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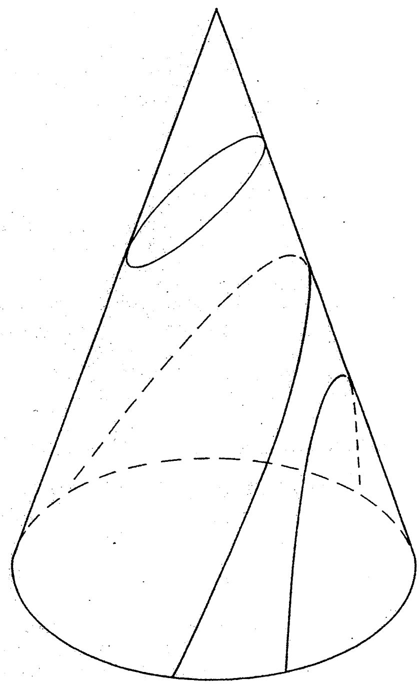

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Function is a mathematics magazine addressed principally to students in the upper forms of schools, and published by Monash University.

It is a "special interest" journal for those who are interested in mathematics. Windsurfers, chess-players and gardeners all have magazines that cater to their interests. *Function* is a counterpart of these.

Coverage is wide - pure mathematics, statistics, computer science and applications of mathematics are all included. There are articles on recent advances in mathematics, news items on mathematics and its applications, special interest matters, such as computer chess, problems and solutions, discussions, cover diagrams, even cartoons.

Function does *not* aim directly to help its readers pass examinations. It does guarantee to enrich and inform their mathematical lives, to provide high quality articles and stimulating reading for the mathematically inclined.

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THE FRONT COVER

M.A.B. Deakin, Monash University

If a cone is sliced through, the resulting cross-section is a curve called a *conic section*. Our cover shows three types of conic section: the ellipse, the parabola and the hyperbola. Which type arises depends on the angle between the plane of the cut and the side of the cone.

The Greeks began this study and the theory given above was known to Menaechmus (4th century B.C.), a pupil of Plato's. Euclid wrote extensively on the conic sections but these works have not survived. The slightly later (3rd century B.C.) work of Apollonius has, however, come down to us and it is an impressive body of work. Later work by Pappus of Alexandria (about 300 A.D.) extended the theory developed by Apollonius.

Very little happened to expand on this body of knowledge until the 17th century A.D., however. In that time developments of both a practical and a theoretical nature began to take place.

On the observational side, Kepler in 1609 published his first law of planetary motion: that the planets move in ellipses with the sun at one focus. (The ellipse has the property that there are two points, the *foci*, in its interior, such that light emitted from one is reflected off the ellipse into the other.)

Later (about 1680), Newton showed that this implied that gravity obeys an inverse square law. If we do this calculation in reverse, and ask what orbit a body attracted to a larger one by such a force must follow, we find that the path must be a conic section. That is to say that it could be elliptical, but it could also be parabolic or hyperbolic.

The conic sections in fact belong to a single family of curves, distinguished only (apart from the scale on which they are drawn) by a single parameter called e (standing for "eccentricity" - in an ellipse, it measures the distance of the foci from the centre). If $0 < e < 1$ the conic section is an ellipse, if $e = 1$ we have a parabola, and for $e > 1$ the curve is hyperbolic. The very special case $e = 0$ is a circle - a very particular type of ellipse.

If e is small, the ellipse is nearly circular, and this is the case with the planetary orbits. In the case of Mars, $e \simeq 0.09$ and this corresponds to a "flattening" of 0.4% in the orbit - i.e. its longest diameter is only 0.4% larger than its shortest. It was Mars whose orbit Kepler succeeded in computing and we see from these figures that this was a considerable feat, requiring very precise data. Luckily this was to hand following the work of the Danish astronomer Tycho Brahe, whose pupil Kepler had been.

Only two planets have more eccentric, i.e. more pronouncedly elliptical, orbits than Mars. These are Mercury ($e \simeq 0.20$) and Pluto ($e \simeq 0.25$). However, Kepler had no knowledge of the second of these and the first is very hard to observe as it lies so near the sun. (Copernicus' dying words are said to have been "It grieves me to die without ever having seen Mercury".)

For an ellipse, the eccentricity defines the shape; higher values of e corresponding to longer, flatter ellipses. The first notably eccentric orbit recognised as such was Halley's comet, whose return to visibility this year has excited so much interest.

In 1750, Halley, after whom the comet is named, suspected that it moved in a highly eccentric orbit, which nonetheless remained elliptical. It is now a matter of record that he was correct in this. The comet follows an orbit for which $e \simeq 0.97$.

In fact Halley's comet is one of a quite small number that remain relatively close to the sun. It goes to a farthest point somewhat beyond the orbit of Neptune before swinging back to visit us again.

Comets typically (there are exceptions) have values of e very near 1. This means that their orbits are very nearly parabolas. (It is very unlikely that in any given case e will be precisely equal to 1.) This has been taken to mean that they come from very great distances and may or may not visit us periodically.

To travel these distances requires very great times (e.g. Delavan's comet, 1914, was calculated to have an elliptic orbit that took it round the sun every 24 million years). A lot can happen in 24 million years and really all we can say is that prediction over such eons is a very uncertain business.

The theoretical orbits are altered by interactions with planets and it is thought that this process "captured" Halley's and other close comets perturbing them out of their former long period paths. It is believed also that the hyperbolic orbits are due to the same process of interaction but that here it has accelerated the comets to speeds that will ultimately (unless some further interaction occurs) eject them from the solar system altogether.

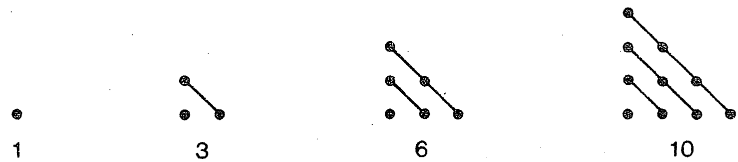
So there is a lot we still don't know and a lot that we do. Perhaps the most surprising thing is that a purely geometric theory, developed out of mathematical curiosity over 2000 years ago, should turn out to be of importance in so different a field of research.

POLYGONAL, PRIME AND PERFECT NUMBERS

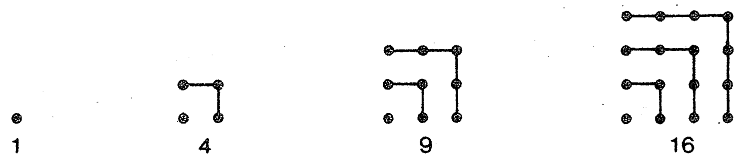
J.C.Stillwell, Monash University

This article is devoted to some miscellaneous topics in Greek number theory which are intriguing but somewhat outside of the mainstream of mathematics.

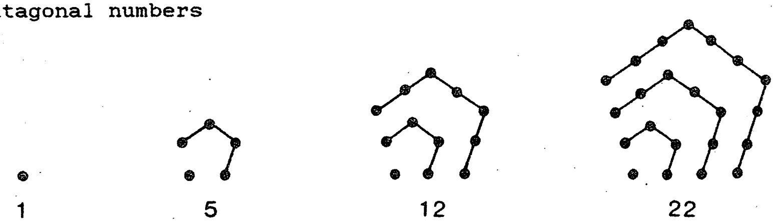
The *polygonal numbers* were studied by the Pythagoreans, and result from a naive transfer of geometric ideas to number theory. One has, e.g., the triangular numbers:



square numbers



pentagonal numbers



etc. From these diagrams it is an easy exercise to calculate an expression for the mth n-agonal number as the sum of an arithmetic series and to show, e.g., that a square is the sum of two triangular numbers. Apart from Diophantus' work, which contains impressive results on sums of squares, Greek results on polygonal numbers were of this elementary type.

