about earlier, the one Archimedes did. And we know the answer to that. It is 1/3.

So although *M1* and *M2* approach the question in different ways, they reach the same answer. This is quite surprising, for best estimate problems are not like questions where we have all the data and only one answer can be correct. Here the answer could very well depend on how we try to cope with the incompleteness of what we're told. We have no right to expect that two different approaches will yield the same estimate. In many cases they don't, but in this instance they do.

In fact it is known that for a very wide range of such cases, the two approaches always yield the same estimate. And so, now that we know this, we can apply whichever method we prefer. But *M1*'s approach is computationally simpler, and so we can use it to bypass the more difficult calculations needed by $M2$. In other words, we can combine the simplicity of $M2$'s theory with the computational convenience of $M1$'s method.

And so, if we like, we can use this result to find areas and volumes much more quickly and easily than Archimedes could. Here's an example. Find the area of the quasi-triangle between the base-line, the vertical line at $p = 1$ and the graph of p^3. The answer is a quarter. How do I know? By estimating the probability of three heads in three tosses of a coin.

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PYTHAGOREAN TRIPLES AND COMPLEX NUMBERS

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A Pythagorean triple is a set of three positive integers representing the lengths of the sides of some right-angled triangle. The very simplest Pythagorean triple is (3,4,5) which was known to the civilisations of ancient times.

Pythagorean triples (x,y,z) are the positive integral solutions of the equation

$$x^2 + y^2 = z^2. \qquad (1)$$

The general solution of equation (1) was known to the Greek mathematician Diophantus (250?-334? A.D.). (See *An Introduction to the Theory of Numbers* by G.H. Hardy & E.M. Wright, 4th ed., Clarendon Press, Oxford, 1960, pp. 190-191.)

Complex numbers are numbers of the form $a + ib$, where $i^2 = -1$ and a and b are real numbers. We say that a complex number is a *Gaussian integer* if both a, b are ordinary (real) integers. a is called the *real* part and b is called the *imaginary* part of $a + ib$. The complex numbers $a + ib$ and $c + id$ are equal only if $a = c$ and $b = d$.

The *modulus* of the complex number $a + ib$, written $|a + ib|$, is defined to be the real number $\surd(a^2+b^2)$.

Here I want to discuss the connection between Pythagorean triples and Gaussian integers. It will be shown that this connection provides a natural way to find all the integer solutions of equation (1) and also that it allows us to give an interpretation of the formula from which those solutions are constructed.

If $w(= u + iv)$ is a Gaussian integer, then so is its square $z(= x + iy)$. In this case we have

$$z = x + iy = (u + iv)^2 = w^2, \qquad (2)$$

and we say that z is a *Gaussian perfect square.*

Note, using $i^2 = -1$, that

$$x + iy = (u+iv)^2 = u^2 + 2iuv + i^2v^2$$
$$= u^2 - v^2 + i(2uv).$$

Hence, equating real and imaginary parts, $x = u^2 - v^2$ and $y = 2uv$. Thus

$$|z|^2 = x^2 + y^2 = (u^2-v^2)^2 + (2uv)^2$$
$$= u^4 - 2u^2v^2 + v^4 + 4u^2v^2$$
$$= u^4 + 2u^2v^2 + v^4$$
$$= (u^2+v^2)^2.$$

In other words, (x,y,u^2+v^2) is a Pythagorean triple, so that *when* $z = x + iy$ *is a Gaussian perfect square,* $(x,y,|z|)$ *is a Pythagorean triple.*

The question now arises: Is the converse also true? To put it more precisely: *If for the Gaussian integer* $z(= x + iy)$ *the set* $(x,y,|z|)$ *gives a Pythagorean triple, is it true that* z *is a Gaussian perfect square?*

If the question is asked in this form, the answer is an immediate "no", because $(9,12,15)$ is a Pythagorean triple, but $9 + 12i$ is not a Gaussian perfect square. In fact (as you can easily check by squaring)

$$(9 + 12i)^{1/2} = \pm(2\sqrt{3} + i\sqrt{3}),$$

and this is clearly not a Gaussian integer.

