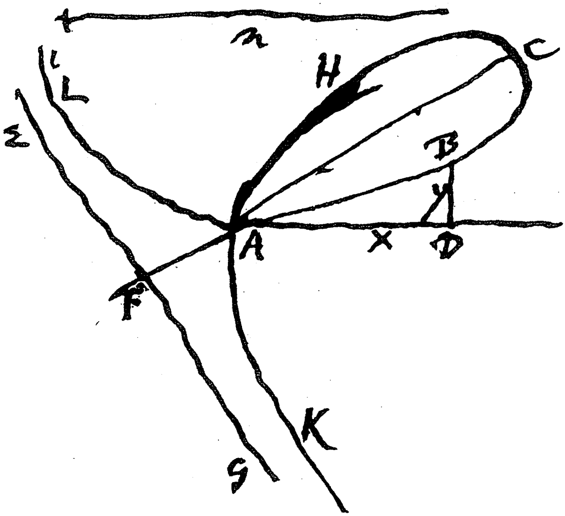
ISSN 0313 - 6825

FUNCTION

Volume 9 Part 3 June 1985



A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University.

Reg. by Aust Post Publ. No. VBH0171

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

The magazine is published five times a year, appearing in February, April, June, August, October. Price for five issues (including postage): $8.00*; single issues $1.80. Payments should be sent to the business manager at the above address: cheques and money orders should be made payable to Monash University. Enquiries about advertising should be directed to the business manager.

*$4.00 for *bona fide* secondary or tertiary students.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

Registered for posting as a periodical - "Category B"

The practical problems involved in the siting of facilities lead to very interesting and important mathematics. Dennis Lindley writes on what is involved in the siting of a tornado shelter to serve two communities. His article also highlights the political difficulties involved in such choices. While mathematics can help in the allocation of scarce resources, it provides little comfort for those who miss out!

John Stillwell writes on Pythagoras' Theorem - a topic that never seems to lose its interest, Joseph Kupka continues his account of measure theory, and there is much more besides.

CONTENTS

The Front Cover. J.C. Stillwell and M.A.B. Deakin	2
Pythagoras' Theorem. J.C. Stillwell	4
Easter. S. Rowe	10
A Visit to Flatland. P.E. Kloeden	13
Mathematical Measure Theory: II. Birth. Joseph Kupka	15
Where to site a Tornado Shelter. Dennis Lindley	19
An Historical Curiosity. J.A. Deakin	24
Squash Mathematics. Stephen R. Clark	25
Letter to the Editor	29
Problem Section	29
Perdix	31

THE FRONT COVER

J.C. Stillwell and M.A.B. Deakin,
Monash University

The curve

$$x^3 + y^3 = 3axy \qquad (1)$$

is known as the *Folium of Descartes*. The graph opposite, a computer-drawn curve prepared by Peter Fox of Monash University, shows its shape very accurately. Descartes did discuss the curve in 1638 and found the leaf-shape - the closed portion to the right of the double point. So the curve is aptly named (the Latin word *folium* means "leaf").

However, because of his neglect of negative coordinates, Descartes misunderstood the rest of the curve and it was the Dutch physicist and mathematician Christiaan Huygens who first produced a correct account. Our cover picture is Huygens' drawing, taken from a letter to the Marquis de l'Hospital (now referred to as l'Hôpital) dated 29 December 1692. These two continued to explore the properties of the curve in subsequent correspondence.

Because Equation (1) contains terms of third order, but not higher, it is said to be of third degree. First degree curves are the simplest possible, namely straight lines; second degree are the next simplest and are conic sections, which were known to the ancient Greeks. The conic sections are the ellipse (of which the circle is a special case), the parabola and the hyperbola. (See *Function, Vol.*7, *Parts* 2,3.)

The notion of the degree of a curve was introduced by Descartes and has proved to be a useful measure of complexity. Curves need to be of degree three or more to exhibit certain features. The folium of Descartes possesses a double point (or self-intersection) at the origin. As will be evident from the previous paragraph, curves of degree one or two do not ever possess such a feature.

As x gets very large, positive or negative, the curve lies closer and closer to the line

$$x + y + a = 0 \qquad (2)$$

which is the *asymptote* to the folium. (The hyperbola, alone of the simpler curves, possesses asymptotes.) The intersections of this line with the axes give a scale to the curve, and it is seen that when $x = \pm 3a$, the limit of our drawing on this page, the folium is already well approximated by its asymptote.

We may also provide a scale by determining the length of the leaf. To do this, put $y = x$ and so reach, from (1),

$$2x^3 = 3ax^2$$

or

$$x = 3a/2.$$

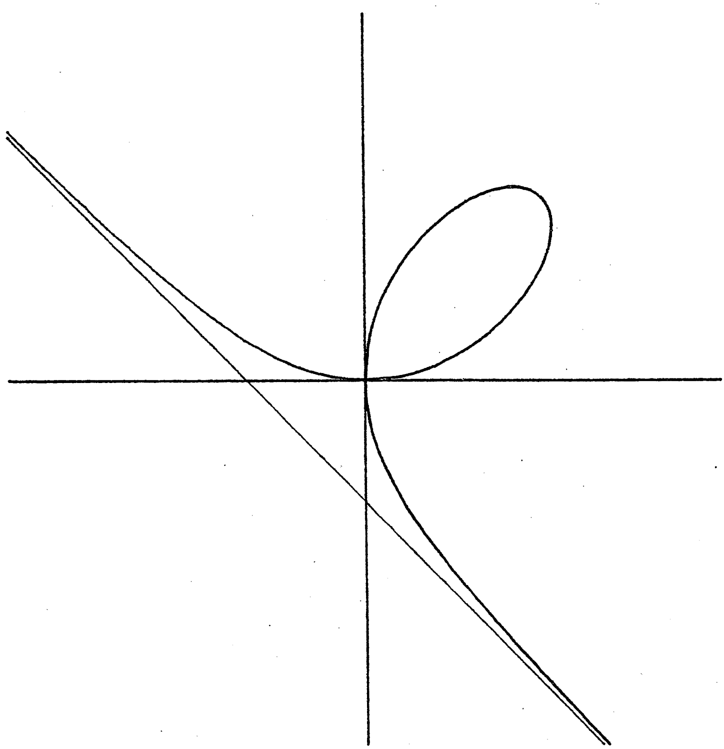
Similarly we find

$$y = 3a/2,$$

and so the length, measured along the line $y = x$, is

$$\sqrt{\left(\frac{3a}{2}\right)^2 + \left(\frac{3a}{2}\right)^2} = \left(\frac{3\sqrt{2}}{2}\right)a .$$

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞



PYTHAGORAS' THEOREM

J.C. Stillwell, Monash University

Introduction.

If there is one theorem which is familiar to all mathematically educated people, it is probably the theorem of Pythagoras. It will probably be recalled as a property of right-angled triangles (Fig.1):

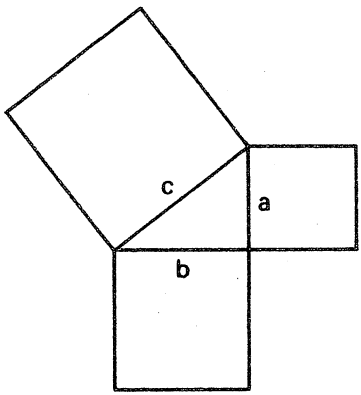


Figure 1.

the area of the square on the hypotenuse equals the sum of the areas of the squares on the other two sides. This may also be recalled in algebraic form

$$c^2 = a^2 + b^2,$$

perhaps along with some interesting numerical examples, such as the triangle with sides $a = 3$, $b = 4$, $c = 5$.

This theorem has in fact held a fundamental position between number and geometry throughout the history of mathematics. Sometimes this has been a position of conflict, as followed the discovery that $\sqrt{2}$ is irrational (see below), and sometimes one of cooperation. It is often the case that new ideas emerge from such areas of tension, resolving the conflict and allowing previously irreconcilable ideas to interact fruitfully.

In the present article I shall illustrate this process with some episodes in the history of Pythagoras' theorem. This is appropriate since Pythagoras' theorem is undoubtedly the first theorem in history to have a really deep and lasting influence on mathematics.

Pythagorean triples.