On the whole, the Greeks seem to have been mistaken in attaching much importance to polygonal numbers. There are no major theorems about them, except perhaps the following two. The first is the theorem conjectured by Bachet [1621] (in his edition of Diophantus' works) that every positive integer is the sum of 4 integer squares. This was proved by Lagrange [1770]. A generalisation, which Fermat [1670] stated without proof, is that every positive integer is the sum of n n-agonal numbers. This was proved by Cauchy [1815], though the proof is a bit of a let down because all but 4 of the numbers can be 0 or 1. The other remarkable theorem about polygonal numbers is a formula proved by Euler [1750] and known as Euler's pentagonal number theorem, since it involves the pentagonal numbers.

[The 4 square theorem and the pentagonal number theorem were both absorbed into a much larger theory around 1830, Jacobi's theory of theta functions.]

The *prime numbers* were also considered within the geometric framework, namely as the numbers with no rectangular representation. A prime number, having no factors apart from itself and 1, has only a "linear" representation. Of course, this is no more than a restatement of the definition of prime, and most theorems about prime numbers require much more powerful ideas, however the Greeks did come up with one gem. This is the proof that there are infinitely many primes, in Book IX of Euclid's *Elements*.

Suppose on the contrary that there are only finitely many primes $p_1,\ldots,p_n$. But then, what about

$$p = p_1 p_2 \ldots p_n + 1 \ ?$$

This number is larger than $p_1,p_2,\ldots,p_n$ yet could have no factors, since it leaves remainder 1 when divided by any prime p_i. Thus p is a prime number different from $p_1,\ldots,p_n$. Contradiction!

A *perfect number* is one which equals the sum of its factors (including 1 but excluding itself). E.g. $6 = 1 + 2 + 3$ is a perfect number. So is $28 = 1 + 2 + 4 + 7 + 14$. Although this is a concept which goes back to the Pythagoreans, only two noteworthy theorems about perfect numbers are known. The first is in Euclid and the second is a converse, proved by Euler [1849]. Euclid concludes Book IX of the *Elements* by proving that if $2^n - 1$ is a prime, then $2^{n-1}(2^n - 1)$ is a perfect number. These perfect numbers are of course even, and Euler proved that every even perfect number is of Euclid's form. *It is still not known whether there are any odd perfect numbers.* In view of Euler's theorem, the existence of even perfect numbers depends on the existence of primes of the form $2^n - 1$, known as Mersenne primes. *It is not known whether there are infinitely many Mersenne primes*, though larger and larger ones seem to be found quite regularly. In recent years, each new world record prime has been a Mersenne prime, giving a corresponding world record perfect number.

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IN OBTUSE ANGLE'S STUDY

This excerpt is by the poet and artist William Blake. It is taken from Chapter 5 of his manuscript fragment *An Island in the Moon*.

Obtuse Angle, Scopprell, Aradobo, & Tilly Lally are all met in Obtuse Angle's study.

'Pray,' said Aradobo, 'is Chatterton a Mathematician?'

'No,' said Obtuse Angle. 'How can you be so foolish as to think he was?'

'Oh, I did not think he was - I only ask'd,' said Aradobo.

'How could you think he was not, & ask if he was?' said Obtuse Angle.

'Oh no, Sir. I did think he was, before you told me, but afterwards I thought he was not.'

Obtuse Angle said, 'In the first place you thought he was, & then afterwards when I said he was not, you thought he was not. Why, I know that - '

'Oh no, sir, I thought that he was not, but I ask'd to know whether he was.'

'How can that be?' said Obtuse Angle. 'How could you ask & think that he was not?'

'Why,' said he, 'it came into my head that he was not.'

'Why then,' said Obtuse Angle, 'you said that he was.'

'Did I say so? Law! I did not think I said that.'

'Did not he?' said Obtuse Angle.

'Yes,' said Scopprell.

'But I meant - ' said Aradobo, 'I - I - I can't think. Law! Sir, I wish you'd tell me how it is.'

Then Obtuse Angle put his chin in his hand & said 'Whenever you think, you must always think for yourself.'

'How, sir?' said Aradobo. 'Whenever I think, I must think myself? I think I do. In the first place - ' said he with a grin.

Continued on page 17.

ARE WE RELATED, YOU AND I?

G.A.Watterson, Monash University

I inherited my genes from my parents. One half of my genes come from my mother, and the other half from my father. I am not exactly like either of them, but I do bear some visible similarities with both of them. You and I are not brothers or sisters ("sibs"), that is, as we do not have the same parents, we are not *very* closely related. But we *are* related, if our ancestry is traced back far enough. Sometime back in the remote past, you and I would have shared a common ancestor.

How far back would we have to go in order to find a common ancestor? Of course if your surname is Watterson, you may be related to me through a quite recent common ancestor on our fathers' side. If you have a different surname, nevertheless we could be related by a common ancestor on our mothers' side, or, for that matter, on our fathers' side, or even on both sides.

Let us investigate this problem using very simple assumptions. The answer we get will not be quite right, because the assumptions are not quite right. But the way of tackling the problem is the way theoretical geneticists study the theory of evolution.

Suppose a gene is chosen from me (say, one of my two genes which determine whether my blood-group is A, B, AB or O). And suppose one of your blood-group genes is also chosen. Your gene and my gene are copies of genes present in the population of our parents' generation. Suppose that that population consisted of N people. There would have been $2N$ blood-group genes amongst them, as each person has two such genes.

If my gene was a copy of one of those $2N$ genes, chosen at random, then my gene would have probability $1/2N$ of being a copy of the *same* gene that your gene is a copy of. [Of course, we happen to know that that didn't happen in our particular case, because I don't have any "sibs". But if you and I were replaced by two randomly chosen people, and if mating had been at random among our parent generation, then the argument would apply more accurately.]

So let us write

$$p = \frac{1}{2N}$$

for the probability that our two genes are related because they are copies of the same gene one generation ago. If, in fact, our two genes are copies of two distinct genes one generation ago, even so those two genes might themselves have a common parent gene (in our grandparents' generation). The probability, that our two genes are *not* related with respect to our parents' generation but *are* related with respect to our grandparents' generation, is

$$(1 - p)p = \left[1 - \frac{1}{2N}\right] \frac{1}{2N} .$$

Arguing in a similar way, we can say that the probability that our two genes are both descended from a common ancestor gene x generations ago, but not more recently, is

$$\underbrace{(1 - p)(1 - p)(1 - p) \ \dots \ (1 - p)}_{x-1 \text{ terms}}p = (1 - p)^{x-1}\, p .$$

Call this probability $P(x)$. We have shown that

$$P(x) = (1 - p)^{x-1}\, p ,$$

which applies for $x=1, 2, 3, \dots$.

The probabilities $P(x)$ form a geometric sequence, because

$$P(1) = p , \quad P(2) = (1 - p)p , \quad P(3) = (1 - p)^2 p , \quad \dots$$

so that each term is $(1 - p)$ times the previous term in the sequence.