However, we can obtain more interesting results if we refine the question somewhat. First define a *primitive* Pythagorean triple as a Pythagorean triple whose members are co-prime (that is to say, have no factors in common, other than $+1$ and -1). It will be seen that $(9,12,15)$ is not a primitive Pythagorean triple as 3 is a common factor of all the members.

Before coming to our revised converse theorem, notice two important properties of primitive Pythagorean triples.

(A) If (a,b,c) is a primitive Pythagorean triple and $c > a$, $c > b$, then one of the numbers a, b is even.

(B) Taking b to be the even element of the triple, then $c + b$, $c - b$ are both perfect squares (and both odd).

Proofs:

(A) Assume that a, b are both odd. Then $a = 2n + 1$, for some integer n, and $a^2 = 4(n^2+n) + 1$, so that if we divide a^2 by 4, the remainder will be 1. Similarly with b^2. So $a^2 + b^2$ will leave a remainder of 2 when divided by 4. But no perfect square has this property (prove this as an exercise). Hence equation (1) could not be satisfied by (a,b,c). If equation (1) is to hold therefore, one of a, b (let us say b) must be even.

(B) $$a^2 = c^2 - b^2 = (c+b)(c-b). \qquad (3)$$

Since b is even and (a,b,c) is a primitive Pythagorean triple, c must be odd. So both $c + b$, $c - b$ are odd. Furthermore, they are co-prime with no divisors (other than ±1) in common. To see that this is so, suppose otherwise. Then

$$c = \frac{1}{2}\Big[(c+b) + (c-b)\Big]$$

and

$$b = \frac{1}{2}\Big[(c+b) - (c-b)\Big]$$

and if $c + b$, $c - b$ have a common divisor d, this will divide both c, b and so will be a common divisor of these numbers, and also, from equation (3), a divisor of a contrary to our initial assumption, unless $d = +1$ or -1.

[The very careful reader will note that this argument breaks down in the case $d = 2$, but here we have already seen that 2 does not divide c.]

Now from equation (3), $(c+b)(c-b)$ is a perfect square and no factors, other than $+1$ and -1, divide both $c + b$, $c - b$. Thus these quantities must themselves both be perfect squares.

We need another preliminary result about complex numbers:

(C) If x and y are positive real numbers, then there exist positive real numbers u and v, say, such that $x + iy = (u+iv)^2$, in other words, such that $u + iv$ is a square root of $x + iy$.
[Remark: the word positive can be removed from this statement, if you use the fact that i is a square root of -1; but we are only interested in the case where x and y are positive, and it will help us to choose u and v as positive.]

Proof:

All we do for a proof is solve the equation $x + iy = (u+iv)^2$, i.e.

$$x + iy = u^2 - v^2 + i(2uv), \qquad (4)$$

for u and v.

Equating real and imaginary parts gives, as before,

$$x = u^2 - v^2 \qquad (5)$$

$$y = 2uv, \qquad (6)$$

whence, again as before,

$$x^2 + y^2 = (u^2+v^2)^2,$$

so that

$$\surd(x^2+y^2) = u^2 + v^2. \qquad (7)$$

From equations (5) and (7) we have

$$2u^2 = \surd(x^2+y^2) + x \qquad (8)$$

and

$$2v^2 = \surd(x^2+y^2) - y. \qquad (9)$$

Since x is positive and $\surd(x^2+y^2) > y$, therefore the right-hand sides of equations (8) and (9) are each positive. Dividing them by 2 and taking positive square roots, we have positive real numbers u and v satisfying $x + iy = (u+iv)^2$, as required.

With these preliminaries behind us, we can now proceed to state and prove the major result.

Theorem:

Let $z = x + iy$ be a Gaussian integer, such that $(x,y,|z|)$ is a primitive Pythagorean triple, and y is even. Then z is a Gaussian perfect square.

