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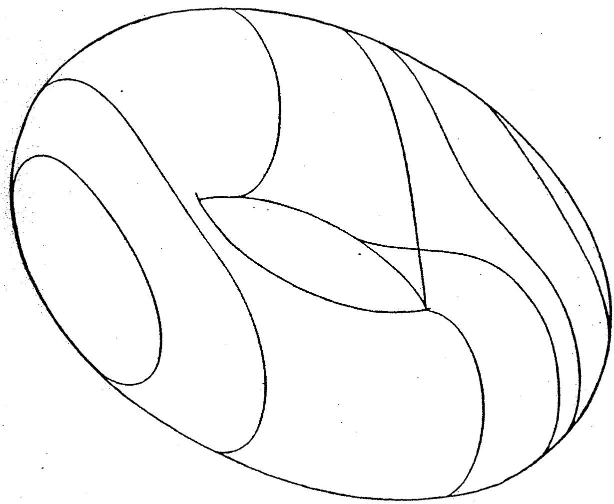

A SCHOOL MATHEMATICS MAGAZINE

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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Clayton, Victoria, 3168.

Alternatively correspondence may be addressed individually to any of the editors at the mathematics departments of the institutions shown above.

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One of *Function's* predecessors was the widely read and influential *Ladies' Diary*, an intellectual journal for women, notable for its inclusion of a mathematical section. Over 100 years ago, mathematics was seen as part of the general culture and no one saw anything strange in a women's magazine discussing mathematics. How times have changed!

The concept of area and its generalisations have generated much important mathematics from the days of Archimedes to the present. Concepts important to calculus and probability theory find their origins in this aspect of mathematical enquiry.

Just two of the stories told in this issue of *Function*.

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THE FRONT COVER

J.C. Stillwell and M.A.B. Deakin,
Monash University

If a circle C is rotated about an axis A that lies in its plane but does not intersect it (Figure 1) the result is a circular tube - a donut-, or ring-shaped object now referred to as a torus. The torus was one of the few surfaces studied by the Greeks of antiquity. (The others were the sphere and its generalisation the spheroid, the cylinder and the cone.) The Greek name for the torus was *spira*.

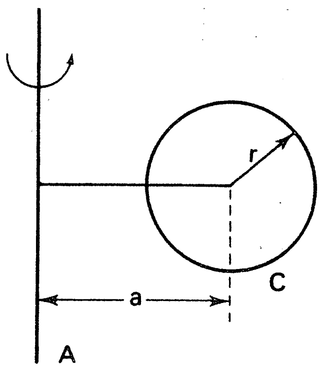


Figure 1

Imagine now a plane slicing through a torus in a direction parallel to the axis A. The intersection will be a curve whose shape depends on the distance between the plane and the axis. Six such curves are shown in perspective on the cover. From left to right they are

(a) a convex, or outwardly bulging, oval
(b) a pinched or squeezed oval
(c) a pair of ovals
(d) a figure eight curve
(e) another pinched oval
(f) another convex oval.

These curves are known as *spiric sections*, and it is believed that they were first discussed by the Greek geometer *Perseus*. Very little is known about Perseus. It is thought that he lived in the third century BC, but even this date is doubtful. We know about him from two rather obscure passages in a work by the later Greek geometer *Proclus*, who relied on the account given by yet another author *Geminus*.

† This is based in part on a discussion in E. Brieskorn and H. Knörrer's *Ebene Algebraische Kurven* (Algebraic Plane Curves) published by Birkhäuser (1981). Birkhäuser will also be publishing the English translation of this work, prepared by J.C. Stillwell.

If r is the radius of the circle C, and if a is the distance between the axis and the centre of this circle, and k is the distance from the axis to the plane intersecting the torus, then we have convex ovals (a), (f) if

$$a \leqslant k < a + r.$$

This one Proclus listed as the second curve "broad in the middle". Pinched ovals (b), (e) are produced if

$$a - r < k < a$$

(Proclus' third type "narrow in the middle"). If

$$k = a - r,$$

the figure eight curve (d) is produced. This was Proclus' first type, which he called the "hippopede", or "horse-fetter".

Finally, if

$$0 \leqslant k < a - r,$$

a pair of ovals results.

Quite what Perseus discovered of all this is a matter of some doubt. One authority, Ivor Bulmer-Thomas, believes that he ignored the last possibility and the case $a = k$ (absorbing it as we do into the case $a < k$) and thus found three types out of a possible five. Proclus wrote "Three lines upon five sections finding, Perseus made offering to the gods therefor". Other different interpretations have been advanced to explain this obscure passage.