The story of Pythagoras' theorem begins long before Pythagoras, in Babylonia around 1800 BC. Clay tablets surviving from this period, inscribed in the Babylonian cuneiform script, indicate that the Babylonians knew the theorem, and also knew ways of finding instances in which the sides a,b,c of the triangle are integers. Such triples (a,b,c) for example (3,4,5), (5,12,13), (8,15,17), are known as *Pythagorean triples*. (See *Function, Vol.6, Part* 3.)

We now know that the general formula for generating Pythagorean triples is

$$a = (p^2 - q^2)r, \quad b = 2pqr, \quad c = (p^2 + q^2)r.$$

It is easy to see that $a^2 + b^2 = c^2$ when a,b,c are given by these formulae, and of course a,b,c will be integers if p,q,r are. Even though the Babylonians did not have the advantage of our algebraic notation, it is plausible that this formula, or the special case

$$a = p^2 - q^2, \quad b = 2pq, \quad c = p^2 + q^2$$

(which gives all solutions a,b,c without common factor as well as *some* with a factor) is the basis for triples they listed. Less general formulae have been attributed to Pythagoras himself (c. 500 BC) and Plato while a solution equivalent to the general formula is given in Euclid's *Elements*. As far as we know, this is the first statement of the general solution, and proof that it is general.

The *Elements* were written around 300 BC, so this gives a date at which the problem of Pythagorean triples reached a certain maturity. However, important as the *Elements* are for other topics we shall bypass their treatment of Pythagorean triples. A more satisfactory approach, for our purposes, emerges from the work of Diophantus around 250 AD, and we shall describe it in the next section.

Rational points on the circle.

We arrive at the equation to the unit circle, in a way which would have been comprehensible to the Greeks of Euclid's era, by constructing a right-angled triangle OAP as shown in Figure 2. Applying Pythagoras' theorem to this triangle, which has sides x,y and hypotenuse 1, we get

$$x^2 + y^2 = 1.$$

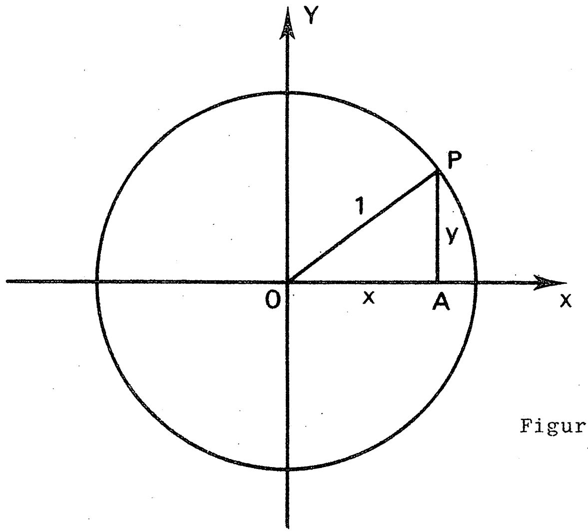


Figure 2.

Now any Pythagorean triple (a,b,c) satisfies

$$a^2 + b^2 = c^2$$

or

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

and hence gives a point $\left(\frac{a}{c}, \frac{b}{c}\right)$ on the unit circle with fractional, i.e. rational, coordinates. We call such a point a *rational point*. Conversely, any rational point $\left(\frac{a}{c}, \frac{b}{c}\right)$ on the circle (and of course any rational point can be written in this form by taking a common denominator) gives a Pythagorean triple (a,b,c).

Thus the problem of finding Pythagorean triples is the same as finding rational points on the circle, or of finding rational solutions of the equation $x^2 + y^2 = 1$.

Such problems are now called *Diophantine*, after Diophantus, who was the first to deal with them seriously and successfully. *Diophantine equations* have acquired the more special connotation of equations for which integer solutions are sought; however, Diophantus himself sought only rational solutions. [There is an interesting open problem which turns on this distinction. In 1970, the mathematician Matiasevich proved that there is no algorithm for deciding which polynomial equations have integer solutions. It is not known whether there is an algorithm for deciding which polynomial equations have *rational* solutions.]

Most of the problems solved by Diophantus involve quadratic or cubic equations, usually with one obvious trivial solution. Diophantus used the obvious solution as a stepping stone to the non-obvious, but no account of his method survived. It was ultimately reconstructed by Fermat and Newton in the 17th century, and we shall say more about its general form later. At present, we need it only for the equation $x^2 + y^2 = 1$, which is an ideal showcase for the method in its simplest form.

A trivial solution of this equation is $x = -1$, $y = 0$, which is the point Q on the unit circle (Fig.3).

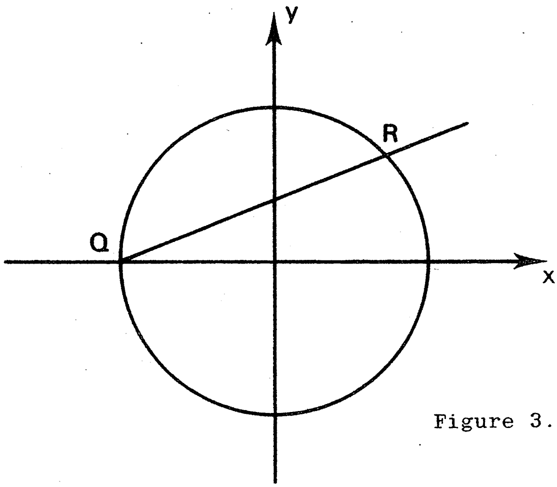


Figure 3.

In fact a line through Q, with rational gradient t,

$$y = t(x + 1) \tag{1}$$

will meet the circle at a second rational point R. This is because substitution of $y = t(x + 1)$ in $x^2 + y^2 = 1$ gives a quadratic equation with rational coefficients and one rational solution (namely $x = -1$), hence the second solution must also be a rational value of x. But then the y value of this point will also be rational, since t and x will be rational in equation (1). Conversely, the line through Q and any other rational point R on the circle will have rational slope. Thus by letting t run through all rational values, we find all rational points $R \neq Q$ on the unit circle.

What are these points? We find them by solving the equations we have just discussed. Substituting $y = t(x + 1)$ in $x^2 + y^2 = 1$ gives

$$x^2 + t^2(x + 1)^2 = 1$$

or

$$x^2(1 + t^2) + 2t^2x + (t^2 - 1) = 0.$$

This quadratic equation in x has solutions $-1, \frac{1-t^2}{1+t^2}$. The non-trivial solution $x = \frac{1-t^2}{1+t^2}$, substituted in (1), gives $y = \frac{2t}{1+t^2}$. So that $(1-t^2, 2t, 1+t^2)$ is a Pythagorean triple.

Right-angled triangles.

It is high time we looked at a Pythagoras' theorem from the traditional point of view, as a theorem about right-angled triangles, however we shall be rather brief about its proof. It is not known how the theorem was first proved, but probably simple considerations of area were involved, perhaps arising from reflection on floor tiling patterns. Just how easy it can be to prove Pythagoras' theorem is shown by the following figure.

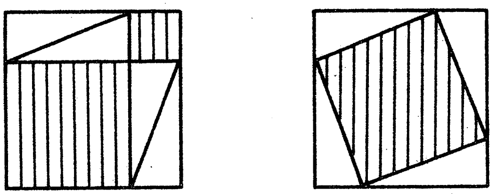


Figure 4.

Each big square contains four copies of the given right-angled triangle. Subtracting these four triangles from the big square leaves, on the one hand, the squares on the two sides of the triangle (first picture). On the other hand (second picture), it also leaves the square on the hypotenuse.

This proof, like the hundreds of others that have been given for Pythagoras' theorem, rests on certain geometric assumptions. It is in fact possible to transcend geometric assumptions by using numbers as the foundation for geometry, and when this is done Pythagoras' theorem plays a different role, becoming true almost by definition, as an immediate consequence of a more sophisticated definition of distance, chosen so as to make the theorem hold.

The development of a geometry based on number is a beautiful example of the reconciliation of conflicting ideas. In the next section, we shall see how the conflict first arose.

Irrational numbers.

We have mentioned that the Babylonians, although aware of the geometric meaning of Pythagoras' theorem, devoted most of their attention to the whole number triples it brought to light, the Pythagorean triples. Pythagoras and his followers were even more devoted to whole numbers. It was they who discovered the role of numbers in musical harmony - the fact that dividing a vibrating string in 2 raises its pitch by an octave, dividing in 3 raises the pitch another fifth, and so on. This great discovery, the first clue that the physical world might have an underlying mathematical structure, inspired them to seek numerical patterns, which to them meant *whole number* patterns, everywhere. Imagine their consternation, then, when they realised that Pythagoras' theorem implied the existence of irrational quantities!