Proof:

Let $z = w^2$, where $w = u + iv$. We have to show that u, v are real integers.

We are in the situation of Result (C), above, and with the same notation, except that we have now called $x + iy$ the complex number z, and so also $|z| = \surd(x^2+y^2)$. Hence, from adding and subtracting, respectively, (6) from (7), we have

$$(u+v)^2 = u^2 + v^2 + 2uv = |z| + y \qquad (10)$$

and

$$(u-v)^2 = u^2 + v^2 - 2uv = |z| - y. \qquad (11)$$

We now apply Property (B) to the primitive Pythagorean set $(x,y,|z|)$, taking $a = x$, $b = y$ and $c = |z|$.

Property (B) then tells us that the integers $|z| + y$ and $|z| - y$ are each perfect squares, i.e. each squares of integers. Consequently, from (10) and (11) it follows that $u + v$ is an integer and that $u - v$ is an integer.

Property (B) also states that $|z| + y$ and $|z| - y$ are each odd integers. Hence, since the square of an uneven integer is even, $u + v$ and $u - v$ are each odd. Thus

$$(u+v) + (u-v) \quad \text{and} \quad (u+v) - (u-v)$$

are each even, i.e.

$$2u \text{ and } 2v$$

are each even numbers. Hence u and v are integers.

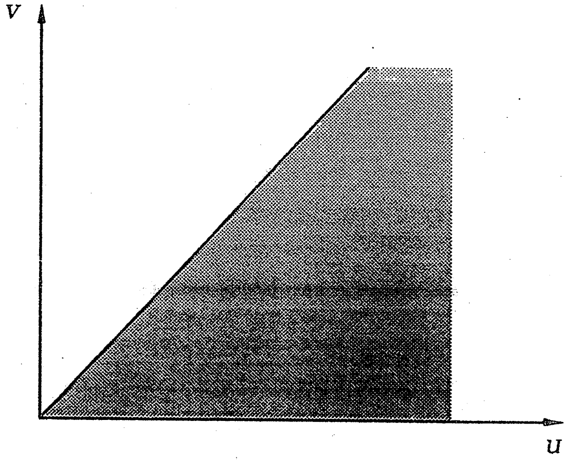
This completes the proof.

The theorem provides a way of constructing *all* the primitive Pythagorean triples from the set of Gaussian perfect squares, by so arranging the triples that the *imaginary* component of the Gaussian square is the *even* element of the triple.

In practice, it suffices to find the squares of the Gaussian integers in the region

$$u > 0, \quad 0 < v < u$$

shown as the shaded region in the diagram.



Squaring Gaussian integers outside this region yields no new triples.

The table below lists the triples obtained by squaring the Gaussian integers $u + iv$ in the region $0 < u \leq 10$, $0 < v < u$, where u, v are *co-prime*, and one is even, so that for

$$x + iy = (u+iv)^2,$$

x and y are co-prime and y is even.

w	$z = w^2$	$(x,y,\lvert z\rvert)$
$2+i$	$3+4i$	(3,4,5)
$3+2i$	$5+12i$	(5,12,13)
$4+i$	$15+8i$	(15,8,17)
$4+3i$	$7+24i$	(7,24,25)
$5+2i$	$21+20i$	(21,20,29)
$5+4i$	$9+40i$	(9,40,41)
$6+i$	$35+12i$	(35,12,37)
$6+5i$	$11+60i$	(11,60,61)
$7+2i$	$45+28i$	(45,28,53)
$7+4i$	$33+56i$	(33,56,65)
$7+6i$	$13+84i$	(13,84,85
$8+i$	$63+16i$	(63,16,65)
$8+3i$	$55+48i$	(55,48,73)
$8+5i$	$39+80i$	(39,80,89)
$8+7i$	$15+112i$	(15,112,113)
$9+2i$	$66+36i$	(77,36,85)
$9+4i$	$65+72i$	(65,72,97)
$9+8i$	$17+144i$	(17,144,145)
$10+i$	$99+20i$	(99,20,101)
$10+3i$	$91+60i$	(91,60,109)
$10+7i$	$51+140i$	(51,140,149)
$10+9i$	$19+180i$	(19,180,181)