Earlier, I said that you and I are definitely related. Under the above assumptions, I can prove that statement. Let us agree that we *are* related if our two genes *do* have a common ancestral gene x generations ago, for some $x=1$ or 2 or 3 or ... . The probability of that is

$$P(1) + P(2) + P(3) + \dots$$
$$= p[1 + (1 - p) + (1 - p)^2 + \dots] .$$

If we agree that we can go back into the past for an infinitely long period, this series is an infinite geometric series, whose sum is

$$p \frac{1}{1 - (1 - p)} = p \frac{1}{p} = 1 .$$

So the probability that our genes (and we) are related *sometime* in the past is 1.

You might have noticed in the above argument that I have assumed that the population remained of size N people every generation. If so, the same value of p applies for the probability that two genes have the same parent gene, whatever generation we are talking about. Actually, if the population size were smaller and smaller as we go back in the past, p would get larger and larger and our chances of having a common ancestor fairly recently would increase. Of course the limiting value which we found above, 1, cannot increase! It is the highest any probability can be.

We may have to go back a very large number of generations before your gene and mine shared a common ancestor. The average ("mean" or "expected") number of generations whould be

$$\begin{aligned} \text{average} &= 1\times P(1) + 2\times P(2) + 3\times P(3) + \ldots. \\ &= p[1 + 2(1-p) + 3(1-p)^2 + \ldots.] \quad \text{generations} . \end{aligned}$$

Here, we have considered the possible numbers 1, 2, 3, ... , multiplied them by the probabilities of their being the case, and summed the result. To work out the answer, you will recognise that

$$1 + 2x + 3x^2 + \ldots$$

is the derivative

$$\frac{d}{dx}(x + x^2 + x^3 + \ldots)$$

which also equals

$$\begin{aligned} \frac{d}{dx}(1 + x + x^2 + x^3 + \ldots) &= \frac{d}{dx}\left[\frac{1}{1-x}\right] \\ &= \frac{1}{(1-x)^2} . \end{aligned}$$

So with x replaced by $1 - p$, we find

$$\text{average} = p\,\frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p} = 2N .$$

This result shows that, if the population always contained, say, N=15,000,000 people, then on average it would be 30,000,000 generations ago before we would find a common ancestor for two of the genes at present in the population. And if we take 20 years (say) as being one generation, then we sould have to go back 600,000,000 years to find a common ancestor. Of course it may have happened more recently, or even more remotely in the past, than this average value.

Incidently, some scientists say that the present human race has been a species, separate from other species, for about one million years. So we might have to go back to some earlier life-form to find our common ancestor gene! Other people argue that all humans descend from Adam, so we need not go further back than him for find our common ancestor gene.

PARALLEL COMPUTING: BREACHING THE FRONTIER OF TIME†

Anthony Maeder, Monash University

Computers are usually thought of as extremely fast electrical machines capable of performing millions of calculations or instructions every second. This behaviour so greatly exceeds human abilities that it may seem pointless to consider ways of making computers run even faster. However, there are some tasks where attaining a particularly high speed of computation is essential because only a short period of time is available until the answer is needed. For example, a program controlling the flight path of a spacecraft must solve sets of complicated mathematical equations based on the craft's current spatial position in time to make a course correction before the craft has moved too far from that position. This urgency occurs in other control situations such as production lines, chemical factories and nuclear reactors. High speeds of computation are also required when processing large amounts of data, for instance the radar or satellite information used in weather forecasting. A few years ago, it was not possible to produce an accurate 24 hour forecast based on all the available data for the Northern Hemisphere in less than a day! This situation has now changed due to the use of parallel computing.

One way of speeding up computers is to design them to do several steps of a computation simultaneously. If certain steps do not affect each other at all, they can be done at the same time in different parts of the computer and the results combined later. Consider the simple problem of adding 10 numbers together. A conventional computer would add the first two, then add the third to that sum, the fourth to the new sum and so on, making the final sum available after 9 steps. In a parallel computer capable of simultaneous additions of different numbers, 5 pairs of numbers could all be added in one step, pairs of their sums added together when available and so on, reaching the answer after only 4 steps. This form of parallel computing is probably the most powerful but has some disadvantages. It is often difficult to break up a computation easily into parallel sections. In many cases, a whole new way

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† See also *Function*, *Vol*.5, *Part* 1, p.13

of doing the computation (an algorithm) must be designed to allow parallelism to be used effectively. This is because the original algorithms were designed with conventional computers in mind, where the grouping of steps is less important. Another disadvantage is that many electrical components are needed to build computers able to perform multiple instructions simultaneously on different data elements, adding to the complexity and cost of the machine.

In view of these difficulties, compromises have often been made when constructing new parallel computers. Such machines are suited only to a particular type of parallel behaviour and will be very fast if algorithms of that type are used. The most common parallel computer is the *vector* or *array processor*. This is based on the idea that many computations consist of the same operation applied to a set of several data items, achieved by stepping through the operation with each data item in turn on a conventional computer. A vector processor would apply the operation to each element of the data set at once. As the name suggests, these computers are well suited to vector and matrix arithmetic, which forms a large part of mathematical computing. The previous example of a parallel algorithm will work on a vector processor by first forming two vectors, each containing 5 data items, adding them and then continuing the process with shorter vectors. Of course, the shorter the vectors used, the less effective use is being made of the computer's processing power. Another kind of compromise is made when computer instructions are overlapped rather than each one being completed before the next one commences. In the example of adding 10 numbers one after another, the method used by the computer to add the first pair of numbers might calculate the rightmost digit of the sum initially and then form the next digits. In this case work could begin on adding the sum to the third number before the sum had been completely worked out! Computers based on this principle are called *Pipeline* machines, because they usually have several partly completed operations 'in the pipeline' at any moment in time.

A number of very powerful computers based on these and similar ideas are now being produced. Some of these are so much faster than any conventional computer that they are termed *supercomputers*. They are usually used for long mathematical calculations, such as simulating a natural phenomenon or producing successive frames of an animated movie, which would occupy conventional machines for hours or even days. Two supercomputers, the Cray and the Cyber 205, are in everyday use in Australia for just these purposes!

What is the next step? A lot of research effort has been put into automatically finding parts of programs which can be computed in parallel and now some experimental computers which work on this principle have been built. The existence of parallel computing has aided the development of artificial intelligence, a process requiring the evaluation of a large number of statements of factural information. A new generation of computers based on *parallel logical* rather than *mathematical operations* is now being built. While there is still a long way to go before parallel computing is commonly available and useful, it has certainly served its original purpose for various types of computations: to break through the barriers of time.

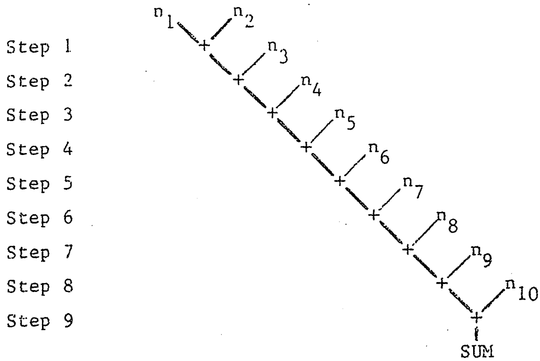

Figure 1. Serial addition of 10 numbers

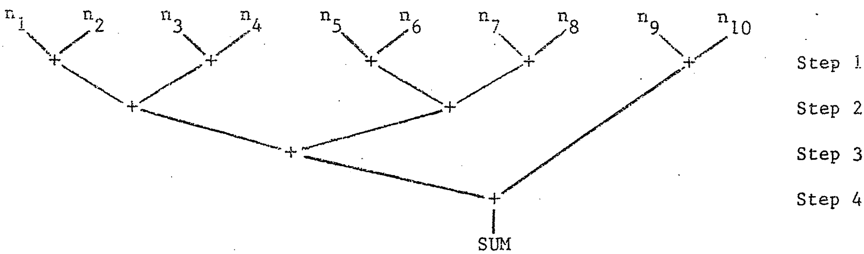

Figure 2. Parallel addition of 10 numbers

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix} + \begin{bmatrix} n_6 \\ n_7 \\ n_8 \\ n_9 \\ n_{10} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{bmatrix} \qquad \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} + \begin{bmatrix} s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} s_6 \\ s_7 \end{bmatrix} \qquad s_5 + s_6 + s_7 = \mathrm{SUM}$$

Figure 3. Vector addition of 10 numbers

THE GROWTH OF FERN FRONDS†

Stephen Murphy, Year 6, Wesley College

Ferns are common in nature. Most live in moist shady situations and some live entirely in water. Usually they are found in regions of high rainfall.

