FUNCTION

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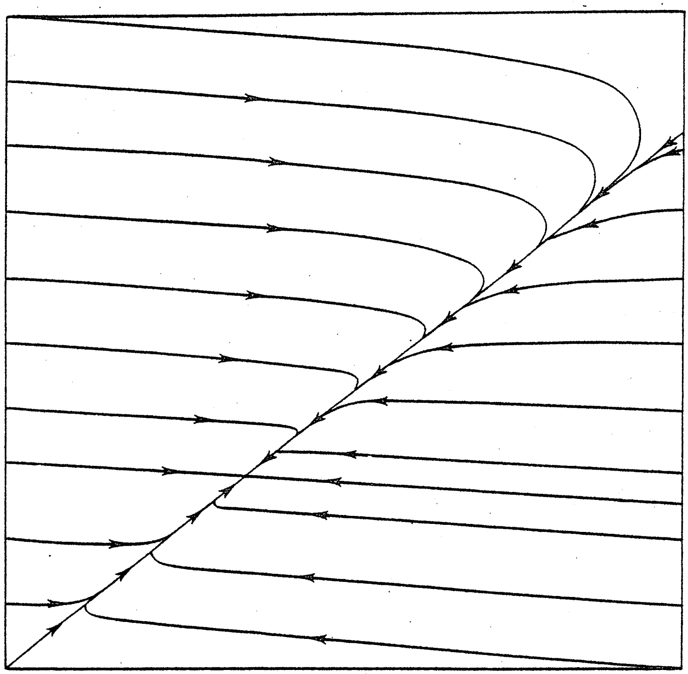

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CONTENTS

The front cover.		34
The square root of zero.	Michael A.B.Deakin	37
Clear thinking about limits.	Alan Pryde	44
Simple rational numbers.	John Mack	48
The Japanese Abacus		54
What are the odds that $\binom{n}{r}$ is even?	Marta Sved	55
Perdix		61

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THE FRONT COVER

The British epidemiologist and pioneer of tropical medicine, Sir Ronald Ross (1857–1932), is best remembered today for his life's work on the transmission of malaria, for which he was awarded the Nobel Prize in Physiology and Medicine in 1902. It was Ross who showed conclusively that malaria is transmitted by mosquitoes and who analysed that transmission with a view to control or eradication of this virulent disease.

Ross was a considerable mathematician as well as being a medical researcher (he also wrote quite meritorious poetry). Among the studies he made was the development of a mathematical model of malaria and its transmission. This led to what are now called the *Ross Malaria Equations*. Our front cover shows the so-called *trajectories* of these equations.

Let a population of mosquitoes live in the same locality as a population of humans and let x be the proportion of mosquitoes infected with malaria and y the population of humans so infected. In given time, a proportion R of the infected humans recover and a proportion M of the mosquitoes die of malaria. If each human suffers B bites in this given time and each mosquito delivers b bites in this same time, then in the long run

$$x = \bar{x} = \frac{Bb-RM}{B(M+b)} \qquad (1)$$

$$y = \bar{y} = \frac{Bb-RM}{b(B+R)}$$

provided these quantities are positive.

If $x \neq \bar{x}$, $y \neq \bar{y}$, then we may plot x, y on a graph as shown in Figure 1. The values of x, y follow the arrows until the values $\bar{x}$, $\bar{y}$ are achieved. Note that for realism we assume

$$0 \leq x \leq 1 \ , \quad 0 \leq y \leq 1 \ .$$

The diagram shows the case for which $\bar{x} = 0.35$, $\bar{y} = 0.30$.

If $\bar{x} < 0$, or $\bar{y} < 0$, then the situation $x = \bar{x}$, $y = \bar{y}$ cannot be achieved and the arrows show that in the long run $x = 0$, $y = 0$. This corresponds to there being no malaria present – a much wished-for situation.

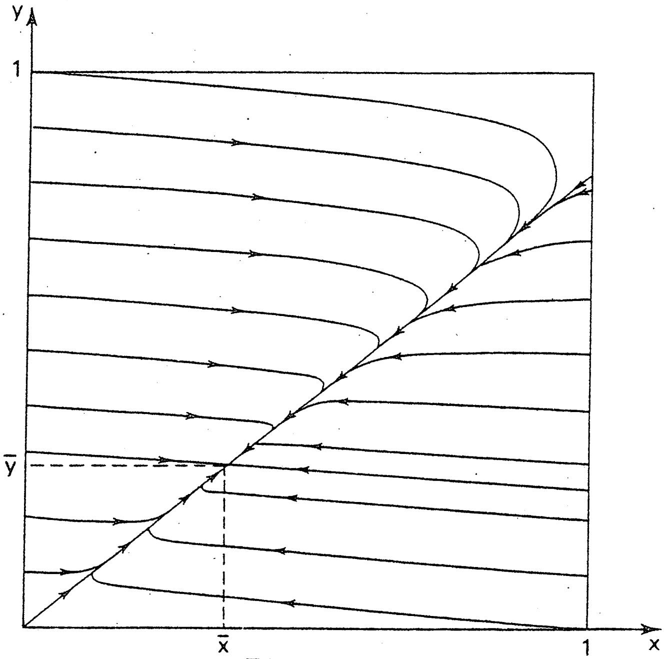

Figure 1.

Equations (1) show us that if $Bb > RM$, then both $\bar{x}$, $\bar{y}$ are positive and the situation is that of Figure 1 which corresponds to a situation of endemic malaria. If, on the other hand, $Bb < RM$, then both $\bar{x}$, $\bar{y}$ are negative and so the malaria is eradicated.

In order to eradicate malaria therefore we need to achieve a situation in which

$$RM > Bb \quad . \tag{2}$$

That is to say we need to combine several factors:

Increase the recovery rate of the humans,

Increase the mortality of the mosquitoes,

Decrease the rate at which mosquitoes bite humans.

Inequality (2) shows an important point. It is not necessary to eliminate *all* mosquitoes, nor to cure *every* human case of malaria, nor need all biting be prevented. As long as Inequality (2) continues to apply, malaria cannot spread in the population. This led to considerable hopes that malaria might be eradicated, as smallpox has been.

Unfortunately, matters not taken into account in the analysis complicate matters. Malaria plasmodia are becoming resistant to the drugs (derivatives of quinine) used to keep the value of R high, and mosquitoes have become resistant to the pesticides used to keep the value of M up. It has also been discovered that malaria infects other animals besides humans and this corresponds (in essence) to higher values of b and lower values of R than we might at first suppose.

Nonetheless the analysis has been very helpful, and continues to be helpful, in malaria control programs.

In 1923, the *American Journal of Hygiene* devoted a special supplement to the Ross Malaria Equations. Most of this was written by Alfred J. Lotka, a mathematician at the Johns Hopkins University in Baltimore. (We tend to forget that malaria was endemic in Baltimore and in many other parts of the United States until well into this century.) Lotka produced a very full study of the equations in the course of this work, and Figure 1 as well as the simplified form of this that appears on the cover) is based on one of his diagrams.

* * * * *

He studied and nearly mastered the six books of Euclid since he was a member of Congress.