Finally, let me remark that equations (5), (6), (7) together with the restrictions placed on u, v give the classical method for obtaining primitive Pythagorean triples: *all primitive Pythagorean triples are of the form* $(u^2-v^2, 2uv, u^2+v^2)$, *for positive integers* $u > v$. (See, for example, the book by Hardy & Wright, referred to above.) The interest of the method outlined above lies in the interpretation it gives of the classical formulae.

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ALTERNATIVE METHODS OF MULTIPLICATION

Garnet J. Greenbury, Brisbane

The alternative method of dividing 263 845 by 64, "*Arithmetic – 16th Century Style*" by Margaret Jackson, *Function*, Volume 13, Part 1, February 1989, reminds me of alternative methods of multiplication by 4, 5, 6, ..., 12.

Here is *Multiplication by 4.*

Rule:

1. Subtract the right-hand digit of the given number from 10, and add five if that number is odd.
2. Subtract each digit of the given number in turn from 9, and five if the digit is odd, and add the integral part of half the neighbour.
3. Under the zero in front of the given number, write half the neighbour of this zero, less 1.

Example:

$\underline{0365187} \times 4$ 8	7 from 10 is 3; add 5 because 7 is odd: total 8
$\underline{0365187} \times 4$ 48	8 from 9 plus half 7: total 4
$\underline{0365187} \times 4$ 748	1 from 9, plus 5, plus half of 8: total 17 Carry 1
$\underline{0365187} \times 4$ 0748	5 from 9, plus 5, plus half of 1, plus 1: total 10 Carry 1
$\underline{0365187} \times 4$ 60748	6 from 9, plus half of 5, plus 1: total 6
$\underline{0365187} \times 4$ 460748	3 from 9, plus 5, plus half of 6: total 14 Carry 1
$\underline{0365187} \times 4$ 1460748	Half of 3, less 1, plus 1: total 1

Can you explain?

* * * * *

APPROXIMATIONS TO π

There is an interesting article in the February 1988 issue of *Scientific American*. It is by J.M. and P.B. Borwein and concerns Ramanujan and π. Ramanujan (see *Function*, Volume 1, Part 3) was a phenomenally gifted and very largely self-taught South Indian mathematician who rose from poverty to a fellowship in the British Royal Society in the space of a brief (33-year) life early this century.

In 1914 he produced a series of approximations to π (in fact via its reciprocal $1/\pi$) of which the simplest is

$$\pi \simeq \frac{9801}{1103\sqrt{8}} = 3.141592730.$$

[Cf. $\pi = 3.1415926535\ldots$. The difference is less than one part in 10^7.]

Borwein and Borwein themselves have produced an elaboration of Ramanujan's formulae. Their simplest approximation reads

$$\pi \simeq \frac{5280(236,674 + 30,303\sqrt{61})^{3/2}}{12(1,657,145,277,365 + 212,175,710,912\sqrt{61})}$$

which is beyond most standard calculators, but which gives π to 24 decimal places.

The Borweins claim that their formulae for π are the most accurate currently available. Of course, simpler (though less spectacular) formulae exist. See, for example, *Function*, Volume 4, Part 1.

* * * * *

PERDIX

Here is the problem paper set in the first Asian Pacific Mathematical Olympiad which I promised, in the last issue, to bring to *Function* readers. The event was a success and there are plans to involve more countries next year, in addition to the pioneer four of Australia, Canada, Hong Kong and Singapore. Details of results are not to hand at the time of going to press.

THE 1989
ASIAN PACIFIC
MATHEMATICAL OLYMPIAD

Tuesday, 14 March, 1989

Time allowed: 4 hours
NO *calculators are to be used.*
Each question is worth seven points.