The irrational enters geometry in a way that cannot be ignored - as the diagonal of the unit square. It follows immediately from Pythagoras' theorem that the diagonal has length $\sqrt{2}$, so the Pythagoreans were obliged to consider, as they saw it, the fractional value

$$\sqrt{2} = \frac{m}{n},$$

where m,n are integers. They argued as follows. We can assume m,n have no common factor. Squaring gives

$$2 = \frac{m^2}{n^2},$$

$$2n^2 = m^2,$$

which means that m^2 is even, and hence so is m. Say,

$$m = 2p.$$

Then $$2n^2 = m^2 = 4p^2,$$

hence $$n^2 = 2p^2,$$

which similarly implies n is even, contrary to the hypothesis that m,n have no common factor. In other words, the assumption that $\sqrt{2}$ is rational leads to an absurdity and hence cannot be true.

The shock of this discovery was traumatic. Legend has it that the first Pythagorean to make the result public was drowned at sea. It led to a split between the theories of number and space which was not really healed until the 19th century (if then, some mathematicians would add). The Pythagoreans could not accept $\sqrt{2}$ as a number, but no one could deny that it was the diagonal of the unit square. Consequently, geometrical quantities had to be treated separately from numbers, or rather, without mentioning any numbers except rationals. And so it was. Greek geometers developed ingenious techniques for precise handling of arbitrary lengths in terms of rationals, known as the *theory of proportions* and the *method of exhaustion*.

When these techniques were reconsidered in the 19th century by Dedekind, he realised that they provided a reconciliation of number and space, after all. But that is another story.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

EASTER

S. Rowe, Student,

Swinburne Institute of Technology

The date of Easter is very important in the Christian ecclesiastical calendar. It governs events over almost a third of each year, from Septuagesima Sunday (nine weeks before Easter Sunday) to Trinity Sunday (eight weeks after).

The date of Easter Sunday is derived astronomically. At first Easter was synchronized with the Jewish Passover, but this, although accepted by the Eastern Church, was rejected by the Church in Rome. In the year 325 it was decreed at the Council of Nicaea that Easter should be celebrated on the same date by all Christians. The date was decided to be the first Sunday following the first full moon on or after the vernal (or spring) equinox. The vernal equinox, in simplified terms, occurs when the Sun passes the point above the Earth's equator, from north to south. At the time of the Council of Nicaea, the vernal equinox was assumed to be fixed at March 21.

The old Julian calendar had a year that was too long. By the sixteenth century the vernal equinox was actually occurring on March 11, not March 21. Thus the celebration of Easter would inevitably move toward the summer season. To stop this, Pope Gregory XIII introduced his revised calendar to maintain the Easter celebration in the spring season and to maintain a better approximation of the solar year.

Today Easter Sunday is calculated as the Sunday following the first full moon after the vernal equinox, which occurs on March 21. It can thus fall as early as March 22 or as late as April 25. Passover is also governed by the vernal equinox full moon, but while Easter intentionally falls after the full moon, Passover coincides with the full moon. Consequently, Passover cannot begin on Easter Sunday.

To determine the date of Easter, the following calculations need to be made. Let

x = the year in question

---

† This article is based on one which appeared in the April Bulletin of the Lions Club of Belgrave.

Divide	by	Quotient	Remainder
the year x	19	-	a
the year x	100	b	c
b	4	d	e
$b + 8$	25	f	-
$b - f + 1$	3	g	-
$19a + b - d - g + 15$	30	-	h
c	4	i	k
$32 + 2e + 2i - h - k$	7	-	ℓ
$a + 11h + 22\ell$	451	m	-
$h + \ell - 7m + 114$	31	n	p

Then we get

n = number of the month (3 = March, 4 = April, etc.)

$p + 1$ = the day of the month upon which Easter Sunday falls.

The extreme dates of Easter are March 22 (as in 1818 and 2285) and April 25 (as in 1886, 1943 and 2038).

The above works for the Gregorian calendar which was introduced in 1582. To determine the date of Easter under the Julian calendar, the following routine needs to be used.

Divide	by	Quotient	Remainder
the year x	4	-	a
the year x	7	-	b
the year x	19	-	c
$19c + 15$	30	-	d
$2a + 4b - d = 34$	7	-	e
$d + e + 114$	31	f	g

Then we get

f = number of the month (3 = March, 4 = April, etc.)

$g + 1$ = the day of the month upon which Easter Sunday falls.

The date of the Julian Easter has a periodicity of 532 years. For instance, we find April 12 for the years 179, 711 and 1243.

The computer printout below gives the date of easter for 1980 and each of the following 25 years.

```
EASTER SUNDAY IN 1980 IS ...      APRIL 6
EASTER SUNDAY IN 1981 IS ...      APRIL 19
EASTER SUNDAY IN 1982 IS ...      APRIL 11
EASTER SUNDAY IN 1983 IS ...      APRIL 3
EASTER SUNDAY IN 1984 IS ...      APRIL 22
EASTER SUNDAY IN 1985 IS ...      APRIL 7
EASTER SUNDAY IN 1986 IS ...      MARCH 30
EASTER SUNDAY IN 1987 IS ...      APRIL 19
EASTER SUNDAY IN 1988 IS ...      APRIL 3
```

EASTER SUNDAY IN 1989 IS ...	MARCH 26
EASTER SUNDAY IN 1990 IS ...	APRIL 15
EASTER SUNDAY IN 1991 IS ...	MARCH 31
EASTER SUNDAY IN 1992 IS ...	APRIL 19
EASTER SUNDAY IN 1993 IS ...	APRIL 11
EASTER SUNDAY IN 1994 IS ...	APRIL 3
EASTER SUNDAY IN 1995 IS ...	APRIL 16
EASTER SUNDAY IN 1996 IS ...	APRIL 7
EASTER SUNDAY IN 1997 IS ...	MARCH 23
EASTER SUNDAY IN 1998 IS ...	APRIL 12
EASTER SUNDAY IN 1999 IS ...	APRIL 4
EASTER SUNDAY IN 2000 IS ...	APRIL 23
EASTER SUNDAY IN 2001 IS ...	APRIL 8
EASTER SUNDAY IN 2002 IS ...	MARCH 31
EASTER SUNDAY IN 2003 IS ...	APRIL 20
EASTER SUNDAY IN 2004 IS ...	APRIL 11
EASTER SUNDAY IN 2005 IS ...	MARCH 27
EASTER SUNDAY IN 2006 IS ...	APRIL 16
EASTER SUNDAY IN 2007 IS ...	APRIL 8
EASTER SUNDAY IN 2008 IS ...	MARCH 23
EASTER SUNDAY IN 2009 IS ...	APRIL 12
EASTER SUNDAY IN 2010 IS ...	APRIL 4
EASTER SUNDAY IN 2011 IS ...	APRIL 17
EASTER SUNDAY IN 2012 IS ...	APRIL 8
EASTER SUNDAY IN 2013 IS ...	MARCH 31
EASTER SUNDAY IN 2014 IS ...	APRIL 20
EASTER SUNDAY IN 2015 IS ...	APRIL 5
EASTER SUNDAY IN 2016 IS ...	MARCH 27
EASTER SUNDAY IN 2017 IS ...	APRIL 16
EASTER SUNDAY IN 2018 IS ...	APRIL 1
EASTER SUNDAY IN 2019 IS ...	APRIL 21
EASTER SUNDAY IN 2020 IS ...	APRIL 12
EASTER SUNDAY IN 2021 IS ...	MARCH 28
EASTER SUNDAY IN 2022 IS ...	APRIL 17
EASTER SUNDAY IN 2023 IS ...	APRIL 9
EASTER SUNDAY IN 2024 IS ...	MARCH 31
EASTER SUNDAY IN 2025 IS ...	APRIL 13

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

THE AMERICAN SCENE

Mathematics today is a tale of two cultures. It is the best of times, and the worst of times. Mathematics enrolments are at all time highs, yet the number of undergraduate majors are near record lows. Applications of mathematics permeate science and society, yet students are generally unable to apply mathematics to complex problems. Demand for mathematics teachers exceeds supply, yet salaries remain uncompetitive. It is an age of wisdom and an age of foolishness.