He began a course of rigid mental discipline with the intent to improve his faculties, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring. - *Abraham Lincoln* (Short Autobiography, 1860)

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THE SQUARE ROOT OF ZERO

Michael A.B. Deakin, Monash University.

If we confine ourselves to the study of real numbers, then it is impossible to ascribe a square root to a negative number. In particular, the number −1 has no square root. But we have come to accept that we can have another, richer number system, that of the so-called complex numbers, in which −1 has not just one but two square roots

$$(\pm\, i)^2 = 1.$$

(We need to have two square roots to achieve consistency with real arithmetic, e.g. $(\pm\, 2)^2 = 4$, etc.)

Complex numbers are the numbers of the form $a + ib$, where a and b are real numbers.

At first, complex numbers were regarded with some suspicion, but in due course they became accepted.

Complex algebra leads to very elegant theorems and results and in some ways is simpler than real algebra. All the familiar rules apply, including the following:

> If c_1 and c_2 are complex numbers and $c_1 c_2 = 0$, then either $c_1 = 0$ or $c_2 = 0$.

In complex algebra, every quadratic equation has exactly two roots (although in some cases these may be equal), every cubic equation has exactly three roots (again with possible equalities), every quartic equation has ..., and so on. These properties are not shared by real numbers.

In 1873, the British mathematician Clifford considered a different algebra, whose elements were of the form $a + \varepsilon b$ where a, b are real numbers and $\varepsilon^2 = 0$. [ε is the Greek letter epsilon.] True, there does not seem to be the same need for these as there is for the complex numbers, as 0 already has a perfectly good square root : itself.

However these new numbers (they are called *dual numbers*) have some applications in mechanics (which is what led Clifford to consider them) and have found other uses since.

First let us see how they work. Let

$$c_1 = a_1 + \varepsilon\, b_1$$

$$c_2 = a_2 + \varepsilon\, b_2$$

be two dual numbers. Then it is required that $c_1 = c_2$ when and only when $a_1 = a_2$ and $b_1 = b_2$. Also we can add them

$$c_1 + c_2 = (a_1 + a_2) + \varepsilon(b_1 + b_2) \tag{1}$$

to form a third dual number, or we could subtract them

$$c_1 - c_2 = (a_1 - a_2) + \varepsilon(b_1 - b_2). \tag{2}$$

Then we might multiply them

$$c_1 c_2 = a_1 a_2 + \varepsilon(a_1 b_2 + a_2 b_1) + \varepsilon^2 b_1 b_2$$

$$= a_1 a_2 + \varepsilon(a_1 b_2 + a_2 b_1) , \tag{3}$$

since $\varepsilon^2 = 0$.

Addition and multiplication obey all the usual rules of algebra bar one: the product of non-zero dual numbers can be zero.

$$(\varepsilon\, b_1)(\varepsilon\, b_2) = 0$$

for all b_1, b_2.

[It might at first sight seem that this equation tells us that $\varepsilon = 0$, but it does not. It is ε^2 that is zero. $\varepsilon\, b_1$ and $\varepsilon\, b_2$ are two dual numbers, distinct (unless b_1 or b_2 is zero) from zero.]

Let us investigate the division of dual numbers. Put

$$\frac{a_2 + \varepsilon\, b_2}{a_1 + \varepsilon\, b_1} = \frac{c_2}{c_1} = c_3 = a_3 + \varepsilon\, b_3$$

and seek to find a_3, b_3.

From the definition of a quotient, we must have

$$a_2 + \varepsilon\, b_2 = (a_1 + \varepsilon\, b_1)(a_3 + \varepsilon\, b_3)$$

$$= a_1 a_3 + \varepsilon(a_1 b_3 + a_3 b_1).$$

Then

$$a_2 = a_1 a_3$$

$$a_3 = \frac{a_2}{a_1} \quad ,$$

unless $a_1 = 0$. We also have

$$b_2 = a_1 b_3 + a_3 b_1$$

$$= a_1 b_3 + \frac{a_2 b_1}{a_1} \quad .$$

So

$$b_3 = \frac{a_1 b_2 - a_2 b_1}{a_1^2} \quad ,$$

again unless $a_1 = 0$.

What we have just done is to show that, except in the case of $a_1 = 0$, we have

$$\frac{a_2 + \varepsilon\, b_2}{a_1 + \varepsilon\, b_1} = \frac{a_2}{a_1} + \varepsilon\, \frac{a_1 b_2 - a_2 b_1}{a_1^2} \quad . \qquad (4)$$

A shorter way of working out the result of dividing c_2 by c_1, that you might prefer, is as follows:

$$\frac{c_2}{c_1} = \frac{a_2 + \varepsilon\, b_2}{a_1 + \varepsilon\, b_1} = \frac{(a_2 + \varepsilon\, b_2)(a_1 - \varepsilon\, b_1)}{(a_1 + \varepsilon\, b_1)(a_1 - \varepsilon\, b_1)}$$

$$= \frac{a_2 a_1 + \varepsilon\, (a_1 b_2 - a_2 b_1) - \varepsilon^2 b_2 b_1}{a_1^2 - (\varepsilon\, b_1)^2}$$

$$= \frac{a_2 a_1}{a_1^2} + \frac{\varepsilon(a_1 b_2 - a_2 b_1)}{a_1^2}$$

since $\varepsilon^2 = 0$,

which reduces to (4).

Formula (4) does not work in the case $a_1 = 0$. We may not divide by any number of the so-called *pure dual* form $\varepsilon\, b$, just as in real algebra we may not divide by zero.

The simplest way to convince oneself that dual algebra is consistent is to note a straightforward correspondence between the dual number $a + \varepsilon b$ and the 2×2 matrix

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad . \tag{5}$$

It is possible, and indeed not difficult, to go through the algebra of such matrices and check all the formulae given above. For example ε corresponds to

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and this matrix, multiplied by itself, gives

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which corresponds to 0. This gives a clear system in which ε,0 are manifestly different.

It is easy to see how Equations (1), (2), (3) translate into this new matrix notation. To check Equation (4), note that as we go over to matrices

$$\frac{a_2 + \varepsilon b_2}{a_1 + \varepsilon b_1} \quad \text{corresponds to} \quad \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}^{-1}$$

(or we could write this matrix product the other way around).

Now, remembering that $a_1 \neq 0$,

$$\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}^{-1} = \frac{1}{a_1^2} \begin{pmatrix} a_1 & -b_1 \\ 0 & a_1 \end{pmatrix}$$

and so

$$\begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix}^{-1} = \frac{1}{a_1^2} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} a_1 & -b_1 \\ 0 & a_1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} a_2 a_1 & a_1 b_2 - a_2 b_1 \\ 0 & a_2 a_1 \end{pmatrix}$$

and this corresponds to Equation (4).