The largest species are the tree ferns which may reach a height of eight metres. The leaves of some tree ferns are more than three metres in length.

The leaf, or *frond*, is the largest part of most ferns. It is usually the only part of the plant to be seen above the soil. Tree ferns are unusual in that the stem is erect and trunk-like, bearing a thick crown of leaves at the top.

Ferns have spores on the undersides of their leaves. When released, these spores germinate in four to eight weeks if the conditions are moist and cool. When they germinate, the result is a small flat plant, often resembling a heart in its shape. After a few weeks this small, heart-shaped plant dries up as the fern develops.

A furled frond grows out from the *crown*, as it is called. In time this will develop into the stem of the frond, with other, feather-shaped, "frondlets" attached to it. These are referred to as the *secondary rachis*. At first, these too develop as spirals along the length of the crozier.

This, the growth of the spiral crozier, is the first of three growth stages I identified. It gives way to the second and most rapid stage as the frond unwinds, so that the frond increases in length while the crozier (the curled portion) gets smaller and smaller until the unwinding is complete.

There is then a third stage when the frond is completely uncurled. In this stage some tip growth takes place before the frond stops growing altogether.

I studied the growth of a tree fern during the spring and the autumn and found that the autumn growth was much quicker than the spring growth, but that the resulting leaves were not as long.

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† Article based on a project selected as a finalist in this year's BHP Science Prize.

For a short period, I studied day-night patterns of growth and found that tree fern fronds continue to grow at night, but day growth is much faster.

The sort of growth I found is illustrated in Figure 1. This plots the spring growth of four different fronds on the same tree fern. They all grow fairly slowly for the first

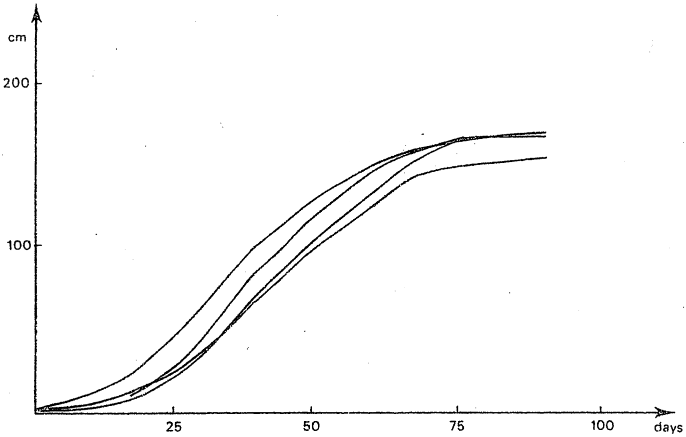

Figure 1.

twenty days and then rapidly until about Day 60. After seventy-five days the growth had almost come to a halt with the secondary rachis fully grown at about 180-190 cm in length.

Figure 2 shows an idealised form of this growth so that the various stages are readily identifiable. Also drawn on the same diagram is the radius of the crozier. This goes to zero at the end of the second stage.

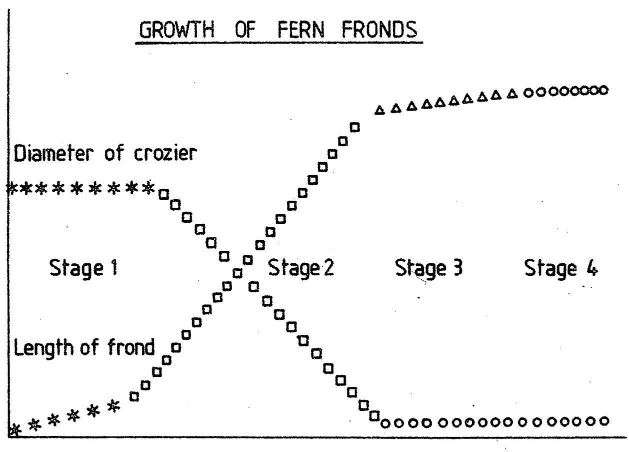

Figure 2.

Figure 3 shows a frond in the second stage of growth. For easier visibility the secondary rachis have been removed. In Figure 4, I show the shape of the frond at this time. This was drawn by carefully tracing around the frond and then using a French curve to "even out" the appearance of the diagram.

To simulate the shape of the crozier at this stage, I used the turtle graphics package to produce a computer-drawn equiangular spiral. This gives a good fit as Figure 5 shows. (Compare its shape with that shown in Figure 4.)

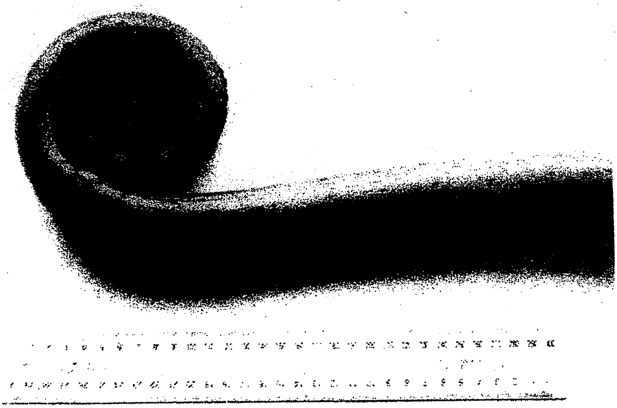

Figure 3.

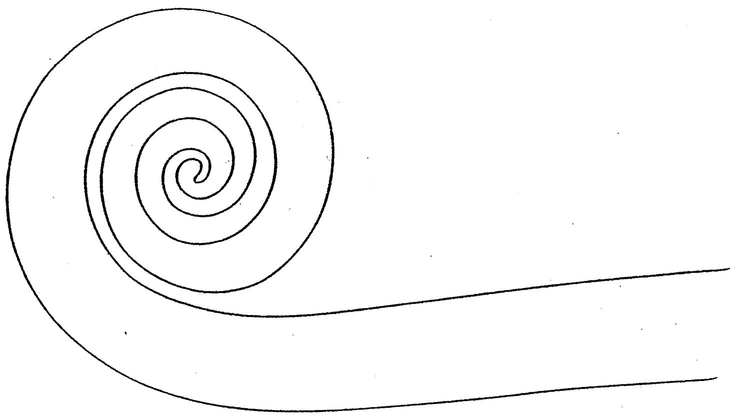

Figure 4.

Figure 5.