Lynn Arthur Steen, President of the Mathematical Association of America, writing in their newsletter *Focus*, *Vol*.5, No.2 (March-April 1985).

A VISIT TO FLATLAND

P.E. Kloeden, Murdoch University

Frequent exposure to science fiction and TV fantasies such as "The Hitchhiker's Guide to the Galaxy" has made most of us familiar with such higher-dimensional terms as "hyperspace" and "space-time", even if we can't really say quite what they are. To a modern mathematician and physicist they are now common ideas, but this has only been so for a few decades. Indeed, it was really only with the introduction of the theory of relativity by Albert Einstein at the beginning of this century that most people became aware of four-dimensional space-time. In view of this, the delightful little book "Flatland" written by Edwin A. Abbott in the early 1880's was indeed remarkably imaginative and ahead of its time. While it may not seem so utterly way out to the modern reader, it is still good fun to read.

Flatland is a two-dimensional world, a surface, inhabited by planar beings of simple geometric shapes. The lowest, uncouth, classes are sharply acute angled isosceles triangles, while the more refined classes comprise the more polygonally shaped beings, with the noblest of all being almost circular. Women of all classes, alas, are merely thin parallelograms. This social hierarchy mimics that of the Victorian Era in which Abbott lived, but is presented as a scathing parody of that society - Abbott was ahead of his times in more than just mathematical imagination.

The story "Flatland" is told in first person by a planar mathematician A. Square, who has been imprisoned for life for making heretical suggestions that the two-dimensional world in which they live may be part of a three-dimensional space. The story has two parts, the first being a brief description of Flatland, its inhabitants, customs and history, while the second part describes Mr Square's encounters with creatures from the three-dimensional world and his subsequent woes in trying to explain his experiences to his planar peers.

One of the big problems in the daily lives of Flatlanders was how to identify one another, since a triangle, square, polygon or circle all look like straight line segments when viewed side on from within their plane. Mistaken identity in a class conscious society could lead to many embarrassing social problems. Our friendly square recounts how in the far historical past Flatland was racked by the Chromatic Sedition, in which the lower classes, and women too, painted themselves in various hues to give the impression of manysidedness and hence to improve their social image. This was of course too much for the haughty polygons of manysides and a bloody civil war ensued. As a result all colour was banned, on the penalty of death, from Flatland except "at our University in some of the highest and most esoteric classes ... [where] the sparing use of Colour is still sanctioned for the purpose of illustrating some of the deeper problems of mathematics".

The real purpose of Mr Square's memoir is to recount his discovery of a third dimension in space land to describe the violent reaction this sparked in his fellow Flatlanders. He first tells of a dream in which he discovered a one-dimensional Lineland, and of the difficulty he had in describing to its inhabitants his own world. After this dream, he himself experiences a visit from a three-dimensional sphere, who appears a most perfect Holy Circle in intersection with Flatland. Ironically, Mr Square has almost exactly the same difficulty in comprehending this strange visitor as the Linelanders had with him. Finally the sphere convinces him and takes him high above Flatland from where he can see everything in Flatland, even the intestines of his fellow Flatlanders. Our hero Mr Square returns to Flatland a convert and apostle for the "Gospel of Three Dimensions", which leads to his ultimate incarceration.

I can heartily recommend that you read this little book of scarcely one hundred pages. It requires no deep mathematical background, just a familiarity with everyday geometrical shapes† and a good imagination.

Reference.

E.A. Abbott, "Flatland", A Romance in Many Dimensions by A. Square.

Many editions, put out by several different publishers, exist. The most accessible is the Dover Reprint in paperback.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

MATHEMATICS, BEAUTY AND NATURE

It can hardly be an accident that Nature betrays her partiality for the beauty of mathematical reasoning.

C.N. Yang, Nobel Prize-
Winner in Physics

---

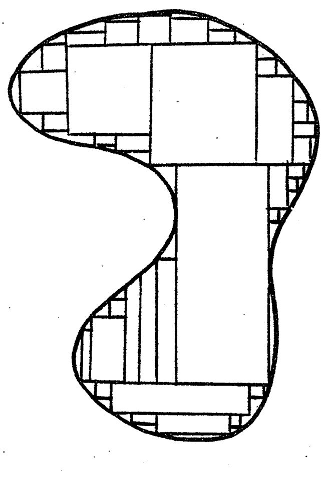
†In two and three dimensions, of course!

MATHEMATICAL MEASURE THEORY: II. BIRTH

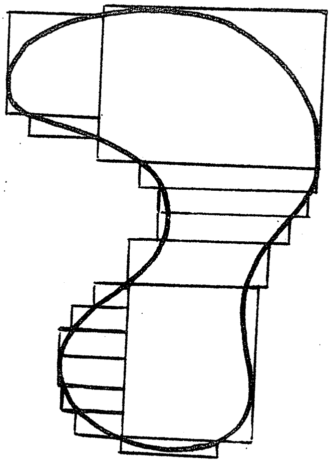
Joseph Kupka, Monash University

Mathematical measure theory traces its modern origins to the work of Peano in the 1880's. Before this time, people had *imagined* that they understood the concept of the area of a plane region. Moreover, it was widely felt that *every* subset of the plane did in fact *have* an area. If the subset was "civilized", then its area could be obtained at once from the Fundamental Theorem of Calculus. If not, then the area could at least be estimated, somehow, by a paved area. The precise nature of this estimation was not clear in the general case, but the problem very likely held little interest for the traditional mathematician. It was the modern inclination to study the *totality* of objects which possess some interesting property that led Peano to consider area, in its own right, as distinct from integral.

Although Peano drew his inspiration from Newton's "wallpaperings", both "inner" and "outer", and from the work of Riemann, his definition of area would certainly have been claimed by the Greeks to be a restatement, in updated language, of precisely their idea. Let *s* be a subset of the plane. Peano considered the *totality* of rectangular pavings of *s* (and not just pavings consisting of long, thin strips) which lay *entirely within* *S*, thus:



The area covered by such a paving would have to be equal to or less than the area A of S. It follows that the *maximum* (or, where this does not exist, the "generalized maximum" or *supremum*) of all such paved areas constitutes a possible underestimate of A, a number which is equal to or less than A. This number is called the *inner content* of S and is denoted $c_*(S)$. At the same time Peano considered the totality of rectangular pavings which *completely cover* S, as at right:

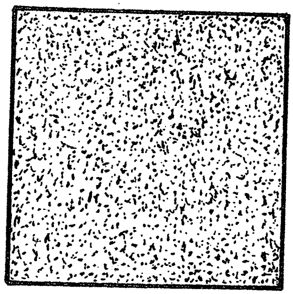


The *minimum* (or, where this does not exist, the *infimum*) of the areas covered by such pavings constitutes a possible overestimate of A, a number which is equal to or greater than A. This number is called the *outer content* of S and is denoted $c^*(S)$. So we have $c_*(S) \leq A \leq c^*(S)$. If it happens that $c_*(S) = c^*(S)$, then A must exactly equal the common value of $c_*(S)$ and $c^*(S)$. This was Peano's *definition* of the area A of S. It is also called the *content* of S and is denoted $c(S)$. If it happened that $c_*(S) < c^*(S)$, then, so far as Peano was concerned, the set S did not *have* an area. It was not *measurable*. Peano's work was independently rediscovered and extended by Jordan in the 1890's (in particular, to a corresponding definition of *volume*), and Jordan's name is now attached to the general notions of content.

If every set S had a content in the sense of Peano and Jordan, the story might have ended with them. But it is very easy to describe a set S for which $c_*(S) < c^*(S)$, and yet which people felt *ought* to have an area. Let us say that a collection or family of objects 0 is *countable* if these objects may be indexed by whole numbers, thus: $0_1, 0_2, 0_3, \ldots$. (Two different objects must receive different suffices.) Every *finite* collection of objects is thus countable, and some infinite collections are countable as well.† Let us now take, say, a square s with area 1, and riddle it with "bulletholes", thus:

†See *Function, Vol.2, Parts* 1, 2.

Let us make each bullethole exceedingly tiny, a single point in fact. But let there be a countable infinity of them. With some care we may arrange the bulletholes so evenly throughout S that no rectangle in S, however small, fails to contain a bullethole. You will ask: Can any of S be left over after such a murderous fusillade? The answer is yes: An *uncountable* infinity of points of S will remain. This is known from a celebrated argument of Cantor, the first prominent mathematical set theorist. From here it is not hard to see that our riddled square will have inner content 0 and outer content 1, hence no area. The set B of bulletholes which we have removed from also has inner content 0 and outer content 1. (One can already imagine the traditional mathematician recoiling in fright and horror.)