For a suitable choice of torus, the figure eight curve becomes the *lemniscate of Bernoulli*, named after the seventeenth century mathematician James Bernoulli. This curve, whose equation is

$$(x^2 + y^2)^2 = (x^2 - y^2)(a^2 + r^2) \tag{1}$$

in the notation used above, was the cover illustration for *Function Vol.*1*, Part* 4 (1977).

The convex ovals specialise in this case to the so-called Cassini ovals, named after the astronomer Cassini (1625 - 1712) who also gave his name to a prominent division in the rings of Saturn. Cassini opposed Newton's theory of gravitation and believed that planets travelled along Cassini ovals, not ellipses as Kepler had proposed. Needless to say, this theory is now discredited.

Equation (1), containing powers up to, but not exceeding, the fourth, is an example of a fourth degree equation. Curves whose equations are fourth degree equations are called *curves of degree four* and their investigation goes back to the seventeenth century. The spiric sections are all curves of degree four.

PROPERTIES OF A CYLINDRICAL COIL

Neil S. Barnett,

Footscray Institute of Technology

Some time ago a radio technician approached me regarding the following problem.

A difficulty had arisen in connection with the use of a cylindrical antenna. For reasons I can't now recall he needed to know the inter-relationship between the antenna length (ℓ), cylinder diameter (D), wire length (L), the number of coils (x) and the so-called pitch (p). Figure 1 illustrates these dimensions.

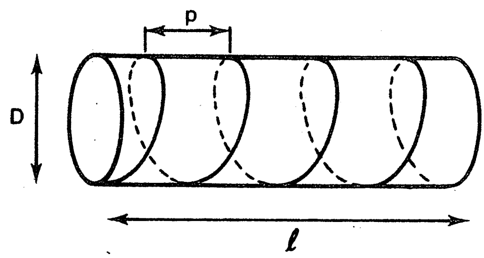
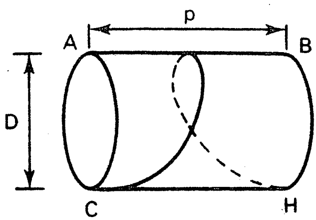


Figure 1

One relationship is quite trivial, namely $\ell = xp$ (assuming that the antenna is made up of equally spaced complete coils).

To relate the wire length (the total length of wire used in the coils) to the other dimensions we need, for a start, the length of wire used in a single coil. To calculate this, consider a hollow cylinder with a single coil drawn around the outside, as illustrated in Figure 2.



(*A* vertically above *C*)

Figure 2.

Imagine, now cutting along *AB* and opening out the cylinder; this gives rise to Figure 3.

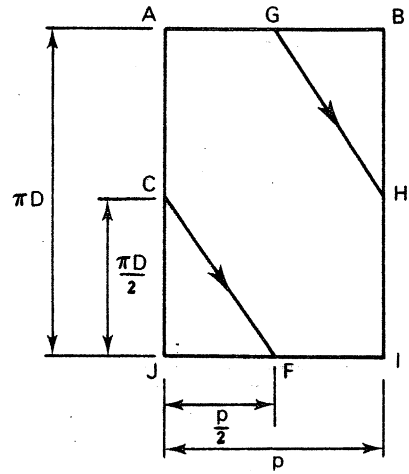


Figure 3.

Lines *GH* and *CF* are the two parts that combine to form the single coil when the cylinder is re-constructed.

If you find this hard to visualise, then actually do it, in reverse: cut out a rectangular strip of paper (say 5cm×12cm) and rule lines joining the mid points of adjacent sides as in Figure 3 (i.e. draw *CF* and *GH*). Press hard then turn the strip over and go over the impressions of these lines on the reverse side. Label the corners *A*, *B* and the point *C* on both sides of the paper. Now bend *IJ* to meet *AB* to form a cylinder and you will observe that the drawn lines on one side of the paper strip form a single coil round the outside of the cylinder.

From Figure 3 the geometry to obtain the length of wire used in a single coil is straightforward.

Note that *CF* and *GH* are the same length. By Pythagoras'

Theorem $CF = \left(\frac{p^2}{4} + \frac{\pi^2 D^2}{4}\right)^{\frac{1}{2}}$. The same result could also be obtained by cutting Figure 2 along CH to obtain Figure 4.