Here is the programme used to produce Figure 5.

```
10   HIRES2,2
15   FORJ=20TO15STEP-5
20   F=1:PI=3.14
30   AL=85*PI/180
40   B=COS(AL)/SIN(AL)
50   FORI=0TO1080STEP5
60   TURTLE1:BYE"TURN90:TUP:C=C+1
70   RA=I*PI/180
80   R=J*EXP(B*RA)
90   T=I-I*2:TURNT:MOVER:X=TPOS(X):Y=TPOS(Y)
100  IFF=1THEN120
110  DRAWPX,PY,X,Y,1
120  PX=X:PY=Y:F=0
130  NEXTI
140  NEXTJ
150  DRAWX,Y,X+30,Y,1
155  DRAW174,100,181,100,1
160  FILLX+5,Y-5,10
```

Finally, Figure 6 shows the variation in growth between day and night. I measured day and night growth from Day 10 to Day 15. During this time day growth was larger than night growth, except for Day 13 when they were the same.

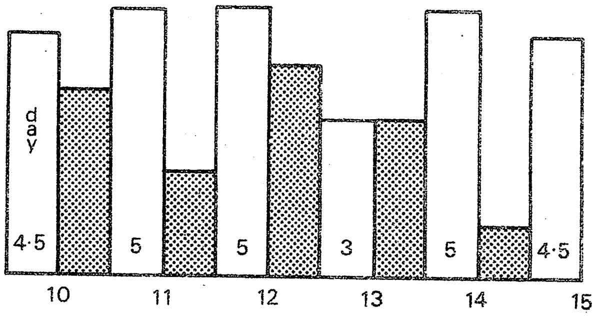

Figure 6.

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Continued from page 6.

'Poo! Poo!' said Obtuse Angle. 'Don't be a fool.'

Then Tilly Lally took up a Quadrant & ask'd, 'Is not this a sun-dial?'

'Yes,' said Scopprell, 'but it's broke.'

At this moment the three Philosophers enter'd, and low'ring darkness hover's over the assembly.

'Come,' said the Epicurean, 'let's have some rum & water, & hang the mathematics! Come, Aradobo! Say something.'

Then Aradobo began, 'In the first place I think, I think in the first place that Chatterton was clever at Fissie Follogy, Pistinology, Aridology, Arography, Transmography, Phizography, Hogamy, Hatomy, & hall that, but, in the first place, he eat every little, wickly - that is, he slept very little, which he brought into a consumsion; & what was that that he took? Fissie or somethink, - & so died!'

So all the people in the book enter'd into the room, & they could not talk any more to the present purpose.

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PASCAL'S TETRAHEDRON†

Bruce Henry, Victoria College, Rusden

Pascal's Triangle is well known to senior school mathematics students:

$$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

etc.

Each row has a 1 on each end and each intermediate number is the sum of the two numbers above it. Alternatively, we can think of the Triangle as growing out of an infinite sea of zeros:

$$\begin{array}{ccccccccccccc} &0&&0&&0&&0&&0&&0& \\ 0&&0&&0&&1&&0&&0&&0 \\ &0&&0&&1&&1&&0&&0& \\ 0&&0&&1&&2&&1&&0&&0 \\ &0&&1&&3&&3&&1&&0& \\ 0&&1&&4&&6&&4&&1&&0 \end{array}$$

etc.

Then any number is the sum of the two numbers above it.

The $(n+1)$th row gives the coefficients of descending powers of x in the expansion of $(x + y)^n$. For example, $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, where 1, 3, 3, 1 is the fourth row of Pascal's Triangle. This is usually why we use the Triangle - to help us to expand $(x + y)^n$. The relation between the rows is easily seen - given one row, to get the coefficient of $x^r y^{n-r}$ in the next row, we must multiply the term in $x^{r-1}y^{n-r}$ by x and the term in $x^r y^{n-r-1}$ by y and add the two together. This corresponds to adding the two coefficients immediately above the one required in the Triangle.

† See also *Function*, *Vol*. 8, *Part* 1, pp.25-26.

Further, the rth term of the $(n + 1)$th row of Pascal's Triangle is $\begin{bmatrix} n \\ r-1 \end{bmatrix}$. This is a direct consequence of the $(n + 1)$th row giving the coefficient of $(x + y)^n$.

We will investigate the relationships between the coefficients of $(x + y + z)^n$, in the hope of getting another useful pattern.

We have
$$\begin{aligned}(x + y + z)^0 &= 1 \,, \\ (x + y + z)^1 &= x + y + z \,, \\ (x + y + z)^2 &= x^2 + 2xy + 2xz + y^2 + 2yz + z^2 \,.\end{aligned}$$

Let us arrange the coefficients like this:

T_0

$$\begin{array}{c} 1 \end{array}$$

T_1

$$\begin{array}{ccc} & 1 & \\ 1 & & 1 \end{array}$$

T_2

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 2 & & 1 \end{array}$$

In order to put the right coefficients on the right terms, imagine the top corner of the triangle as the "x" corner, the lower left one as the "y" corner and the other one as the "z" corner. The top 1 is the coefficient of x^2; as we go down a row, the power of x decreases by 1 and that of y and z increases by 1, so that this row gives the coefficients of xy and xz. The last row has the coefficients of y^2, yz and z^2 in that order. Notice that we could just have easily have started from the y corner - that moving one row away decreases the power of y by one and increases the powers of x and z by one. The whole triangle of numbers is symmetrical in the way the numbers are applied to the powers of x, y and z as well as about the centre.

Since
$$\begin{aligned}(x + y + z)^3 = \; & x^3 \\ & + 3x^2y + 3x^2z \\ & + 3xy^2 + 6xyz + 3xz^2 \\ & + y^3 + 3y^2z + 3yz^2 + z^3 \,,\end{aligned}$$

T_3

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 6 & & 3 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

And you can check for yourself that

T_4

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 4 & & 4 & & & \\ & & 6 & & 12 & & 6 & & \\ & 4 & & 12 & & 12 & & 4 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Before reading further you might like to try to find a way to deduce T_i when T_{i-1} is known. This is the kind of thing that REAL mathematics is about - trying to find a pattern, use it to get new results and prove these results. And after all, this is what we do with Pascal's Triangle.

The pattern here becomes very straightforward if we visualize each of the T's as a layer in a tetrahedron. I expect you have already noticed that the edge of each T is a row of Pascal's Triangle; now we are talking about a tetrahadron of numbers whose three upper faces are each a portion of Pascal's Triangle and the base is the last T_i we are investigating. So now we can get the edge numbers of the next T - it is the interior numbers that are difficult. But each is just the sum of three numbers in the layer above it - the three numbers which are nearest to it. In the same way as we add the two numbers directly above a number in Pascal's Triangle (and these are the two numbers nearest to it), in Pascal's Tetrahedron we must add three numbers in the layer above, and these will be the nearest ones.

Again it is useful to float each layer in a sea of zeros, for example:

```
0     0  0  0  0  0  0  0     0
   0  0  0  0  1  0  0  0  0
0  0  0  0  3  1  3  0  0  0  0
   0  0  3  2  6  2  3  0  0
0  0  1  1  3  2  3  1  1  0  0
   0  0  0  0  0  0  0  0  0
```

The layer above fits over the circles containing the numbers - it is easy to see that each number in the lower layer is the sum of the three numbers which surround it. (Some lines indicate the numbers added.)

If the pattern is correct, you should be able to write down T_5:

T_5

```
                1
           5         5
       10       20       10
    10     30        30      10
  5     20      30       20      5
1     5     10       10      5      1
```

We can justify this pattern in the same way as we justified it in the case of the ordinary Pascal's Triangle; to get a certain coefficient, certain coefficients from the layer above must be multiplied by x, y or z and added together. You can quickly check which numbers these are.