But now B is a countable set. Many people felt that any countable set *should* have an area, and that this area *should* be zero. It can be shown that if a rectangle is subdivided into countably many smaller rectangles, then its area is always the sum of the areas of the smaller rectangles, even when there are infinitely many of these smaller rectangles. Likewise the set B may be subdivided into countably many *single points*. A single point has zero area by anybody's definition. So B *ought* to have zero area as well, and, therefore, the riddled square *ought* to have area equal to 1. In this way intuition called out for an extension of the Peano-Jordan definition of area. But it was not immediately clear how to proceed.

The spark of inspiration came from Emile Borel around the turn of the century:

Use infinitely many paving stones.

More specifically, cover an arbitrary set S with infinitely many nonoverlapping rectangles instead of just finitely many as Peano and Jordan had done. (However, because of the nonoverlap, this infinity would have to be countable.) An infinity of stones, most of them exceedingly tiny, could undoubtedly percolate more effectively down through the various nooks, crevices, cracks, potholes, or other irregularities in the shape of S. Consequently one should be able to achieve a better fit between S and the region covered by the stones.

Borel's idea was seized upon and developed into a proper mathematical theory by his student Henri Lebesgue. The *Lebesgue outer measure* $m^*(S)$ of a planar set S is defined by exact analogy to the outer content of S, except that infinite pavings are used as well as finite ones. This increases the number of paved areas under consideration, and so the minimum (or infimum) of these areas has to be correspondingly *smaller*, i.e. $m^*(S) \leq c^*(S)$.

Lebesgue hoped that $m^*(S)$ would serve as a "true" area measure of S. But this hope needed to be justified on rational grounds. Otherwise $m^*(S)$ would have been an "abstract nonsense" of little enduring interest. The preliminary evidence was encouraging. If S already had an area A from some earlier definition, then, in all cases, it could be *proved* that $A = m^*(S)$. This even included the case in which the set S "extends out to infinity" and which therefore, by definition, cannot be completely covered with only finitely many rectangles. Moreover, the outer measure of the riddled square was 1, and the outer measure of the set B of "bullet-holes" was 0, as intuition had demanded.

BUT - there was one tiny problem. It was not clear for the Lebesgue outer measure, nor was it true, that the whole was going to be equal to the sum of its parts.

Lebesgue dealt with this problem in the spirit of Peano and Jordan by creating a new notion of *inner measure* $m_*(S)$ of S. However, this notion was not analogous to the inner content $c_*(S)$ and it is now archaic.

Our present notions follow a different path, opened up by Caratheodory. Caratheodory developed a new and much more abstract notion of measurability - created for the sole purpose of *forcing* the whole to be equal to the sum of its parts.

The greater abstractness of the Lebesgue theory gave to it an aura of remoteness, of detachment from reality. The scientist can readily accept $c^*(S)$ as the area of S when $c^*(S) = c_*(S)$. But can we really say that $m^*(S)$ is the true area of S when S is "measurable" in this very technical sense? To this question the mathematician responds with the serene air of one who knows that he is going to win the argument: "What do you *mean* by the 'true area' of S?" he says. The scientist does not know. Neither does the mathematician. All the mathematician can do is to adduce evidence in the form of mathematical theorems and examples which will persuade the scientist to accept Lebesgue's measure as the *definition* of area. The positive evidence is strong. Area is countably additive on rectangles. This can be *proved*. If the scientist will believe, on this evidence, that area is countably additive in general, then it can again be *proved* that the Lebesgue measure of a measurable set is precisely the area of that set. There is no negative evidence. No one has found an example of a measurable set (and this includes all of the sets whose areas had previously been defined) for which any sort of intuition demands an area other than the Lebesgue measure of that set. Nor has anyone found an example of a nonmeasurable set for which there is *any* intuitive idea about what its area should be.

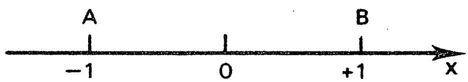
The notion of Lebesgue measure has been extended to three and more dimensions and by giving a more abstract nature to the "paving stones". In this way the modern theory of abstract measure was born.

WHERE TO SITE A TORNADO SHELTER

Dennis Lindley, Minehead, England

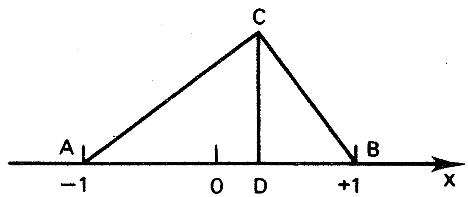
Two small and equal communities decide to build themselves a communal tornado shelter. The countryside around the communities is flat and virtually all places are possible sites for the shelter. A reasonable mathematical description of the situation is to think of the countryside as a plane containing two points, A and B, describing the positions of the two communities. Without loss of generality we can take as the x-axis in the plane the line joining A and B and suppose the origin to be the midpoint of the segment AB. And choosing our scale suitably we can suppose A is at $x = -1$ and B at $x = +1$. Figure 1 shows such a representation.

Figure 1.



Our problem is where to put the shelter, its position being described as (x,y) in the coordinate system we have used. (The y-axis is naturally perpendicular to AB, passing through 0.) We first show that only a very few places are reasonable sites.

Figure 2.



Consider any point in the plane *not* on the line segment AB, such as C in Figure 2. Obviously both communities want the shelter to be as near to themselves as possible. The point D, in Figure 2, on the line segment AB, is nearer, *both* to A and to B, than is C. (D may be the foot of the perpendicular from C to the x-axis but clearly there are other possibilities.) Consequently D is a better position for the shelter than C.

The argument holds for any C not on AB, so that the only sites we need consider are those on AB. None of these can be ruled out, as C has just been ruled out, because any move of D nearer to A will make it further from B. The strong feature of the above argument is that D is nearer to *both* communities than is C. Now the problem is to find D, or equivalently to find x, $-1 \leqslant x \leqslant +1$, such that $(x,0)$ is the best place.

So far quite weak assumptions have been made. To make further progress we need to be more precise. The dominant considerations in choosing the site will be the distances the two communities have to travel to reach the shelter and whether these distances can be covered in the interval between receipt of the tornado warning and the arrival of the tornado. If the shelter is too far away the community will be exposed to the full force of the wind: otherwise they will be safe in the shelter. To model this, introduce S as the distance that a community will be able to travel between the warning and the arrival of the tornado. We suppose that S is a continuous random variable having probability distribution function $P(s) = Pr(S \leqslant s)$ and probability density function $p(s)$ given by

$$p(s) = \frac{dP(s)}{ds}.$$

Of course $P(0) = 0$, $P(s)$ increases with s, $\lim_{s\to\infty} P(s) = 1$, and

$$P(s) = \int_0^s p(u)du.$$

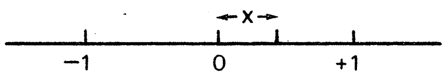
One other assumption has to be made. This concerns how serious it is for a community to be caught outside the shelter when the tornado arrives. There are three possibilities

(a) both communities are in the shelter,
(b) one is in and one is outside the shelter,
(c) both are outside the shelter.

The seriousness of these will be assigned values 0, 1 and 2 respectively: the units will not matter. Equivalently 1 is the loss of one community: both communities lost counts double. There are obviously other possibilities.

Now suppose the shelter is at x (see Figure 3)

Figure 3.



Then the probability that the community at A will be outside the shelter is the probability that the most it can cover is $1 + x$: this probability is $P(1 + x)$. For B the similar value is $P(1 - x)$. Each of these events will cause a loss of 1. So the loss expected with a shelter at x is

$$P(1 + x) + P(1 - x). \tag{1}$$

(In amplification: an expected loss is obtained by multiplying each possible loss by the probability of that loss and adding over the possible values. Here each community can either lose zero or one. The first of these contributes zero to the expected loss, the only term is therefore one times the probability.) There are cogent reasons for choosing x to minimize the expected loss and so we now have a little problem in calculus, namely to minimize (1) with respect to x.