One use for dual numbers is in the analysis of small errors. Suppose, for instance, a rectangle has length a_1 and width a_2 and that the length a_1 has associated with it a (relatively) small error $b_1 \geqq 0$. That is to say, if L is the true length, then

$$a_1 - b_1 \leq L \leq a_1 + b_1 \quad . \qquad (6)$$

Similarly, there is a relatively small error $b_2 \geqq 0$ in the measured width a_2 :

$$a_2 - b_2 \leq W \leq a_2 + b_2 \quad . \qquad (7)$$

The true area LW thus satisfies

$$(a_1 - b_1)(a_2 - b_2) < LW < (a_1 + b_1)(a_2 + b_2) \; .$$

I.e. $a_1 a_2 - (a_2 b_2 + a_2 b_1) + b_1 b_2 \leq LW \leq a_1 a_2$
$+ (a_1 b_2 + a_2 a_1) + b_1 b_2$.

Remember now that b_1 , b_2 are relatively small, and so $b_1 b_2$ is *very* small when compared to $a_1 a_2$. For many practical purposes we may simply neglect the $b_1 b_2$ and write

$$a_1 a_2 - (a_1 b_2 + a_2 b_1) < LW < a_1 a_2 + (a_1 b_2 + a_2 b_1). \qquad (8)$$

If we now compare Inequality (8) with Inequalities (6) and (7) and the way these were set up, we see that we can say that (to a good approximation) the area is $a_1 a_2$ with a relatively small error of $a_1 b_2 + a_2 b_1$.

This may be done very efficiently using dual algebra. Write the length as $a_1 + \varepsilon\, b_1$ and the width as $a_2 + \varepsilon\, b_2$. Then the area is $(a_1 + \varepsilon\, b_1)(a_2 + \varepsilon\, b_1)$ and we can multiply this out to give $a_1 a_2 + \varepsilon\,(a_1 b_2 + a_2 b_1)$, obtaining the estimate of the area and its associated error both in the one calculation.

Dual algebra was initially proposed by Clifford as a means of dealing with certain problems in mechanics and in geometry. This work was taken up by Study in Germany (1903)and by Kotel'nikov in Russia (1895-1899). The best English summary of this work is to be found in Chapters 2, 3 of Brand's *Vector and Tensor Analysis* (New York : 1947) but it is rather too technical to go into here. In particular, it considers the situation where the a,b occurring above are not numbers but *vectors*. (Clifford's original study began, in point of fact, with an even more complicated case.)

Another recent application of dual algebra concerns the foundations of calculus. Newton's original version of this made use of so-called *infinitesimal* quantities: numbers that, seen from some points of view were zero, but from other points of view (for example when we wished to divide by them) were not. This rather unsatisfactory situation was brought to a focus by the philosopher (and bishop) Berkeley in his book *The Analyst* (1734).

Berkeley accepted the *utility* of calculus while pointing to logical inadequacies in its *foundations*. This unsatisfactory state of affairs remained until the work of Cauchy (1789-1857) who succeeded in providing a logically watertight basis for the calculus, by avoiding entirely all reference to infinitesimals. This has remained the standard approach to calculus to this day.

Nevertheless, as Abraham Robinson wrote in his book *Non-Standard Analysis* (Amsterdam : 1966), "in spite of this shattering rebuttal [Bishop Berkeley's argument], the idea of infinitely small or *infinitesimal* quantities seems to appeal naturally to our intuition." Robinson set himself the task of reinstating the infinitesimal, and he succeeded in doing so in a series of researches undertaken in the 1950's.

Beginning with an idea first mooted, but not elaborated, by Newton's great contemporary Leibniz, he succeeded in producing an algebra that included ordinary (real and complex) numbers and infinitesimal quantities in one unified whole. (Much as the system of complex numbers includes real and imaginary numbers in a different such whole.) The approach is now termed *non-standard analysis* and it provides an alternative (but equally rigourous) approach to the foundations of calculus.

Robinson's account is technical and difficult, although it succeeds in its aim of reinstating the infinitesimal as a valid concept. More recently (1982) Dawn Fisher suggested the use of dual algebra. In her version infinitesimals are numbers of the form εb, i.e. neither real numbers nor zero. Whether this much simpler version can do all that Robinson's can is not entirely clear. This is a matter time may perhaps clear up.

Interestingly, although in his initial paper Clifford did treat dual algebra incorporating the element ε, for which $\varepsilon^2 = 0$, he spends much more time on a related system. This is the so-called *duo* or *duplex* algebra involving numbers of the form $a + \omega b$, where $\omega^2 = 1$. [ω is the Greek letter omega.] Again, require that $a_1 + \omega b_1 = a_2 + \omega b_2$ only if $a_1 = a_2$ and $b_1 = b_2$. Duplex algebra proceeds very like of Equations (1), (2) need not detain us at all, while the analogue of Equation (3) is the slightly more complicated

$$(a_1 + \omega\, b_1)(a_2 + \omega\, b_2) = (a_1 a_2 + b_1 b_2) + \omega(a_1 b_2 + a_2 b_1) \tag{9}$$

To divide duplex numbers, proceed as we did in setting up Equation (4). The result is (multiply top and bottom of the fraction on the left by $a_1 - \omega\, b_1$)

$$\frac{a_2 + \omega\, b_2}{a_1 + \omega\, b_1} = \frac{a_2 a_1 - b_2 b_1}{a_1^2 - b_1^2} + \omega\, \frac{a_1 b_2 - a_2 b_1}{a_1^2 - b_1^2} \,. \tag{10}$$

This result holds as long as $b_1 \neq \pm a_1$. We may not, in other words, divide by multiples of $1 \pm \omega$. The product of two non-zero duplex numbers can be zero. This is seen in the equation

$$(1 + \omega)(1 - \omega) = 0 \,. \tag{11}$$

Duplex numbers are less used nowadays than dual numbers, but they find some applications in the theory of relativity and they have a ready interpretation in ordinary algebra : $\alpha + \omega\, b$ corresponding to $a \pm b$, $a - \omega\, b$ to $a \mp b$.

Finally we may mention that both duplex and complex numbers have matrix interpretations similar to that of Matrix (5). We have the following correspondences :

$$a + i\, b \quad \text{to} \quad \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$a + \varepsilon\, b \quad \text{to} \quad \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$$

$$a + \omega\, b \quad \text{to} \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

(Can you see the pattern underlying these matrices?)

CLEAR THINKING ABOUT LIMITS

Alan Pryde, Monash University

Mathematics, ever so slightly abused, can be used to obtain some interesting conclusions. Consider for example the following problem: given a non-negative number L, find all solutions $x \geq 0$ of the equation

$$\sqrt{(x + \sqrt{(x + \sqrt{(x + \dots)}}} = L \quad . \qquad (1)$$

It is probably fairly clear what the left side of equation (1) means - there is an infinite number of square roots to be taken, with x added each time - but to be precise we may proceed as follows. Let $a_1 = \sqrt{x}$, $a_2 = \sqrt{(x + a_1)}$, $a_3 = \sqrt{(x + a_2)}$, ..., $a_n = \sqrt{(x + a_{n-1})}$ for $n > 1$. Then the problem is to find x such that

$$\lim_{n\to\infty} a_n = L \, . \qquad (2)$$

Some might argue, however, that we can solve (1) without mentioning limits. Indeed, squaring (1) we obtain $x + L = L^2$ so that the solution is $x = L^2 - L$.