We also seek a relationship between these numbers within one layer of Pascal's Tetrahedron and the binomial coefficents. In T_5 you will notice that the numbers in each row have a common factor which is the first (non-zero) number in the row. This is $\binom{5}{n-1}$ where n is the number of the row. With the common factor removed, the layer becomes

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

and these numbers are easily recognized as the binomial coefficients as displayed in Pascal's Triangle! We can therefore expect the rth number of the nth row of the kth layer of the Tetrahedron to be $\begin{bmatrix} k \\ n-1 \end{bmatrix} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}$

This formula can be checked by a consideration of the binomial coefficients and the way they operate in the expansion of $(x + y + z)^n$.

So we have an easy way to get the coefficients in the expansion of $(x + y + z)^n$. I wonder what happens in $(x + y + z + w)^n$?

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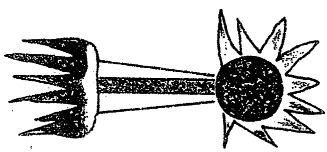
LETTERS TO THE EDITOR

HALLEY IN 1066?

The illustration below, taken from the Bayeux tapestry,



is commonly and erroneously believed to represent Halley's comet.

For a start, it looks nothing like Halley's comet, or any other comet for that matter. Furthermore, Halley's comet has a period of 75 years, whereas

$$(1986 - 1066)/75 = 12.2\dot{6} \ ,$$

a number which is clearly not integral, as it should be were the usual explanation correct.

Simple inspection of the picture shows that what is depicted is a rocket. We now know that the visitors in the rocket (c.f. *Chariots of the Gods*) were induced to side with the invading Normans rather than the defending Saxons. This has affected permanently the history of these British Isles.

Dai Fwls ap Rhyll,
Llanfairpwllgwyngyll-
gogerychwyrndrobwyll-
llantisiliogogogoch,
Wales.

ELEGANT POWERS

$$5^2 = 4^2 + 3^2$$

$$6^3 = 5^3 + 4^3 + 3^3$$

$$15^4 = 14^4 + 9^4 + 8^4 + 6^4 + 4^4$$

$$12^5 = 11^5 + 9^5 + 7^5 + 6^5 + 5^5 + 4^5$$

$$28^6 = 23^6 + 22^6 + 21^6 + 20^6 + 18^6 + 16^6 + 15^6 + 13^6 + 12^6 + 9^6 + 7^6 + 6^6 + 5^6 + 4^6 + 2^6 + 1^6$$

Relations of numbers such as those above probably don't have more importance than possibly somewhat stimulating your curiosity: what do you think of these relations? Are there any more? Which are the simplest? I will just make a few observations.

The first one is more than well-known. You probably know more of these triads from Pythagoras' proposition. But the second is also worth remembering. The divisions of 15 and 12 can quickly be checked by a calculator. But you will need quite a bit of paper for checking 28. There aren't so many divisions of higher powers. Or should we say: we haven't found those yet? There exists a division of 1827 into 127 eighth powers and one of 9339636 into 90 with ninth powers.

Which division is the "nicest"?

Here you can use various yardsticks:

1. Keep the numbers into which you divide as small as possible, but avoid doubles. For example:

$13^2 = 12^2 + 4^2 + 3^2$ instead of $13^2 = 12^2 + 5^2$.

2. Look for a division with the least possible numbers. For example: $a^4 = 315^4 + 272^4 + 120^4 + 3^4$ can be shorter. How large is a ?
   This also occurs in fifth powers:

$$b^5 = 133^5 + 110^5 + 84^5 + 27^5 .$$

   This division was found in 1966 (aided by a computer). It disproved the supposition of Euler who believed that the number of terms must be at least as large as the exponent. So in the example there are not 5 terms but only 4.

3. Use only consecutive numbers:

$$c^2 = 24^2 + 23^2 + \ldots + 2^2 + 1^2$$
$$d^2 = 28^2 + 27^2 + \ldots + 19^2 + 18^2$$
$$e^3 = 14^3 + 13^3 + 12^3 + 11^3 .$$

   You can find a number of divisions yourself by using the following formulae:

$$1^2 + 2^2 + \ldots + n^2 = ((2n + 3).n + 1).n/6$$
$$1^3 + 2^3 + \ldots + n^3 = ((n + 2).n + 1).n^2/4 .$$

However, if you don't limit yourself, then there's the possibility that you'll come across various possibilities. Thus:

$$65^2 + 63^2 + 16^2 = 60^2 + 25^2 = \ldots \text{ or}$$
$$108^3 = 106^3 + 38^3 + 24^3 = 89^3 + 82^3 + 15^3 = \ldots$$

NEGATIVE AND FRACTIONAL EXPONENTS

It isn't so difficult to make a division with fractional exponents. Try it:

$$18^{1/2} = 8^{1/2} + 2^{1/2} \quad \text{or} \quad 54^{1/3} = 16^{1/3} + 2^{1/3} \quad \text{etc.}$$

The negative exponent is a bigger problem. For the exponent -1 I know the division:

$$2^{-1} = 3^{-1} + 6^{-1} .$$

Who knows any other divisions with exponent -1 ? Are there divisions with exponent -2, -3, etc.?

S. van den Horst,
Gemeentelijke
Scholengemeenschap
Woensel,
Eindhoven, Nederland.

THE FIVE TWOS

$1 = 2 + 2 - 2 - 2/2$
$2 = 2 + 2 \div 2 - 2 - 2$
$3 = 2 + 2 - 2 + 2/2$
$4 = 2 \times 2 \times 2 - 2 - 2$
$5 = 2 + 2 + 2 - 2/2$
$6 = 2 + 2 + 2 + 2 - 2$
$7 = 22 \div 2 - 2 - 2$
$8 = 2 \times 2 \times 2 + 2 - 2$
$9 = 2 \times 2 \times 2 + 2/2$
$10 = 2 \div 2 + 2 + 2 + 2$
$11 = 22 \div 2 + 2 - 2$
$12 = 2 \times 2 \times 2 + 2 + 2$
$13 = (22 + 2 + 2)/2$
$14 = 2 \times 2 \times 2 \times 2 - 2$
$15 = 22 \div 2 + 2 + 2$
$16 = (2 \times 2 + 2 + 2) \times 2$
$17 = (2 \times 2)^2 + 2/2$
$18 = 2 \times 2 \times 2 \times 2 + 2$
$19 = 22 - 2 - 2/2$
$20 = 22 + 2 - 2 - 2$
$21 = 22 - 2 + 2/2$
$22 = 22 \times 2 - 22$
$23 = 22 + 2 - 2/2$
$24 = 22 - 2 + 2 + 2$
$25 = 22 + 2 + 2/2$
$26 = 2 \times (22/2 + 2)$

Garnet J. Greenbury,
Taringa, Queensland.

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ABOLISHING DISTINCTIONS

No sharp dividing line can in fact, be drawn between 'pure' and 'applied' mathematics. There should not be a class of high priests of undulterated mathematical beauty, exclusively responsible to their own inclination, and a class of workers who serve other masters. Class distinctions of the kind are at best the symptons of human limitations that keep most individuals from roaming at will over broad fields of interest.

R. COURANT

PROBLEM SECTION

We begin by giving solutions to the rest of the problems posed last year.

SOLUTION TO PROBLEM 9.4.1.