A minimum is typically found by differentiating and equating the result to zero. If we do this here and remember that $dP(s)/ds = p(s)$ we easily find

$$p(1 + x) - p(1 - x) = 0 \tag{2}$$

and solutions to this equation are candidates for the site of the shelter. We must however remember to check (a) whether solutions are truly minima (and not maxima), (b) if there are several local minima which is the least, and (c) to look at the end points $x = \pm 1$ since minima there are not necessarily found by differentiation.

Let us try a few cases. The simplest distribution is

$$p(x) = \lambda e^{-\lambda x},$$

for some positive λ, with

$$P(x) = 1 - e^{-\lambda x}. \tag{3}$$

Equation (2) is

$$e^{-\lambda(1+x)} = e^{-\lambda(1-x)}$$

or on taking logarithms

$$-\lambda x = +\lambda x$$

with unique solution $x = 0$ irrespective of λ. This is easily seen to be a minimum and to give a value of (1) less than at $x = \pm 1$. So we conclude that the best place for the shelter is at the origin, namely mid-way between the two communities. At this point some of you will say that is obvious without any mathematics. But let us consider other cases.

The form $p(x) = \lambda e^{-\lambda x}$ is not very realistic. The exponential decreases with x and hence so does $p(x)$ implying that a very small distance of travel is more likely than a large one. This is not true: usually the warning time is quite appreciable and the most probable value is about 15 minutes. Therefore a more sensible distribution of distance would have the general shape illustrated in Figure 4, rising to a maximum and then falling as s increases. A simple

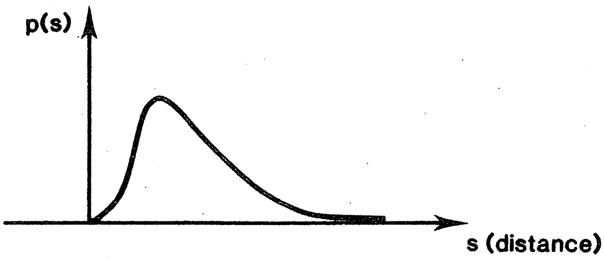


Figure 4.

function with this property is

$$p(s) = \lambda^2 s e^{-\lambda s} ,$$

for some positive λ. Equation (2) is then

$$\lambda^2 (1 + x) e^{-\lambda(1+x)} = \lambda^2 (1 - x) e^{-\lambda(1-x)}$$

or

$$e^{2\lambda x} = \left(\frac{1 + x}{1 - x}\right) . \qquad (4)$$

Now the roots are less clear. An obvious one is $x = 0$ as before: but are there others? To consider this take logarithms of both sides of (4) and consider whether

$$\log\left(\frac{1 + x}{1 - x}\right) - 2\lambda x$$

equals zero. The derivative of this function is

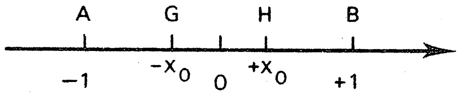
$$\frac{2}{1 - x^2} - 2\lambda$$

and there are two possibilities (excluding the intermediate case $\lambda = 1$)

(a) $\lambda < 1$ and the derivative does not vanish in $-1 \leqslant x \leqslant 1$, or

(b) $\lambda > 1$ and the derivative vanishes twice in $-1 \leqslant x \leqslant 1$.

The first case, $\lambda < 1$, means that (4) has only one root, so this must be the obvious one, $x = 0$. Hence again the optimum place for the shelter is midway between the communities: (It is easy to show that $x = 0$ is a minimum, not a maximum, and is less than at the extreme values $x = \pm 1$.) The case $\lambda > 1$ is more subtle. Now there are 3 roots: one is our old friend $x = 0$ but there are two others at $x = \pm x_0$ for some x_0. It is easy to verify that $x = 0$ is now a local maximum! So that the mid-point is worse than nearby points. It is equally easy to see that $x = \pm x_0$ are both true minima. Consequently there are two, equally good, best sites for the shelter as shown in Figure 5 by the points G,H.



A numerical example may prove instructive. If $\lambda = 1\frac{1}{2}$ the roots are at $x_0 = 0{\cdot}859$ so that H is near B (at $x = 1$) and G is near A (at $x = -1$). To put the shelter at H means that the community at B will almost certainly be saved and that at A will not. To place it midway at $x = 0$ would place both communities in jeopardy. From the viewpoint of an outsider interested in saving the total number of people our mathematics has done a good job in finding G and H. From the viewpoint of either of the two communities our solutions are not so apposite, B desiring H and A going for G. We conclude with a few comments.

1. Notice how it was possible by a very elementary argument to rule out almost all possible sites, namely those not in AB. This is very common. In economics there are many cases where possible values could lie in a region yet all can be ruled out except those on the edge. These are often called Pareto optimal. The difficulty is to select amongst the few left: with us those in AB; in economics, those on the edge.

2. The quantity λ has a simple interpretation. $p(x)$ remember is the probability of being able to travel a distance x between the warning and the arrival of the tornado. The expected distance is therefore $\int_0^\infty xp(x)dx$ using the concept of expected value explained above and replacing summation by integration. With $p(x) = \lambda^2 xe^{-\lambda x}$, the distribution that caused the surprising roots when $\lambda > 1$, it is easy to see that the expectation is

$$\lambda^2\int_0^\infty x^2 e^{-\lambda x}dx = \lambda^{-1}\int_0^\infty u^2 e^{-u}du \qquad \text{with} \quad u = \lambda x.$$
$$= 2\lambda^{-1}.$$

Denote this by μ. Hence $\lambda > 1$ means that $\mu < \frac{1}{2}$. So that if the communities could only expect to cover *less* than half the distance between their homes and the midway site, it would be better to site the shelter near to one of the communities, at G or H.

3. When the mean is less than $\frac{1}{2}$ this problem illustrates in a rather simple situation the conflict between society and the individual. Looked at from society's viewpoint, where the communities are two individual groups, G or H are clearly the best sites: but from the viewpoint of one individual B, G is clearly bad. The concept of minimizing an expected loss

continued on p.28.

AN HISTORICAL CURIOSITY

J.A. Deakin,

Shepparton College of TAFE

Did you know that you can use your "sine tables" to find the product of two numbers? Neither did I until I read of a process called "prosthaphaeresic multiplication" in a recent publication (1), in which multiplication of numbers is carried out by the addition of trigonometric functions. My curiosity aroused, I consulted some references, and pass on to readers the results of my investigations.

The process known as prosthaphaeresis was in use for approximately 100 years prior to the invention of logarithms, and requires only the use of a table of sines (or cosines) for carrying out a multiplication. The method is based on the identity

$$\sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)] .$$

The angles A and B, whose sines, omitting the decimal point, are equal to the numbers to be multiplied, and can be found from a table of sines. Then $\cos(A - B)$ and $\cos(A + B)$ can be found from the same table, and half the difference of these gives the required product (2,3). An example, using modern notation, illustrates the procedure.

E.g. to find the product $83{\cdot}96 \times 236{\cdot}2$.

Since $83{\cdot}96 = 0{\cdot}8396 \times 10^2$ and $236{\cdot}2 = 0{\cdot}2362 \times 10^3$, we have $83{\cdot}96 \times 236{\cdot}2 = 0{\cdot}8396 \times 0{\cdot}2362 \times 10^5$. Now $0{\cdot}8396 = \sin 57°6'$ so that $A = 57°6'$ and $0{\cdot}2362 = \sin 13°40'$ so that $B = 13°40'$. Hence $A - B = 43°26'$ and $A + B = 70°46'$ and

$$\begin{aligned}\sin A \sin B &= \tfrac{1}{2}(\cos 43°26' - \cos 70°46')\\ &= \tfrac{1}{2}(\sin 46°34' - \sin 19°14')\\ &= \tfrac{1}{2}(0{\cdot}7262 - 0{\cdot}3294)\\ &= 0{\cdot}1984\end{aligned}$$

so that $83{\cdot}96 \times 236{\cdot}2 = 0{\cdot}1984 \times 10^5 = 19840$. A calculator gives 19831 for the product.

This historical curiosity illustrates the fact that in Mathematics, as in science, theories and methods become unpopular and are neglected, so that work passes into obscurity and is only brought to the notice of latter-day mathematicians by historians. With the now universal pocket calculator, will articles be written in the future in which the use of logarithm tables will be cited as an historical curiosity? Already slide rules are museum pieces, since they are no longer manufactured.