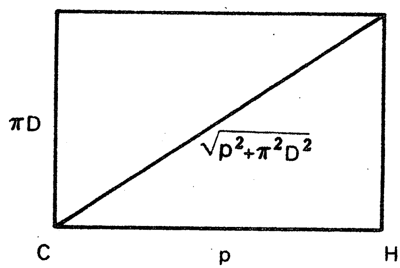


Figure 4.

Thus if x is the number of coils in the antenna,

$$L = x(p^2 + \pi^2 D^2)^{\frac{1}{2}}$$
$$= \frac{\ell}{p}(p^2 + \pi^2 D^2)^{\frac{1}{2}}. \qquad (1)$$

It would be expected that L and ℓ be almost equal when D is very small - we say that we would expect L to approach ℓ as D approaches 0 i.e. $L \to \ell$ as $D \to 0$. Putting $D = 0$ into (1) we obtain $L = \ell$ consistent with this expectation.

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FROM THE INDIAN SCRIPTURES

This beautiful well in the form of mathematics excels even the orb of the full moon (with all its digits) for it increases in form even when it is tasted (multiplied) by the learned ones, unlike the moon, who decreases in form when tasted by the Gods; and also this science of mathematics has both wings or sides (of an equation in a primary division) quite distinct, unlike the moon, who is hardly seen on the first day of both the fortnights of a month. What is the good of saying too much? Whatever there is in three worlds moving and non-moving - all that cannot be understood without the help of mathematics.

A DIARY FIT FOR EVERY LADY'S TOILET, AND GENTLEMAN'S POCKET[†]

Gilah C. Leder, Monash University

In the eighteenth century illiteracy was still widespread in England. It is commonly believed that in the 1750's, approximately 60 per cent of adult males and perhaps 40% of adult females were literate. As "writing was taught to those who could read 'competently well', and figures were taught only after the art of writing had been mastered", (Schofield 1968, p.316) levels of numeracy would have been considerably lower still. The prevailing attitude that only the leisured classes should be educated and have leisure pursuits and that anything which distracted the labouring classes from their labours was dangerous, can be used as one very rough indicator of those with and without education. Schools that provided elementary education were few and grossly inadequate. For girls the educational picture was particularly dismal. Typically, girls born into aristocratic families received their education from a governess in their own home, while the daughters of successful tradesmen and merchants were more likely to be sent to boarding school where the main emphasis was placed on accomplishments such as singing, dancing, painting and needlework. In addition they might be taught a little French, reading, writing and sufficient arithmetic for them to be able to keep household accounts.

It is against this background that John Tipper launched (in 1704) a new periodical, the *Ladies' Diary*. The scope of the publication was set down in the preface of the first issue.

[†] This quotation has been taken from the front page of the *Ladies' Diary* published in 1753. The word *toilet* is archaic. We would say *handbag*.

THE LADIES' DIARY,

FOR THE YEAR OF OUR LORD

1840,

Being Bissextile or Leap Year,

DESIGNED PRINCIPALLY FOR THE AMUSEMENT AND INSTRUCTION OF THE

FAIR SEX.

WITH AN APPENDIX OF CURIOUS AND VALUABLE MATHEMATICAL RESEARCH, FOR THE USE OF STUDENTS.

The Hundred and Thirty-seventh Almanack published of this Kind.

Virtue and taste, with female softness join'd,
All that subdues and captivates mankind,
In Britain's matchless fair resplendent shine;
They rule love's empire by a right divine:
Justly their worth the enraptured world admires,
Whom fair VICTORIA's bright example fires.

LONDON:

PRINTED FOR

THE COMPANY OF STATIONERS

And sold by G. GREENHILL, at their Hall, Ludgate street.

[*Price One Shilling, stitched.*]

PRINTED BY C. ADLARD, BARTHOLOMEW CLOSE.

> The Woman's Almanack is ... a book designed on purpose for the diversion and use of the fair sex, which shall contain (besides those things common to other almanacks) something to suit all conditions, qualities and humours. The ladies may here find their essences, perfumes and unguents; the waitingwomen and servants, excellent directions in cookery, pastry, and confectionery; the married shall have medicines for their relief, and instructions for the advancement of the families; the virgin directions for love and marriage...; mothers shall have rules for the education of their children, and those that delight in gardening, painting, or music, shall not want assistance to advance their pleasures; in sum, the ingenious shall have something exalted to exercise their wit, and the meanest some subjects adapted to their level ... . To conclude, nothing shall here be, but (what all women ought to be) innocent, modest, instructive, and agreeable.