QUESTION 1

Let $x_1, x_2, \ldots, x_n$ be positive real numbers, and let

$$S = x_1 + x_2 + \ldots + x_n .$$

Prove that

$$(1+x_1)(1+x_2)\ldots(1+x_n) \leq 1 + S + \frac{S^2}{2!} + \frac{S^3}{3!} + \ldots + \frac{S^n}{n!}.$$

QUESTION 2

Prove that the equation

$$6(6a^2 + 3b^2 + c^2) = 5n^2$$

has no solution in integers except $a = b = c = n = 0$.

QUESTION 3

Let A_1, A_2, A_3 be three points in the plane, and for convenience, let $A_4 = A_1$, $A_5 = A_2$. For $n = 1$, 2 and 3, suppose that B_n is the midpoint of A_nA_{n+1}, and suppose that C_n is the midpoint of A_nB_n. Suppose that A_nC_{n+1} and B_nA_{n+2} meet at D_n, and that A_nB_{n+1} and C_nA_{n+2} meet at E_n. Calculate the ratio of the area of triangle $D_1D_2D_3$ to the area of triangle $E_1E_2E_3$.

QUESTION 4

Let S be a set consisting of m pairs $(a.b)$ of positive integers with the property that $1 \le a < b \le n$. Show that there are at least

$$4m\cdot\frac{(m - \frac{n^2}{4})}{3n}$$

triples (a,b,c) such that (a,b), (a,c) and (b,c) belong to S.

QUESTION 5

Determine all functions f from the reals to the reals for which

(1) $f(x)$ is strictly increasing,
(2) $f(x) + g(x) = 2x$ for all real x,

where $g(x)$ is the composition inverse function to $f(x)$. (Note: f and g are said to be composition inverses if $f(g(x)) = x$ and $g(f(x)) = x$ for all real x.)

Australian 1989 Olympiad Team

This year, because of the inception of the Asian-Pacific Mathematical Olympiad, a new procedure was introduced to select the Mathematical Olympiad team. In previous years the team was selected on the basis of performance in the Australian Mathematical Olympiad. This year the selection process was in two stages: part were chosen by taking those who had performed exceptionally well in the Australian Mathematical Olympiad. The intention was to choose 2 or 3, if possible, at this stage. This year 3 were so chosen. For the choice of the rest of the team, performance in the Asian Pacific Mathematical Olympiad, and results of tests taken at the IBM training school in April, were also taken into account.

The team for 1989 is:

Danny Calegari,	Melbourne Church of England Grammar School
Kevin Davey,	St Kevin's College, Victoria
Philip Gleeson,	St Joseph's College, New South Wales
Mark Kisin,	Melbourne Church of England Grammar School
Alan Offer,	Caboolture High School, Queensland
Brian Weatherson,	Mazenod College, Victoria.

Reserve:

Christopher Eckett, Brisbane Church of England Grammar School.

Government sources are again providing no financial or other help towards the process of training and selecting a team and they provide no help towards the expenses of team members, this year the cost of travelling to attend the International Mathematical Olympiad in West Germany.

Lobby your M.P. if you want to help mathematics.

So far as I know, Australia is the only country in the world where Government funds are not given, and indeed given generously, to support its Mathematical Olympiad team.

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Problem 13.2.1 (taken from Workman's *The tutorial arithmetic* (as revised by Geoffrey Bosson, 6th ed., 1963; and sent to us by J.C. Barton).

Prove that, if the sum of any two square numbers is equal to a third square number, then one of the three numbers must be divisible by 5.

Problem 13.2.2 (taken from the *Argus* in the 1930's, from its Education Column).

A man who had no watch was about to leave for a friend's home when he noticed that his clock had stopped. He went to the home of his friend, and after listening to a wireless programme for a couple of hours, returned home and set his clock. How could he do this with any degree of accuracy without knowing beforehand the length of the trip from his friend's place?

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> The mathematician, carried long on his flood of symbols, dealing apparently with purely formal truths, may still reach results of endless importance for our description of the physical universe.
>
> Karl Pearson