Begin with a sequence of integers

$$n_1, n_2, n_3, n_4, \ldots .$$

These are divided by another integer m (say) to give remainders

$$2, 4, 8, 5, \ldots .$$

The proposer, Garnet A. Greenbury, asked for the next number in this second sequence.

Perdix, formerly a regular columnist with *Function*, wrote to us on this problem.

For any n, there exist integers k, m such that

$$n - 2 = km$$

$$(n + 2) - 4 = km$$

$$(n + 6) - 8 = km$$

$$(n + 3) - 5 = km .$$

This can be seen merely by factorising $n-2$. If this is prime, either $k = n-2$, $m=1$ or $k=1$, $m = n-2$.

So, take

$$n_1 = n, \quad n_2 = n + 2, \quad n_3 = n + 6, \quad n_4 = n + 3 .$$

Provided that $m > 8$, 2, 4, 8, 5 will be the remainders on division by m.

Now $(n + s - 2) - s = km$
and this gives

$$n_5 = n + s - 2$$

with a remainder of s, as long as $s < m + 1$.

The proposer tells us that the sequence he had in mind was: 2, 4, 8, 16, 32, ... , with 11 as the divisor. This gave remainders: 2, 4, 8, 5, 10, 9, 7, 3,, 6, 1, ... The other questions asked relied on this interpretation. A solution along these lines was sent in by David Shaw of Geelong West Technical School.

SOLUTION TO PROBLEM 9.5.1.

I have N weights of 1 kg, 2 kg, ..., N kg respectively. Remove the weight m kg and require

Sum of (weights < m kg) = Sum of (weights > m kg) .

For what values of N, m is this possible?

The requirement boils down to this.

$$1 + 2 + \ldots + (m - 1) = (m + 1) + (m + 2) + \ldots + N .$$

This, using the sum formula for an arithmetic progression, gives

$$\frac{(m - 1)m}{2} = \frac{N(N + 1)}{2} - \frac{m(m + 1)}{2}$$

or

$$m^2 = N(N + 1) .$$

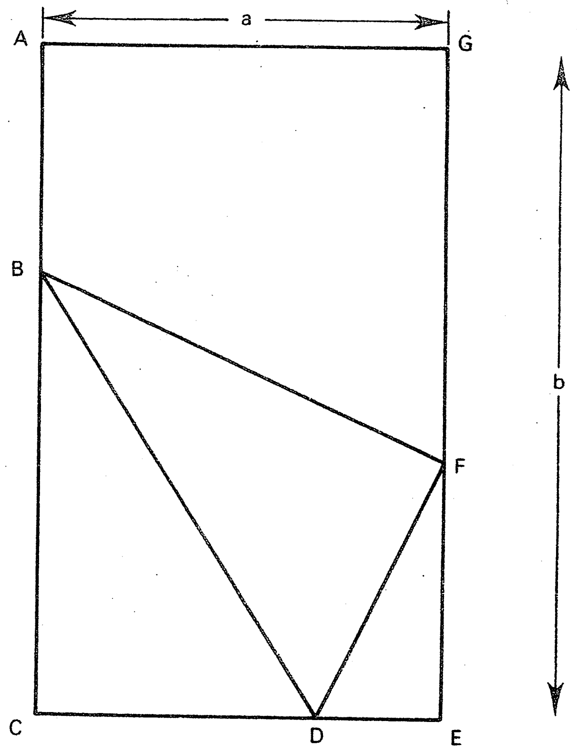
Since

$$N^2 < N(N + 1) < (N + 1)^2 ,$$

$N(N + 1)$ always lies between two adjacent square numbers and so cannot itself be square. The condition cannot be achieved.

SOLUTION TO PROBLEM 9.5.2.

In the diagram opposite, representing a folded piece of paper, how must the fold be made to maximise the length BD?



Let $\measuredangle\ CBD = \theta$. Then $\measuredangle\ BFG = 2\theta$. Also $BC = BF$ and we wish to maximise BD which equals $BF/\cos\theta$. But $BF \sin 2\theta = a$. So $BF = a/\sin 2\theta$, and we must minimise $a/\sin 2\theta \cos\theta$, i.e. we need to *maximise* $\sin 2\theta \cos\theta$, which is $2 \sin\theta(1 - \sin^2\theta)$. This expression is maximised when $\sin\theta = 1/\sqrt{3}$.

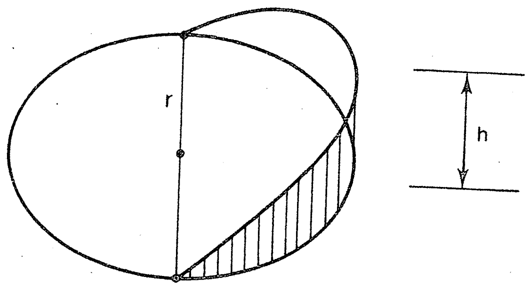
Note that the answer is independent of the shape and size of the paper.

SOLUTION TO PROBLEM 9.5.3.

Two cylinders of equal radius r meet at right angles in such a way that their axes of symmetry intersect. Find the volume of the region common to both cylinders.

Domenica Turkiewicz, then of Year 12, All Hallows' School, Brisbane, wrote:

"It is very hard to draw this figure but the common volume is made up of eight pieces like this. The diameter of the circle shown is $2r$ and so is the diameter of the corresponding circle for the other cylinder. So $h = r$.

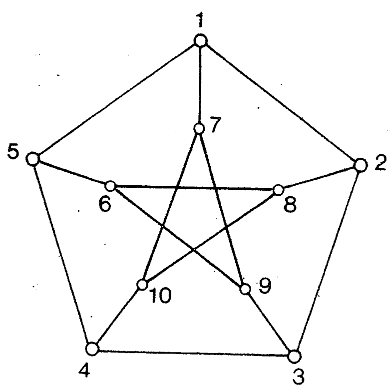


"The volume of such a figure is $\frac{2}{3}r^3$. So the volume V of the common region is eight times this, or $\frac{16r^3}{3}$."

Finally, here are some new problems. Both come from the 1985 School Mathematics Competition.

PROBLEM 10.2.1.

The dots in the diagram represent towns which a travelling salesperson has to visit. Two dots (towns) are joined by a line if there is some direct road-link between them.



A *circuit* is a trip through successive dots along the specified lines which starts and ends at the same dot and which visits any given dot at most once.

An *m-circuit* is a circuit that involves m dots. Hence the circuit 1, 2, 3, 4, 5, 1 which starts and ends at 1 and goes through the dots 2, 3, 4, 5 is a 5-circuit.

For convenience the travelling salesperson would like to find a 10-circuit so that each town would be visited once and only once on any sales trip.

Show that no 10-circuit exists.

For what values of m do m-circuits exist? (Give reasons for your answer.)

PROBLEM 10.2.2.

What is the smallest value of α so that $F(x) = 7x^{11} + 11x^{7} + 10\alpha x$ is divisible by 77 for every value of x?

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PERDIX

Several outstanding results were achieved in the Australian Mathematical Olympiad Examination this year and we have the strongest hopes for the success of our team at Warsaw.

The team selected is:

Adrian Chen, Prince Alfred College, Adelaide, S.A.
David Hogan, James Ruse High School, Carlingford, N.S.W.
Ross Jones, Rosney College, Rosney, Tasmania.
Catherine Playoust, Loreto Convent, Kirribilli, N.S.W.
Ben Robinson, Narrabundah College, Kingston, A.C.T.
Terence Tao, Blackwood High School, Eden Hills, S.A.