It appears that Tipper wanted the *Ladies' Diary* to be educational as well as entertaining. Already in the second issue he wrote:

> There is nothing more strange and surprising to some persons than the various motions and appearances of the sun and moon. I shall endeavour to explain the motions of these two great lamps of heaven, and that by an instrument very familiar to the female sex; I mean by the rim of an ordinary spinning wheel.

It is not clear what prompted Tipper to insert two arithmetical questions in the *Diary* for 1707. As master of Bablake school and a mathematician of considerable ability he was certainly qualified to do so. Perhaps he was given the idea by a reader, Mr John White, of Rutterly, in Devon, who enclosed the two arithmetic enigmas used in the Diary of 1707 when he sent in his answers to two enigmas posed in the *Ladies' Diary* of the previous year. It is also possible that Tipper responded to the main theme of the influential "Essay on the usefulness of mathematical learning", published in 1700, that with the growing relevance of mathematics to an ever widening field of activities the study of mathematics should be readily accessible to more people. The inclusion of some mathematics problems in his annual publication may have been an attempt to meet this need. The first two questions give an indication of the level of mathematical proficiency involved.

> 1. In how long time would a million of millions of money be in counting, supposing one hundred pounds to be counted every minute without intermission, and the year to consist of 365 days, 5 hours, 45 minutes?
>
> 2. If to my age there added be one half, one third, and three times three; six score and ten the sum you'd see, pray find out what my age may be?

Tipper's innovation seems to have been welcomed by the *Diary's* purchasers, for more mathematical questions were inserted in all subsequent issues. In fact, by 1710 most of the domestic articles were omitted so that the *Diary* contained predominantly mathematical questions and poetical enigmas.

The emphasis of the *Ladies' Diary* on the mathematical problems was continued by Henry Beighton, who became its second editor in 1714. He justified this bias in his editorial comments in the issue of 1718:

> I believe that the *Diary* has the good fortune to fall into a multitude of hands which mathematical books seldom or never would ... Foreigners would be amaz'd when I show them no less than 4 or 500 several letters from so many several women, with solutions geometrical, arithmetical, algebraical, astronomical and philosophical.

Subsequent editors also retained the *Diary's* mathematical sections. For well over one hundred years, from the modest two problem beginning in 1707 until the final issue in 1840, the *Ladies' Diary* made contemporary mathematics readily available to women and men with some education. It is worth recalling that throughout this long publication history, the word *Ladies'* was retained in the title, presumably because women continued to buy the *Ladies' Diary*. This impression is heightened by the inclusion, periodically, of problems which seemed to be written particularly for a female readership.

One question is reproduced here:

> Dear Ladies fair, I pray declare,
> In Dia's page next year,
> When first it was I 'gan to pass
> My time upon this sphere.
> My age so clear; the first o' the year
> In years, in months, and days
> With ease you'll find, by what's subjoin'd*
> Exact the same displays.

$$\left.\begin{aligned} {}^*xy + z &= 238 \\ xz + y &= 158 \\ x + y + z &= 39 \end{aligned}\right\}$$ where x = the year, y = the months, and z = the days of my age, the 1st January, 1795.

(Question 984, 1795, submitted by Miss Nancy Mason)

However, while the mathematical content of successive issues generally reflected the increasing mathematical sophistication of the eighteenth and nineteenth centuries, it continued to be the somewhat simpler arithmetic and algebra questions which were addressed to women. Questions that had direct application to navigation, to engineering, to physics and other sciences, i.e., questions with practical applications and which improved "youth in numbers and fitted them for business" were stated without this bias. Thus one is left with the impression that women participated predominantly in those areas of mathematics consistent with leisure pursuits rather than in those with vocational overtones.

The general importance attached to the mathematical content of the *Ladies' Diary* is perhaps best illustrated by the publication of two sets of books. In 1775 Charles Hutton, professor of mathematics at the Royal Military Academy and editor of the *Ladies' Diary* from 1774 to 1818, published the *Diarian Miscellany*. Its subtitle indicated that the book combined "all the useful and entertaining parts, both mathematical and poetical, extracted from the *Ladies' Diary*, from the beginning of that work in 1704, down to the end of the year 1773".