References:

1. Davis, P.J. and Hersh, R., The mathematical experience, p.19, Penguin, Harmondsworth, 1983.

2. Hobson, E.W., A treatise on plane and advanced trigonometry, Dover, New York, 1957.

3. Glaisher, "On multiplication by a table of single entry", Philosophical Magazine, 1878.

[This curiosity was the basis for Problem 3.3.3. Eds.]

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

SQUASH MATHEMATICS

Stephen R. Clarke,

Swinburne Institute of Technology

In a previous article† we saw how to evaluate chances of players winning matches, and the mean length of matches. In this article we see how to extend that for games like squash, volley ball and badminton, where a point is scored only if a player wins a rally which he has served.

But first consider a game up to n points where the winner of any rally scores a point and then serves for the next rally. [In fact this is how American squash is played.] Consider two players Alice and Bill. Let a be the probability that A wins the next point when A is serving, so $1 - a$ is the probability that B wins the point when A is serving. Similarly let b, $1 - b$ be the probabilities that B will win, lose the point when B is serving.

†See *Function, Vol.9, Part 2.*

Let $P(i,n)$ be the probability that A will win the game when the score is (i,j) and A is serving, and $P'(i,j)$ be the probability of A winning the game when the score is (i,j) and B is serving. Since if a player wins a point, he serves the next point we have:

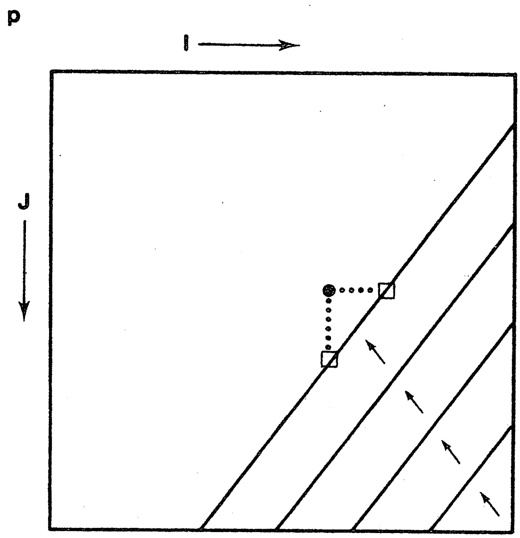
		a	A wins, Score $(i+1,j)$ A still serving so probability now $P(i+1,j)$
Score (i,j) A serving	$P(i,j)$		
		$1-a$	A loses, Score $(i,j+1)$, B now serving so probability now $P'(i,j+1)$

A similar diagram can be drawn if B is serving. So

$$P(i,j) = aP(i+1,j) + (1-a)P'(i,j+1)$$
$$P'(i,j) = (1-b)P(i+1,j) + bP'(i,j+1)$$
$$P(i,n) = P'(i,n) = 0$$
$$P(n,j) = P'(n,j) = 1 \qquad 0 \leqslant i,\ j < n \ . \qquad (1)$$

While these equations are slightly more complicated than we had before, they can still be solved iteratively, beginning at $i = j = n - 1$ and working down to $i = j = 0$, the probabilities at the beginning of the game. The following Microsoft Basic Program does this.[†]

[†]Observe that the table $P(I,J)$ is being filled in the manner suggested by the diagram:



```
10 DIM P(20,20),D(20,20)
20 INPUT "length of game n";N
30 INPUT "Probability of A, B winning serve";A,B
40 FOR I = 0 TO N-1
50 P(I,N)=0:D(I,N)=0:P(N,I)=1:D(N,I)=1
60 NEXT I
70 FOR I = N-1 TO 0 STEP -1
80 FOR J = N-1 TO 0 STEP -1
90 P(I,J) = A*P(I+1,J) + (1-A)*D(I,J+1)
100 D(I,J) = (1-B)*P(I+1,J) + B*D(I,J+1)
110 NEXT J
120 NEXT I
130 PRINT P(0,0),D(0,0)
140 END
```

Now in Australian squash, badminton and volley ball, you only win a point when you win a rally that you serve. Thus a point becomes a series of rallies, with serve alternating until a server wins a rally [or effectively until someone wins two rallies in a row]. The previous model applies, provided we ensure that a and b are probabilities of winning *points*, not rallies.

For simplicity, we will assume player A has a constant probability p of winning a rally. Thus $q = 1 - p$ is the probability of B winning any rally. In squash, unlike tennis, there is not much advantage in serving. However you could repeat this with different probabilities P_A, P_B of players A and B winning their serve.

Consider a point (i.e. a sequence of rallies) to which A begins serving. If A denotes rally won by A, B won by B, this sequence could be any of

$$A, BB, BAA, BABB, BABAA, BABABB, \ldots$$

These respectively have probabilities

$$p, qq, qpp, qpqq, qpqpp, qpqpqq, \ldots$$

or denoting qp by x

$$p, q^2, xp, xq^2, x^2p, x^2q^2, \ldots . \tag{2}$$

Thus the probability that A wins the point to which she begins serving is

$$a = p + xp + x^2p + \ldots = \frac{p}{1 - x} = \frac{p}{1 - pq}$$

and $$1 - a = q^2 + xq^2 + x^2q^2 + \ldots = \frac{q^2}{1 - x} = \frac{q^2}{1 - pq} .$$

In a similar manner, or by symmetry, the probability that B wins a point to which he begins serving is

$$b = \frac{q}{1 - pq}, \quad \text{so that} \quad 1 - b = \frac{p^2}{1 - pq} .$$

These can now be substituted in the original equations, or more simply an extra line in the program included to calculate a, b from p.

```
30 Input "Probability of A winning rally";P
35 A = P/(1 - P*(1 - P)): B = (1 - P)/(1 - P*(1 - P)).
```

Running this program with $p = q = \cdot 5$, $n = 9$ gives $P(0,0) = 0\cdot535$, $P'(0,0) = 0\cdot465$ which shows a 3% advantage in winning the toss at squash. These values can now be used as a and b with $n = 3$ to simulate the match which gives $P(0,0) = 0\cdot513, P'(0,0) = 0\cdot4857$. Using this method we can calculate chances of winning matches from any position for different players. For example a player who wins only 58% of rallies will win 96% of squash matches.

Note that in the above we have assumed games are played to 9 points. In fact at 8-8, the receiver has the choice of playing to 9 or 10. We could use the program above to determine if $P'(0,0)$ is greater for a 2 point or 1 point game, and so decide the best strategy for the receiver.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

continued from p.23.

is a beautiful solution to society's problem but is less satisfactory to the individual B who partially perceives the situation as a conflict between itself and A. A major mathematical problem, at present unsolved, is that of finding solutions acceptable to individuals in disagreement. The problem becomes even more serious when individuals are replaced by governments and disagreement by conflict. Game theory is a branch of mathematics that has something to say about conflict but is totally inadequate as a model of most conflict situations.

4. Many generalizations of the simple problem are possible. What if the communities are unequal? Or there are more than two of them? There is a continuous generalization to where $\rho(x,y)$ is the population density at (x,y): where should a communal facility be placed? How could one allow for differences in the terrain?

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

WOMEN IN MATHEMATICS UPDATE

A recent article and letter in *Monash Reporter* (May, June 1985) highlight the extent to which young women mathematicians are beginning to penetrate a hitherto almost exclusively male field: Applied Mathematics. Of the postgraduate students enrolled in this field at Monash, sixteen are men and six are women. This gives the women 27% of the places. Still a long way to go till equality is achieved, but if we think that only a few years ago, the expected figure would have been 0%, Monash and the six women mathematicians involved have cause for some pride.

LETTER TO THE EDITOR

In *Function, Vol.9, Part* 1, the decimal expansion of $\frac{100}{81}$ is given and the question is asked what 1·234567890123456789O... equals. The answer to this is not particularly interesting, but what is of interest is the rapidity with which Euclid's algorithm tells us we can't do much factoring.

$$\begin{aligned} 1\cdot 234567890123456789 0\ldots &= 10 \times 0\cdot\dot{1}23456789\dot{0} \\ &= 10 \times \frac{1234567890}{9999999999} \\ &= 10 \times \frac{137174210}{1111111111} . \end{aligned}$$

Euclid's algorithm quickly (and mysteriously?) tells us that there is no further factoring.

$$\begin{aligned} \gcd(1111\ 111\ 111,\ 137\ 174\ 210) &= (1111\ 111\ 111,\ 13\ 717\ 421) \\ &= (101\ 010\ 101,\ 13\ 717\ 421) \\ &= (13\ 717\ 421,\ 4\ 988\ 154) \\ &= (13\ 717\ 421,\ 2\ 494\ 077) \\ &= (2\ 494\ 077,\ 1\ 247\ 036) \\ &= (1\ 247\ 036,\ 5) \\ &= 1. \end{aligned}$$

David L. Dowe,
Monash University.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

PROBLEM SECTION

SOLUTION TO PROBLEM 9.1.2.