This seems to be the end of the matter: the problem is apparently solved. However, experimenting with different choices of the number L, we find this is not the case.

$L = 3$: Then $x = L^2 - L = 6$ and equation (1) becomes

$$\sqrt{(6 + \sqrt{(6 + \sqrt{(6 + \dots)}}} = 3 \, . \qquad (3)$$

Indeed $a_1 = 2.449\dots$, $a_2 = 2.907\dots$, $a_3 = 2.984\dots$, $a_4 = 2.997\dots$, ... a sequence apparently approaching $L = 3$.

$L = \frac{1}{2}(1 + \sqrt{5})$: Then $x = 1$ and equation (1) becomes

$$\sqrt{(1 + \sqrt{(1 + \sqrt{(1 + \dots)}}} = \tfrac{1}{2}(1 + \sqrt{5}) \qquad (4)$$

which is the answer to problem 10.3.2 of this magazine.

$L = 1$: Then $x = 0$ and equation (1) becomes

$$\sqrt{(0 + \sqrt{(0 + \sqrt{(0 \dots}}}) = 1 \, . \qquad (5)$$

$L = 0$: Again $x = 0$, though equation (1) becomes

$$\sqrt{(0 + \sqrt{(0 + \sqrt{(0 + \ldots)}}} = 0 . \tag{6}$$

Now (5) and (6) cannot both be true and so we are faced with a dilemma. Clearly (6) is correct and not (5). Therefore our solution $x = L^2 - L$ is not valid for all choices of $L > 0$, and there must be doubt as to its validity for any particular L. For example, can we be sure that (3) and (4) are correct? On the other hand, the computations that led to the solution $x = L^2 - L$ were valid, weren't they?

One of the features of modern mathematics is its ability to resolve such dilemmas. The problem with our solution $x = L^2 - L$ of equation (1) is that we have made a hidden assumption which is not always a valid one : we have assumed that equation (1) does have a solution for any given $L \geqq 0$. In other words, what we have done is to show that if there is a solution of (1) then it has to be $x = L^2 - L$. In reality therefore, we have only half solved our problem. The more difficult half is to determine the values of L for which there is a solution. To do this requires some interesting, perhaps difficult, mathematics, and the final answer is, I think, rather surprising.

Our aim is to determine those L for which equation (1) is satisfied for some $x \geqq 0$. We do this by proving

$$\text{for each } x \geqq 0 , \quad L = \lim_{n\to\infty} a_n \text{ exists.} \tag{7}$$

Clearly when L exists it is $\geqq 0$.

Having done this, our previous computations show that if L denotes the limit then $x = L^2 - L$ or $L^2 - L - x = 0$. So $L = \frac{1}{2}(1 \pm \sqrt{(1 + 4x)})$ and we must also recall that $L \geqq 0$. If $x = 0$ then $L = 1$ or 0; but as we have already observed, (5) is incorrect, and so $L = 0$. If $x > 0$, since $L \geqq 1$, then $L = \frac{1}{2}(1 + \sqrt{(1 + 4x)})$ which, as x varies, assumes all values $L > 1$. So equation (1) is solvable for $L = 0$ and for each $L > 1$. In each case the solution is $x = L^2 - L$.

Our final task is to prove (7).

A very useful theorem says that if $\{a_n\}$ is a sequence which is both increasing (i.e. $a_n \leqq a_{n+1}$ for all n) and bounded above (i.e. there is a number K such that $a_n \leqq K$ for all n) then lim a_n exists.

We now show that our sequence a_n has these two properties, i.e. that it is increasing and bounded above.

First we show it is increasing. Certainly since $x \geqq 0$, $a_1 = \sqrt{x} \geqq 0$. Then $a_1 = \sqrt{x} \leqq \sqrt{(x + a_1)} = a_2$, since $a_1 \geqq 0$; in turn it then follows that $a_2 = \sqrt{(x + a_1)} \leqq \sqrt{(x + a_2)} = a_3$ since $a_1 \leqq a_2$; and so on: once we have shown that $a_{k-1} \leqq a_k$, for any k, we can then use this to show that $a_k = \sqrt{(x + a_{k-1})}$ $\leqq \sqrt{(x + a_{k+1})}$. This procedure shows that, for all n, $a_n \leqq a_{n+1}$. Thus our sequence is increasing.

To show that $\lim\limits_{n\to\infty} a_n$ exists it remains to show that the sequence a_n is bounded above. We show that in fact, for all n, $a_n \leqq \frac{1}{2}(1 + \sqrt{(1 + 4x)})$. We show this step by step just as we showed that the sequence a_n was increasing. Certainly, for $n = 1$, $a_1 = \sqrt{x} = \frac{1}{2}\sqrt{(4x)} < \frac{1}{2}(1 + \sqrt{(1 + 4x)})$; then, using this result, for $n = 2$, we have

$$a_2 = \sqrt{(x + a_1)} \leqq \sqrt{(x + \frac{1}{2}(1 + \sqrt{(1 + 4x)}))}$$

$$= \sqrt{(\frac{1}{4} + \frac{1}{2}\sqrt{(1 + 4x)} + \frac{1}{4}(1 + 4x))}$$

$$= \frac{1}{2}\sqrt{(1 + 2\sqrt{(1 + 4x)} + (\sqrt{(1 + 4x)})^2)}$$

$$= \frac{1}{2}\sqrt{(1 + \sqrt{(1 + 4x)})^2}$$

$$= \frac{1}{2}(1 + \sqrt{(1 + 4x)}) .$$

Thus $a_2 \leqq \frac{1}{2}(1 + \sqrt{(1 + 4x)})$.

Now if you look at the above calculation you see that the only assumption used in showing that $a_2 \leqq \frac{1}{2}(1 + \surd(1 + 4x))$ is that $a_1 \leqq \frac{1}{2}(1 + \surd(1 + 4x))$.

Hence, using $a_2 \leqq \frac{1}{2}(1 + \surd(1 + 4x))$ it immediately follows that $a_3 = \surd(x + a_2) \leqq \frac{1}{2}(1 + \surd(1 + 4x))$; and so on : we next get $a_4 \leqq \frac{1}{2}(1 + \surd(1 + 4x))$, and similarly then for $a_5, a_6, \ldots$. So, for all n , $a_n \leqq \frac{1}{2}(1 + \surd(1 + 4x))$.

Thus $\frac{1}{2}(1 + \surd(1 + 4x)$ is an upper bound for the sequence a_n .

So, by the theorem we quoted, $\lim\limits_{a_n \to \infty} a_n$ exists.

It is by no means uncommon for mathematicians, or those using mathematics, to approach a problem with the unspoken assumotion that there is a solution to a given equation. This is a small abuse of mathematics which frequently will not produce any difficulties. However, as our example above should demonstrate, the technique is not infallible. To overcome the adverse consequences, like equation (5), deeper arguments and more interesting, though perhaps harder, mathematics may be required.

* * * * *

Bertrand Russell

"Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true."

"Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of *anything*, then such and such another proposition is true of that thing. It is essential not to discuss whether the first proposition is really true, and not to mention what the anything is, of which it is supposed to be true."