Before actually publishing the *Diarian Miscellany* Hutton had tried to assess public demand for his initiative. "Such gentlemen as please to encourage this undertaking are desired to signify it by a line directed to Mr Hutton" (The *Ladies' Diary*, 1771, p.46). Its acceptance seems to have been considerable, for in 1817 the venture was repeated by Thomas Leybourn. The latter also modified some of the original questions. Those which had originally appeared in verse "which in almost every case was bad, and often hardly intelligible, are, generally speaking, changed into plain but perspicuous prose" (Leybourn 1817, vi). It should be noted that the preference of many of the contributors, particularly in the early years, for mathematical problems posed and answered in verse was hardly conducive to mathematical sophistication. Nevertheless, in the prefaces to their books, both authors emphasized the contribution of the *Ladies' Diary* to the study and improvement of mathematics.

They also stressed that the mathematical section of the *Ladies' Diary* was "the result of the joint labour of almost all the mathematicians of eminence, that have appeared in England in the course of the last century" (Leybourn 1817, vi). (Hutton singled out in particular the contributions of William Emerson, John Landen and Thomas Simpson.) Yet the editors never lost sight of their primary target audience; the amateur mathematician. In one issue (the *Ladies' Diary* of 1748) contributors were urged to check carefully the problems they submitted "because if errors are printed they reflect upon the *Diary* as well as the contributors". They were also urged to work on their solutions until they were satisfied with their quality.

Possibly to stress this point, particularly elegant solutions were singled out for praise. In yet another issue (the *Ladies' Diary* of 1791) a contributor using the pseudonym of Dynamicus, was warned that his "ingenious problems and dissertations ... are rather too long, and of too intricate a nature, for the plans of the Diary". He was advised that his contributions might be more suitable for inclusion in a more extensive work like the *Philosophical Transactions*.

The flavour of the mathematical content of the *Ladies' Diary* can perhaps be best conveyed by reproducing some of the problems it contained. They are taken from Leybourn's (1817) compilation of the *Diary's* mathematical sections.

Question 76 (from the *Ladies' Diary*, 1720) proposed by Mr W. Crabb.

> If the side of the face of each of the regular solids (or platonic bodies as they are called) be 29 inches: What is the content of each in wine gallons?

Question 597 (from the *Ladies' Diary*, 1769) proposed by Mr Tho. Sadler.

> Dear Ladies, you with ease may find*
> A matchless hero's name,
> Who was beloved by mankind,
> And mounted up to fame:
> To serve his country boldly dar'd
> Hot sulphur, smoke and fire,
> And long campaigns' fatigue be shar'd,
> To conquer proud Monsieur.

*viz. From the equations

$$\begin{aligned}(w + x + y + z &= 52)\\(wx + yz &= 360)\\(wz + xy &= 280)\\(wy + yz &= 315)\end{aligned}$$

where z, x, y and z denote the places of the letters in the alphabet, comparing the gentleman's name.

Question 1201 (from the *Ladies' Diary* of 1809) proposed by Mr T. Myers, R.M.A.

> What is the area of a right-angled triangle, the radius of its circumscribing circle being twenty, and the area of the inscribed circle a maximum?

The passage of time and the inadequacy of the historical records make it difficult to quantify the full impact of the *Ladies' Diary*. Nevertheless, the *Diary's* contents offer a fascinating overview of popular mathematics in the eighteenth and early nineteenth centuries. Or, in the words of Thomas Leybourn (1817, p.xi) they "exhibit to the public a picture of the taste of the British nation, for the study of Mathematics, which has certainly more cultivators ..., than is commonly supposed". The fact that a number of issues of the Ladies Diary found their way into Australian libraries is suggestive of the priorities of some of our early settlers!

References:

Hutton, C.: 1775, *The Diarian Miscellany*, (3 vols) Robinson and Baldwin, London.

Leybourn, T.: 1817, *The Mathematical Questions proposed in the Ladies' Diary*, (4 vols), Mawman, Oxford.

Schofield, R.S.: 1968, 'The measurement of literacy in pre-industrial England', in J. Goody (Ed.), *Literacy in Traditional Societies*, Cambridge University Press, Cambridge.

NOTE: Selected issues of the *Ladies' Diary* can be found in libraries in a number of capital cities in Australia, as well as in the Fawcett Library in London and the British Museum Library.

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OUR 1985 OLYMPIANS

We now know who will represent us at the International Mathematical Olympiad in Finland. They are listed below and we wish them all well. For the first time we have a girl on the team and we hope that this trend will continue.