The reserve for the team is

Bruce Cox, Sydney Church of England Grammar School, North Sydney, N.S.W.

The IBM May training school for the team will also be attended by:

Daniel Calegari, Melbourne Church of England Grammar School, Melbourne, Victoria.
Kin Yan Chung, Duncraig Senior High School, Duncraid, W.A.
Mitchell Porter, Toowoomba Grammar School, Toowoomba, Queensland,

all three of whom will be eligible for selection for the Australian team in 1987.

Congratulations to all of the above!

I am told that the team selectors had great difficulty this year in making their final selection because of the overall excellence of performance in the Australian Mathematical Olympiad.

* * * * * * * *

Problems

The Perdix column in the last issue of 1985 was devoted to giving a solution to the final problem, number 6, set in the International Mathematical Olympiad (IMO), at Prague, July 5, 1985. The problem is:

For every real number x_1, construct the sequence, x_1, x_2, ... , by setting

$$x_{n+1} = x_n.(x_n + \frac{1}{n}) \qquad (1)$$

for each $n \geq 1$. Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1 \qquad (2)$$

for every n.

The solution offered was presented so as to be understandable to anyone having done mathematics up to final high school year. With more mathematical knowledge much of the long argument for the solution could be omitted because the results established were reasonably standard. There have been several requests for alternative solutions and further discussion.

Let me begin by summarising the solution already given. I shall then give the official solution that was available at the IMO at Prague.

SUMMARY of solution given in *Function*, *Volume* 9, *Part* 5.

Step 1 The discussion may be restricted to C-sequences x_1, x_2, ... , x_n, ... for which $0 < x_1 < 1$.

Step 2 There are two types of C-sequence:
Type I: such that there is a largest term x_k, say, and then

$x_1 < x_2 < \ldots < x_k \geqq x_{k+1} > x_{k+2} > \ldots > x_{k+t} > \ldots$, and $x_n < \frac{k}{k+1}$ for all n.

Type II: such that the sequence steadily increases, i.e. $x_1 < x_2 < \ldots < x_k < \ldots < x_n < \ldots$.

Step 3 Call the set of all numbers x with $0 < x < 1$, that start a C-sequence of type I, the set A; and denote by B the set of the remaining x with $0 < x < 1$. Thus B is the set of numbers starting a C-sequence of type II.

Step 4 A and B are nonempty and there is a number c such that $a \leqq c \leqq b$ for all $a \in A$ and $b \in B$.

Step 5 No number $a \in A$ starts a C-sequence that satisfies the inequalities (2) for every n.

Step 6 If $b \in B$ and $c < b$, then the C-sequence starting with b does not satisfy inequalities (2) for every n.

Step 7 The number c does not lie in A; and hence c is the least number in B.

Step 8 The C-sequence starting with c satisfies the inequalities (2) for every n.

Hence c is the sole number satisfying the conditions of the problem.

OFFICIAL IMO solution

Let $x_1 = x$. Then $x_n = P_n(x)$ where P_n is a polynomial of degree 2^{n-1} with positive coefficients. Note that P_n is increasing[(a)] and convex[(b)] for $x \geqq 0$. Observing that the condition $x_{n+1} > x_n$ is equivalent to $x_n > 1 - \frac{1}{n}$, the problem can be reformulated as follows: there is a unique positive real number t such that $1 - \frac{1}{n} < P_n(t) < 1$ for all $n \geqq 1$. Since P_n is increasing for $x \geqq 0$ and $P_n(0) = 0$, it follows that for each n there are unique values for a_n and b_n such that $P_n(a_n) = 1 - \frac{1}{n}$ and $P_n(b_n) = 1$, respectively. Note that $P_{n+1}(a_n) = (1 - \frac{1}{n})(1 - \frac{1}{n} + \frac{1}{n}) = 1 - \frac{1}{n}$ and $P_{n+1}(a_{n+1}) = 1 - \frac{1}{n+1}$. Since P_{n+1} is increasing, it follows that $a_n < a_{n+1}$. Similarily, $P_{n+1}(b_n) = 1 + \frac{1}{n}$ and $P_{n+1}(b_{n+1}) = 1$, so $b_{n+1} < b_n$. It follows that $[a_n, b_n]$ is a nested sequence[(c)] of closed intervals and its intersection is non-empty[(d)]. Since P_n is convex for $x \geqq 0$ and $P_n(0) = 0$, we find that $P_n(x) \leqq x/b_n$, $0 \leqq x \leqq 1$, in particular $1 - \frac{1}{n} = P(a_n) \leqq a_n/b_n$. Together with the fact that $1 = b_1 > b_2 > b_3 > \ldots$ this means that $b_n - a_n \leqq 1/n$ for all n. Consequently there is only one point t that belongs to all the intervals. This point satisfies $1 - \frac{1}{n} < P_n(t) < 1$ for all n and it does so uniquely. For any point $x \neq t$, either $x < a_n$ or $b_n < x$ for sufficiently large n meaning that $P_n(x) < 1 - \frac{1}{n}$ or else $P_n(x) > 1$.

EXPLANATION OF SOME TERMS

(a) P_n is "increasing" means that if $a < b$ then $P_n(a) < P_n(b)$.

(b) P_n is "convex" (for $x \geqq 0$) means that the graph of P_n is convex downwards.

For example each of the functions x^k, $k = 1, 2, \ldots$, is convex for $x \geqq 0$. Consequently any sum of positive multiples of powers of x is convex for $x \geqq 0$.

(c) $[a_n, b_n]$ denotes the set $\{x \mid a_n \leqq x \leqq b\}$.

A sequence of intervals, or sets, $I_1, I_2, \ldots, I_n, \ldots$ is "nested" if, for all n, $I_n \supseteqq I_{n+1}$.

(d) That the intersection of a nested sequence of closed sets is non-empty is a basic property of the real numbers. See Chinn and Steenrod, referred to below.

A good source for further explanation of the ideas used in the above solution is the book, addressed to High School students, "First concepts of topology" by W.G. Chinn & N.E. Steenrod, New Mathematical Library No.18. Consult this to try and sort out what is involved in the above proof.

In any event Chinn & Steenrod's book is compelling reading and should be read by any aspiring Olympiad competitor.

COMMENTS on the official solution and comparison with Perdix solution.

The official solution finds a sequence

$$a_1 < a_2 < \ldots < a_n < \ldots$$

and a sequence

$$b_1 > b_2 > \ldots > b_n > \ldots$$

such that $a_i < b_j$ for all i and j. Each $a_n \in A$, i.e. starts a type I C-sequence and each $b_n \in B$, i.e. starts a type II C-sequence. This is clear because the defining condition for

a_n, namely, $P_n(a_n) = 1 - \frac{1}{n}$, says that the C-sequence starting with a_n has reached its maximum at the term $P_n(a_n)$, and similarly, since $P_n(b_n) = 1$, the C-sequence starting at b_n does not have all its terms less than $\frac{k}{k+1}$ for any positive integer, k, and so is of type II.

Thus the sequence $a_1, a_2, \ldots, a_n, \ldots$, is a strictly increasing sequence of elements of A, each less than each member of the strictly decreasing sequence $b_1, b_2, \ldots, b_n, \ldots$, of elements of B. Since $b_n - a_n \leq \frac{1}{n}$, these two sequences converge to the same point, namely to the number c, the sequence (a_n) converging from below to c, the sequence (b_n) converging from above to c.

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