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SIMPLE RATIONAL NUMBERS

John Mack, University of Sydney

In these days of calculators and computers, decimal notation reigns supreme upon the screen and moreover, what we see is usually truncated or rounded to say 8, 10 or 12 decimal places unless we are using an exact, multiprecision or infinite precision arithmetic package for our calculations. For 'small' integers, assuming our calculating device is decently accurate, (as most are), then we expect the results of arithmetic operations to be exact until we exceed the display size. It then becomes an interesting problem (and one of extreme importance) to decide whether or not one can use one's calculating device to obtain exact answers to problems in integer arithmetic.

For example, if you have a standard scientific calculator, it is fairly easy to discover the highest power of 2 where size will fit exactly into your screen display. If it is, say, 2^{30}, then the challenge is to work out how, using your calculator, plus pencil and paper, you might find the exact value of 2^{40} without doing 'unnecessary' work.

The problem we are going to discuss here is of a different kind and concerns the fact that the decimal representation of rational numbers lying between 0 and 1 is not always finite. While $\frac{1}{8} = 0.125$ is a terminating expansion, $\frac{1}{3} = 0.333\ldots$ is not. It is fairly easy to prove directly that if x lies between 0 and 1 and has a terminating decimal expansion, then x is a rational number which can be written in the form p/q, where the only possible prime divisors of q are 2 or 5 .

With a little more work, it is possible to show that the decimal expansion of a reduced rational fraction $x = p/q$ will always terminate (remember that a fraction p/q is said to be reduced if p and q have no common factors other than $\pm$ 1), unless some prime other than 2 or 5 is a divisor of q and when this occurs then the decimal expansion of x is infinite and 'eventually periodic'. This means that, possibly after an initial block of digits, the rest of the expansion consists of a certain finite sequence of digits which repeats over and over again. For example,

$$\frac{1}{15} = 0.0666\ \ldots.$$

has an initial block consisting of the single digit 0, followed by the digit 6 which repeats, while

$$\frac{5}{44} = 0.11363636 \ldots$$

has an initial block of length 2 followed by the recurring sequence 36, and

$$\frac{1}{41} = 0.02439024 \ldots$$

has no initial block, just the recurring sequence 02439.

Some experimentation with a calculator will help you to guess when there is and when there is not an initial block.

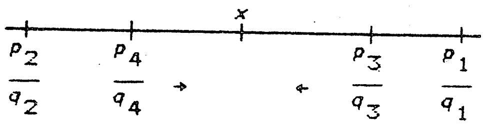
The upshot of all this is that if we are relying on a limited display, decimal representation calculator, then we will often be unable to tell, from what we see, whether or not we actually have a terminating or eventually periodic decimal representation and so unable to tell whether or not we may be dealing with a rational number. This leads to the question "Given a truncated decimal, can we guess whether or not there is a 'simple' rational number that might be close to or equal to it?" By 'simple', we mean one with a denominator small compared with the length of the decimal, for, after all, every finite decimal is a rational number.

For example, suppose we read '0.02439' on our display. This is the rational number 2439/10000, but our question is "Is there a rational number p/q, with q much less than 10000, which could be the exact answer to our problem?" From the example above, we see that, indeed, $\frac{1}{41}$ might well be the actual answer and we could re-examine the steps which led to our calculated answer to see whether $\frac{1}{41}$ is in fact correct.

There is a technique for finding such simple rational numbers. This technique, called the continued fraction method, produces a sequence of rationals with increasing denominators that lie close to a given number. Rather than discuss the technique in general terms, we shall apply it to a specific problem. Afterwards, we'll give some references which contain a fuller account of the method.

Here is the problem. You are at a meeting at which someone, in order to support an argument, says "I have done a survey on this issue and found that 58.6% of those surveyed said 'Yes', while only 31% said 'No', so clearly there is strong support for my proposal." The problem is that the number of persons surveyed is not stated and you would like to know how *small* a sample could be surveyed to produce results as quoted. 58.6% means $\frac{586}{1000}$ or $\frac{293}{500}$, 31% means $\frac{31}{100}$, or $\frac{155}{500}$, so perhaps the survey really covered 500 people, which is an impressive number. But wait a minute. Quoting to 1 decimal place suggests rounding off and so we are looking for percentages in the range 58.55 to 58.64 and 30.95 to 31.04, assuming the speaker is giving accurate estimates. Perhaps there is a denominator q (equal to the survey size) which will produce percentages $100p_1/q$ and $100\,p_2/q$ within the above ranges, with q much less than 500?

To apply the continued fraction method, we need to understand how it works. Given any number x between 0 and 1, it will give us a sequence p_1/q_1, p_2/q_2, ... of rationals, alternately larger than and smaller than x, which close down onto x:



Taking $x = 58.55\% = 0.5855 = 1171/2000$, we obtain its continued fraction as follows:

$$x = \frac{1171}{2000} = \frac{1}{\frac{2000}{1171}} = \frac{1}{1+\frac{829}{1171}}$$

$$= \frac{1}{1+\frac{1}{\frac{1171}{829}}}$$

$$= \frac{1}{1+\frac{1}{1+\frac{342}{829}}}$$

$$= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\frac{829}{342}}}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{145}{342}}}}$$

$$= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\frac{342}{145}}}}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{52}{145}}}}}$$

$$= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\frac{145}{52}}}}}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \frac{41}{52}}}}}}$$

$$= \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\frac{52}{41}}}}}}} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{11}{41}}}}}}}$$

= (after several more such steps)

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\frac{1}{2}}}}}}}}}}$$

The numbers p_n/q_n are found by calculating the values of the intermediate steps:

$$\frac{p_1}{q_1} = \frac{1}{1} = \frac{1}{1}, \quad \frac{p_2}{q_2} = \cfrac{1}{1+\frac{1}{1}} = \frac{1}{2}, \quad \frac{p_3}{q_3} = \cfrac{1}{1+\cfrac{1}{1+\frac{1}{2}}} = \cfrac{1}{1+\frac{2}{3}} = \frac{3}{5},$$

$$\frac{p_4}{q_4} = \frac{7}{12}, \quad \frac{p_5}{q_5} = \frac{17}{29}, \quad \frac{p_6}{q_6} = \frac{24}{41}, \quad \frac{p_7}{q_7} = \frac{89}{152}, \quad \frac{p_8}{q_8} = \frac{113}{193},$$

$$\frac{p_q}{q_q} = \frac{315}{538}, \quad \frac{p_{10}}{q_{10}} = \frac{428}{731}, \quad \frac{p_{11}}{q_{11}} = \frac{1171}{2000}$$ (as it should be!).

The fractions greater than 0.5855, with denominators less than 500, are $\frac{1}{1}$, $\frac{3}{5}$, $\frac{17}{29}$ and $\frac{89}{152}$. Of these, the ones less than 0.5864 are $\frac{17}{29}$ and $\frac{189}{152}$.

We now see if there is a fraction $p_2/29$ lying in the range 0.3095 to 0.3104. Since .3 of 29 is about 9, we try 9/29 and find 9/29 = 0.3103 ... is within the range!