If $A = \frac{1}{1984}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1984})$

and $B = \frac{1}{1985}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1985})$,

which is the larger, A or B?

This problem was solved by Hai Tan Tran of Plympton Park, S.A. and also by Justin Lazar of Malvern, Victoria.

Both solvers reached, with slightly different notation, the equation

$$B = \left(\frac{1984}{1985}\right)A + \frac{1}{1985^2} .$$

Hai Tan Tran now argued: Suppose $B \geqslant A$, then $(1984)(1985)A + 1 > 1985^2 B$ and as B is assumed greater than or equal to A, we would need $A < \frac{1}{1985}$, which is readily shown to be false. Thus $A > B$.

Justin Lazar used the equation in a slightly different way, writing

$$x = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{1984} .$$

He then found $A/x = 1/1984$,

$$B/x = (1/1985) + (1/1985^2 x)$$

$$< (1/1985) + (1/1985^2)$$ (as x is clearly greater than one). This last value he computed and found to be still less than A/x. Thus $A > B$.

Another more general, and hence more informative, approach is possible.

Put

$$A(n) = \frac{1}{n}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n})$$

and ask when is $A(n) > A(n + 1)$?

The condition for this becomes

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \frac{n}{n+1}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) + \frac{n}{(n+1)^2}$$

or

$$\frac{1}{n+1}(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) > \frac{n}{(n+1)^2}$$

or

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} > \frac{n}{n+1} = 1 - \frac{1}{n+1} .$$

Clearly this last statement is true and so the value of $A(n)$ always decreases as n increases.

We would like to give potential solvers more time with the other problems and so give no other solutions this time. Instead, we go straight to this issue's problems.

PROBLEM 9.3.1 (Submitted by Garnet J. Greenbury).

(i) Show that any positive integral power of the product of the first four odd numbers leaves a remainder 1 when divided by 8 or 13.

(ii) Find the set of numbers, such that any one of them when divided into $(5 \times 7 \times 11)^n$, where n is any positive integer leaves a remainder of 1.

(iii) Is there a similar result for $(7 \times 11 \times 13)^n$?

PROBLEM 9.3.2 (Submitted by John Barton from E.W. Hobson's "A Treatise on Plane and Advanced Trigonometry").

Prove that a triangle with angles A, B, C is equilateral if $\cot A + \cot B + \cot C = \sqrt{3}$.

PROBLEM 9.3.3 (Submitted by Hai Tan Tran).

If O is the centre of a circle and M lies on its circumference and if A, B lie outside the circle, show that $AM + MB$ will be maximised if OM bisects $\angle AMB$. Show the same property if this distance is to be minimised.

PERDIX

Some good news! The Federal government, through the personal intervention of Prime Minister Bob Hawke, so we are told, has come to the help of the Australian 1988 Mathematical Olympiad hopes. In the bicentennial year 1988 Australia has offered to act as host country for the International Mathematical Olympiad. The host country pays all expenses of competing teams once they have arrived in the country. Estimated cost in 1988 for Australia will be about $300 000. The offered Federal support will be $150 000.

Australia still remains virtually the only country competing in the International Olympiad whose team's travelling and training expenses are not fully guaranteed by the government.

During January 1986 various mathematics camps will be held, as in previous years. In Victoria there are two:

Somers:: 28 January to 2 February : Education Department Mathematics camp.

Lady Northcote:: 27 January to 1 February : Mathematical Association of Victoria camp.

There is also the

Canberra:: 7 January to 19 January : National Mathematics Summer School.

The National Summer School is a selective camp for highly talented students, while Somers and Lady Northcote are open to all qualifying students subject to their school's recommendation.

For further information consult your teachers or contact the Mathematical Association office, telephone number (03)347 5329.

* * * * *

Now for some more geometry.

Here is our first problem set in three dimensions, rather than just in a plane.

PROBLEM 13. (Question 3 of the 14th U.S.A. Mathematical Olympiad, April 23, 1985: this was one of 5 questions that had to be completed in $3\frac{1}{2}$ hours.)

Let A, B, C and D denote any four points in space such that at most one of the distances AB, AC, AD, BC, BD and CD is greater than 1. Determine the maximum value of the sum of the six distances.

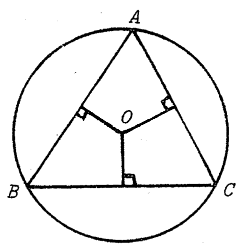
Send me solutions please.

RESULT 9. If two circles touch at a point P then the line joining the centres of the two circles passes through P.

PROBLEM 14. Suppose we have a circle centre A. Now choose any two circles, centres B and C, say, that lie inside and touch the circle centre A, and which also touch each other externally. Show that the length of the perimeter of triangle ABC is independent of the choice of the two circles lying within the circle centre A.

We now state some well-known results about lines associated with a triangle that meet at a point, i.e. that are *concurrent*.

RESULT 10. The perpendicular bisectors of the sides of a triangle are concurrent. The point of concurrency O is called the CIRCUMCENTRE of the triangle.

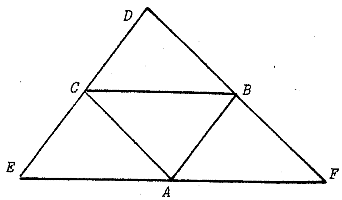


[This is easy to see: for let ABC be a triangle and consider the circle passing through A, B, and C, called the CIRCUMSCRIBING CIRCLE OF ABC. Let its centre be O. Then $OA = OB = OC$. Hence the perpendicular from O to AB bisects AB, i.e. the perpendicular bisector of AB passes through O.]

RESULT 11. The three perpendiculars from vertices to opposite sides of a triangle are concurrent. The point of concurrency H is called the ORTHOCENTRE of the triangle.

We can deduce Result 11 from Result 10 by showing that H is the circumcentre of another triangle. Indeed, let ABC

be a triangle and through each vertex draw a line parallel to the side opposite. The lines through B and C then meet at D, say, through C and A meet at E, say, and through A and B meet at F.

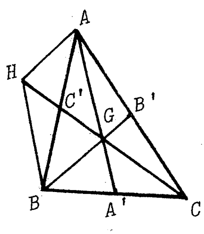


$BCEA$ is a parallelogram, so $BC = AE$; and $BCAF$ is a parallelogram, so $BC = AF$. Thus A is the mid-point of EF. Similarly B is the mid-point of FD and C is the mid-point of DE. The circumcentre, H, say, of DEF is the intersection of the perpendicular bisectors of EF, FD and DE (Result 10). But these perpendicular bisectors are just the lines from the vertices A, B and C perpendicular to the opposite sides.

RESULT 12. The three lines, called MEDIANS, each of which connects a vertex of a triangle to the mid-point of the opposite side, are concurrent. The point of concurrency G is called the CENTROID of the triangle.

[G is the centre of gravity of three equal masses each at one of the vertices; it is also the centre of gravity of a plane sheet, imagined to coincide with the triangle, and of uniform density.]

Using the comment about centres of gravity it is easy to establish that the three medians meet in a point. Here is an alternative geometrical proof.



Let ABC be the triangle and let A', B' be the mid-points of the sides (see diagram) BC, CA respectively. Let AA' and BB' meet at G. Extend CG to H, so that $GH = CG$. Then $B'G$ joins the mid-points of the sides CA, CH of the triangle CAH. Hence $B'G$, and so BG is parallel to AH. Similarly, AG is parallel to HB. Hence $HAGB$ is a parallelogram. But the diagonals of a parallelogram bisect each other. Hence, in particular, if CG meets AB at C', we have $AC' = C'B$. Thus CC' is the third median and it passes through G.

Note the following corollary to the proof.

COROLLARY. $C'G : GC = A'G : GA = B'G : GB = 1:2$.

This follows because GH is bisected at C' and so $C'G = \frac{1}{2} GC$. By symmetry, $A'G = \frac{1}{2} GA$ and $B'G = \frac{1}{2} GB$.