Thus our speaker could have surveyed as few as 29 people and obtained the quoted percentage responses. 29 is not nearly impressive a size as 500 and may well be an unimpressive size in terms of the argument the speaker is supporting!

If you have followed the above calculations, then you would have despaired over calculating the p_n/q_n by successively simplifying the fractions. Fortunately there is an easy way to do it : if we have

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \dots}}}},$$

first calculate $\dfrac{p_1}{q_1} = \dfrac{1}{a_1}$ and $\dfrac{p_2}{q_2} = \dfrac{1}{a_1 + \dfrac{1}{a_2}} = \dfrac{a_2}{a_1 a_2 + 1}$.

Then, miracle of miracles, we can compute p_n/q_n for $n \geq 3$ by the recursive formulae

$$p_n = a_n p_{n-1} + p_{n-2}$$

$$q_n = a_n q_{n-1} + q_{n-2} .$$

Check it out on the above example! If this leads you to believe that it might be true, you can either prove it or find it in an account of continued fractions. This may be found, for example, in H.Davenport's lovely little book *The Higher Arithmetic* published by Hutchinson, (reprinted by Harper and Brothers, New York, 1960), which ought to be in every school library!

Another account can be found in Volume II of C.V.Durell and A.Robson's *Advanced Algebra*, a book that 40 years ago used to be a standard H.S.C. textbook, and so is probably in many school libraries. Continued fractions are first introduced on p.242.

Finally, a note of thanks from the author to Dr Henry Pollack, who recently retired from Bell Research Laboratories in the U.S.A. and who mentioned this nice application of continued fractions to guessing possible sample sizes.

THE JAPANESE ABACUS

An exciting contest between the Japanese abacus and the electric calculating machine was held in Tokyo on November 12, 1946, under the sponsorship of the U.S. Army newspaper, the *Stars and Stripes*. The abacus victory was decisive.

The Nippon Times reported the contest as follows: "Civilisation, on the threshold of the atomic age, tottered Monday afternoon as the 2,000-year-old abacus beat the electric calculating machine by adding, subtracting, dividing and a problem including all three with multiplication thrown in, according to United Press. Only in multiplication alone did the machine triumph."

The American representative of the calculating machine was Pvt. Thomas Nathan Wood of the 240th Finance Disbursing Section of General MacArthur's headquarters, who had been selected in an arithmetic contest as the most expert operator of the electric calculator in Japan.The Japanese representative was Mr Kiyoshi Matsuzaki, a champion operator of the abacus in the Savings Bureau of the Ministry of Postal Administration.

The abacus scored a total of 4 points as against 1 point for the electric calculator. Such results should convince even the most skeptical that, at least so far as addition and subtraction are concerned, the abacus possesses an indisputable advantage over the calculating machine. Its advantages in the fields of multiplication and division, however, were not so decisively demonstrated.

The Abacus Committee of the Japan Chamber of Commerce and Industry says that, in a contest in addition and subtraction, a first grade abacus operator can easily defeat the best operator of an electric machine, solving problems twice as fast as the latter. In multiplication and division the margin of advantage over the electric calculator disappears when there are more than a total of 10 digits involved.

From *The Japanese Abacus*, by Y.Yamazaki, Prentice-Hall, 1965.

* * * * *

WHAT ARE THE ODDS THAT $\binom{n}{r}$ IS EVEN?

Marta Sved, University of Adelaide

You are probably familiar with the expression $\binom{n}{r}$ (read: n choose r), which is the number of ways in which a subcommittee of r members can be chosen out of a committee consisting of n individuals. The symbol $\binom{n}{r}$ is an abbreviation:

$$\binom{n}{r} = \frac{n(n-1)\,..\,(n-r+1)}{1.\,2..\,r}$$

giving the required numbers.

The numbers $\binom{n}{r}$ play a very distinguished role in algebraic expressions. The best known of these expressions is the *binomial* expansion:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + ..+\binom{n}{n-1}ab^{n-1} + b.$$

To make this expression more uniform looking, we define

$$\binom{n}{0} = \binom{n}{n} = 1,$$

which makes sense when we consider that there is exactly one committee with no members and one way of choosing everybody for a subcommittee.

When this is done then the binomial theorem reads

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

You are also likely to be familiar with the *Pascal array* which you obtain when writing down in succession the binomial coefficients occurring in the expansions of $(a+b)^0$, $(a+b)^1$, $(a+b)^2$ and so on. You obtain

1	$n = 0$
1 1	$n = 1$
1 2 1	$n = 2$
1 3 3 1	$n = 3$
1 4 6 4 1	$n = 4$
1 5 10 10 5 1	$n = 5$
1 6 15 20 15 6 1	$n = 6$
1 7 21 35 35 21 7 1	$n = 7$
1 8 28 56 70 56 28 8 1	$n = 8$
1 9 36 84 126 126 84 36 9 1	$n = 9$
1 10 45 120 210 252 210 120 45 10 1	$n = 10$
. . . .	. . . .

This array provides endless fun, if you like to play with numbers. One of the most important properties of the array is expressed by *Pascal's Recursion Formula*:

$$\binom{n+1}{r+1} = \binom{n}{r} + \binom{n}{r+1}.$$

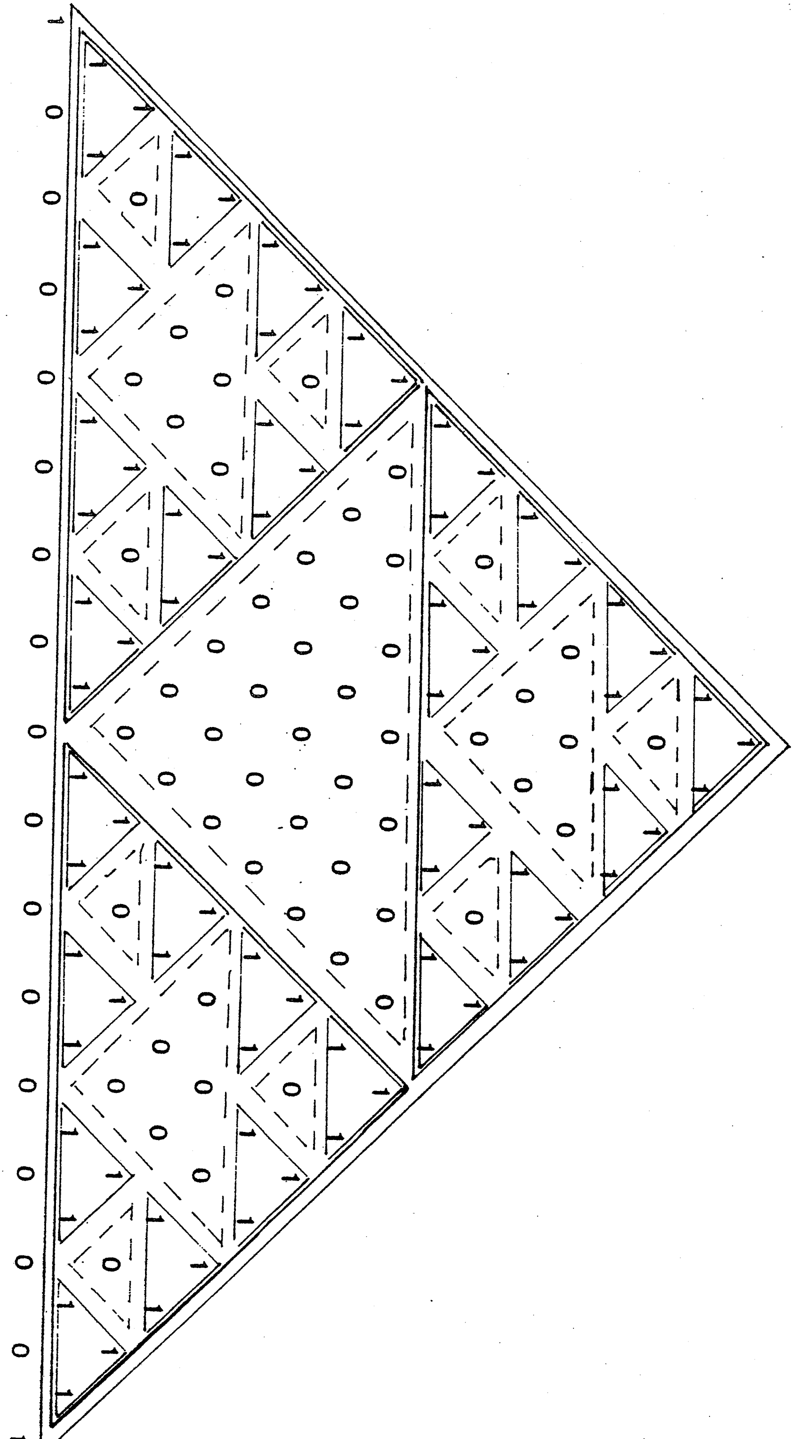
You can check this, (proving is not difficult either), looking at the array and seeing how the entries in each row can be obtained from the entries of the previous row by just adding each time two neighbours. In fact, this is the way of getting *quickly* the coefficients in the expansions of $(a+b)^n$ with increasing n.

The title suggests that we want to deal with the evenness or oddness of these binomial coefficients. A glance at the array already suggests that there is an abundance of even numbers in the Pascal array. Looking for example at rows $n = 2$, $n = 4$, $n = 8$, you find that excepting the first and last numbers in the row, always equal to 1, all the other entries are even. However, when you look at the rows above these rows, that is at rows $n = 1$, $n = 3$, $n = 7$, then you see that all the coefficients are odd (without exception).

There is however a more efficient and also more attractive way of settling the question of evenness and oddness. Instead of writing out the binomial coefficients, we merely state, whether they are even or odd, by writing 1 for the odd coefficients and 0 for the even ones. (Mathematically this means that we write down the *remainders* after dividing each coefficient by 2.) Next we note that we do not even have to look at the Pascal array to obtain our new array, because the Pascal recursion formula gives us an easy rule to obtain each row in succession, since

even + even = even : write $0 + 0 = 0$;
even + odd = odd : write $0 + 1 = 1$;
odd + odd = even : write $1 + 1 = 0$.

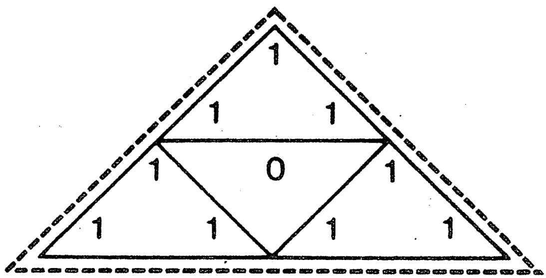
Thus we obtain the *binomial array modulo 2* (see page 57) :



If you have a large enough sheet of paper, you may continue and enlarge the array. It will give you an even better insight into the evenness-oddness problem, and if you paint your 0-Δ's red and your 1-Δ's blue, you can use it as a wallpaper design or pattern for a knitted jersey.

You notice that to the array of ones and zeroes we also added three different types of triangular frames. These may serve as further embellishments, but their real aim is to produce some mathematical results.

Triangles of type  which seem to repeat in a pattern are given a thin frame. Next we note that such triangles which we shall call *cells* seem to cluster on patterns of three:



These *clusters* will be framed by broken lines. Finally, as far as our array goes, we note that these clusters again cluster in threes, giving a more complicated *cluster* which we mark with heavy frames. You may guess that continuing the array you will find that the heavy framed clusters will make similar formations (you may use colours for the new, larger frames) and so you may continue as far as time and paper sheets permits.

What are the conclusions which can be drawn from all this?

Let us first look at the reasons for these regularities. Note that there are unframed spaces filled with zeros, numbering 1 or 6 or 28 in our array. We shall call these spaces *zero-holes*. It follows from our construction that the *cluster arising between the edge of the Pascal array and a zero-hole must be the same as the top-cluster of the same size arising between two edges of the array with its vertex at the apex of the Pascal triangle.* The first zero-hole consisting of a single zero entry is in the row $n = 2$. To its right and left *two cells* appear *identical* to the *cell above the zero entry.* These three cells together form our first cluster. Looking at the last rows of this cluster (replacing the row {1 3 3 1} of the Pascal array), we see that all entries are equal to 1, since this row is the amalgamation of the last rows of two cells.

It follows again from the rule : 1 + 1 = 0 that the row following the last row of the cluster has only zero entries between the two 1-entries at the edges of the whole array. This means that the entries of the Pascal array which this row replaces are even. (They are {4 6 4}. A larger zero hole is now generated consisting of three rows forming an inverted triangle. It replaces

$$\begin{matrix} 4 & & 6 & & 4 \\ & 10 & & 10 & \\ & & 20 & & \end{matrix}$$

of the Pascal array.

Replicas of the cluster above the zero-hole appear on the sides and so we obtain the first heavy framed cluster, with its last row ($n = 7$) consisting of 1's and followed by the next row ($n = 8$) where a new, larger zero-hole, consisting of 7 rows originates. We note now that the first zero-hole begins at $n = 2$, the next *central zero-hole* at $n = 4$, and from the congruency of clusters it follows that at $n = 8$, and generally at $n = 2^k$ new central zero-holes arise, each larger than the preceding one.

We are able to state now that when $n = 2^k$, all binomial coefficients $\binom{n}{r}$ *are even,* provided that $0 < r < n$. Referring to the last rows of clusters, we have

$$\text{if } n = 2^k - 1 \text{ , } \binom{n}{r} \textit{ is odd for all } r \text{ .}$$

These two last results can be proved by algebra, but it is interesting to see how they follow from the geometry of our *modulo 2 array.*

Another consequence, not so well known is that *for* $n < 2^{th}$, *the number of odd binomial coefficients* $\binom{n}{r}$ is 3^k .

Looking again at our picture, we see that for $n < 2$ there are 3 odd binomials, namely

$$\binom{0}{0} = 1 \ , \ \binom{1}{0} = \binom{1}{1} = 1 \ .$$

Since the cluster for $n < 4$ contains three such cells, there are 9 odd binomial coefficients $\binom{n}{r}$ when $n < 2^2$.

Generally : the array for $n < 2^k$ contains 3 clusters congruent to the cluster of entries for $n < 2^{k-1}$, hence 3 times as many 1-s , which signify odd entries in the Pascal array.

Thus : up to $n = 8$, there are $3 \times 9 = 27$ odd binomial coefficients , up to $n = 16$, there are 81 (you can check this), and so on.

We can conclude immediately that the probability of a binomial being even, approaches 1 (it is "almost certain") as the Pascal array increases, although for arbitrarily large n we shall always have completely odd rows, namely when $n = 2^k - 1$.

To be more precise, we find the numbers of even binomial coefficients $\binom{n}{r}$ when $n < 2^k$.

The total numbers of entries in the Pascal array for $n < 2^k$ is

$$N = 1 + 2 + 3 + .. + 2^k = \frac{2^k(2^k + 1)}{2} = 2^{2k-1} + 2^{k-1} \ .$$

Since the number of odd binomials is 3^k, it follows that the number of even coefficients is

$$N_E = 2^{2k-1} + 2^{k-1} - 3^k \ .$$

Hence the probability that in the range considered a binomial coefficient is even is

$$P(E) = \frac{N_E}{N} = 1 - \frac{3^k}{2^{2k-1} + 2^{k-1}}$$

Thus

$$P(E) > 1 - \frac{3^k}{2^{2k-1}} = 1 - 2\left(\frac{3}{4}\right)^k \ ,$$

where for large enough k , $2\left(\frac{3}{4}\right)^k$ can be made as small as we wish. Thus $P(E)$ can be made as near to 1 as we wish by choosing k appropriately large.

PERDIX

The Australian Mathematical Olympiad for 1987 was held on the 3rd and 4th of March. Seventy-one selected competitors took part in the Olympiad, most chosen because of their performances in the Australian Mathematics Competition. Some had shown their ability in other competitions, and a few had been accepted on the strong recommendation of their mathematics teachers.

For success in the Australian Olympiad, 11 gold certificates, 15 silver certificates and 21 bronze certificates were awarded. The remaining competitors all received participation certificates.

How would you have fared in this Olympiad? The papers set are appended. Have a go! Send me any of your solutions, or any queries you have about the questions.

The Australian team for the International Mathematical Olympiad, this time being held in Havana, Cuba, is selected by taking the best six performers in the Australian Mathematical Olympiad. This year's team (in alphabetical order) is:

Chung Kim Yan,	Duncraig Senior High School, Western Australia.
Jonathan Potts,	Brisbane Grammar School, Queensland.
Ben Robinson,	Narrabundah College, Australian Capital Territory.
Luke Seberry,	Sydney Grammar School, New South Wales.
Terence Tao,	Blackwood High School, South Australia.
Ian Wanless,	Phillip College, Australian Capital Territory.

As usual a reserve member of the team was also selected and he is:

Danny Culegari,	Melbourne Grammar School, Victoria.

There is still intensive training ahead for the team. A highlight of this training will be a week in Sydney at the IBM training school. At this school the team, with its reserve, will be joined by eight others, selected also from their performances in the Australian Mathematical Olympiad, and who also are in year 11 or below at school, so that they will still be eligible for selection for the 1988 Olympiad Team.

Those selected to join the school in May in addition to the team and its reserve are:

Geoffrey Bailey,	St Aloysius College. New South Wales.
Robert Gates,	Knox Grammar School, New South Wales.
David Jackson,	Sydney Grammar School, New South Wales.
Mark Kisin,	Melbourne Grammar School. Victoria.
Jeremy Liew,	Duncraig Senior High School, Western Australia.
Martin O'Hely,	Salesian College, Victoria.
Blair Trewin,	Canberra Grammar School, Australian Capital Territory.
Sam Yates,	St Peter's College, South Australia.

The questions in the papers that follow provide difficulties of two kinds. The first kind of difficulty is that of understanding what the problem is. If you are not certain what the circumcircle of a triangle is then you cannot be sure you understand question 1. If you do not know what a prime number is, then you cannot do question 2.

The second kind of difficulty is that of solving the problem once it has been completely understood. Here there are at least two factors that come into most solutions. The first is knowledge. This knowledge can be a knowledge of related mathematical results some of which might be necessary to use in solving the problem. For example, in question 6, you need to be familiar with proofs by mathematical induction to be sure-footed in your solution. Familiarity with the properties of similar triangles helps for question 1.

Let us assume you have all the necessary knowledge. It is then that the main (and intended) difficulty arises: conjure up sufficient ingenuity to put what you know together to lead to a solution.

THE 1987 AUSTRALIAN MATHEMATICAL OLYMPIAD

PAPER I

Tuesday, 3rd March, 1987

Time allowed: 4 hours

NO calculators are to be used.
Each question is worth seven points.

Question 1

GKA is an isosceles triangle with base GK of length $2b$. GA and AK each have length a. Let C be the midpoint of AK and z the circumcircle of the triangle GCK. Let Y be the point on the extension of AK such that if E is the intersection of YG with z then EY is of length $a/2$.
Prove that if x is the length of EC and y is the length of KY then $ay = x^2$ and $xb = y^2$.

Question 2

Let p be a prime number. Show that the integer

$$\binom{2p}{p} - 2 = \frac{2p(2p-1)\cdots(p+1)}{p(p-1)\cdots 1} - 2$$

is a multiple of p.

Question 3

In the country *Patera* there are 20 cities and two airline companies, *Green Planes* and *Red Planes*, to provide communication between the cities. The flights are arranged as follows:

(i) Given any two cities in *Patera*, one and only one of the companies provides direct flights (in both directions and without stops) between the two cities.

(ii) There are two cities A and B in *Patera* such that a journey cannot be made from one to the other (with possible stops) using only *Red Planes*.

Show that, given any two cities in *Patera*, a passenger can travel from one to the other using only *Green Planes*, making at most one stop in some third city.

THE

1987

AUSTRALIAN MATHEMATICAL OLYMPIAD

PAPER II

Wednesday, 4th March, 1987

Time allowed: 4 hours

NO calculators are to be used.
Each question is worth seven points.

Question 4

In the interior of the triangle ABC, points O and P are chosen such that angles ABO and CBP are equal, and angles BCO and ACP are also equal.
Prove that angles CAO and BAP are equal.

Question 5

Let m and n be (fixed) integers greater than 1, m even, and f a real-valued function, defined for all non-negative real numbers, that satisfies the following conditions:

(i) For all $x_1, x_2, \ldots, x_n$,
$f((x_1^m + x_2^m + \cdots + x_n^m)/n) = ([f(x_1)]^m + [f(x_2)]^m + \cdots + [f(x_n)]^m)/n;$

(ii) $f(1986) \neq 1986;$

(iii) $f(1988) \neq 0.$

Prove that $f(1987) = 1.$

Question 6

Prove that for each positive integer n $(n > 1)$,

$$\sqrt{n+1} + \sqrt{n} - \sqrt{2} > 1 + 1/\sqrt{2} + 1/\sqrt{3} + \cdots + 1/\sqrt{n}.$$