Shane Booth	Wanganui Park High School, PO Box 1429, Shepparton, Victoria, 3630.
John Graham	St Ignatius College, Riverview, Tambourine Bay Road, Lane Cove, NSW, 2066.
Alasdair Grant	Melbourne C. of E. Grammar School, Domain Road, South Yarra, Victoria, 3141.
David Hogan	James Ruse Agricultural High School, Felton Road, Carlingford, NSW, 2118.
Andrew Hassell	Christ Church Grammar School, Queenslea Drive, Claremont, WA, 6010.
Catherine Playoust	Loreto Kirribilli, 85 Carabella St, Kirribilli, NSW, 2061.

Reserve:

Andrew Chen	Prince Alfred College, PO Box 571, Norwood, SA, 5067.

MATHEMATICAL MEASURE THEORY
I. ORIGINS

Joseph Kupka, Monash University

1. *What is mathematical measure theory?*

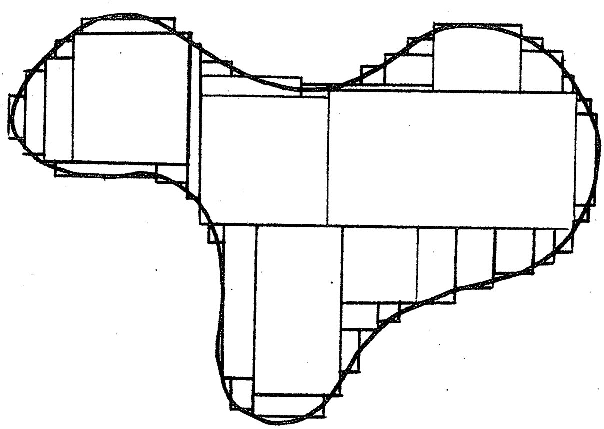
The art and science of measurement - which is the use of a number to describe the amount of some characteristic or quality possessed by an object - constitutes a cornerstone of modern science, and perhaps of all civilized society. Among the earliest qualities which were subjected to systematic measurement were *length*, *area*, and *volume*. Knowledge of the area of a plot of ground can be used to predict its crop yield or to help estimate the size of an enemy force encamped upon it.

Most ancient civilizations developed the basic idea of *units* of measurement. The Egyptian unit of length, for example, was called the *cubit*. The Egyptians lived on relatively flat expanses of ground, and it seems that they mostly subdivided their territory into rectangular plots whose areas could be expressed in whole numbers of square cubits. The ancient Greeks, on the other hand, inhabited hilly terrain with limited amounts of flat land available for agriculture. They needed to ascertain, and so to compare, the amounts of area in pre-existing, irregularly-shaped plots of ground. This accident of geography, coupled with the ever-present military threat posed by the Persian hordes to the east, is held to be largely responsible for the vastly greater sophistication of Greek mathematics over that of the Egyptians. And measurement is a key feature of their mathematics, not in the modern sense of assigning a number to an object, but in the sense of establishing exact relationships between various quantities (mainly lengths and areas). Perhaps the most famous example is the "$a^2 + b^2 = c^2$" relationship of the Pythagorean theorem. The Greeks looked upon this as the equality of two *areas*.

Mathematical measure theory is a branch of modern mathematics which deals with systematic techniques for measuring complicated or irregular objects when the measurements of simple objects are known in advance. Its central idea has a long lineage dating back to a technique invented by the Greeks.

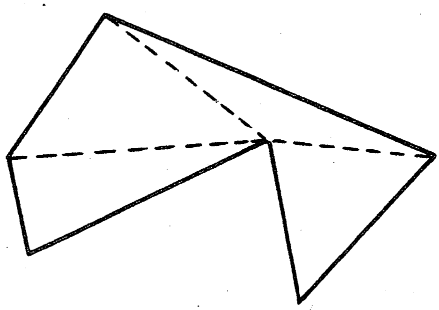
2. *From Euclid to Archimedes: The paving stone technique.*

The first great mathematical treatise, Euclid's *Elements of Geometry*, was written around 300 B.C. It contains many results about the areas of *rectangles* and *triangles* (or, strictly speaking, about the regions enclosed by these figures). These were to be the "simple objects" of area measurement. To obtain the areas of more complicated regions, the Greek mathematicians employed a natural elaboration of the already ancient practice of specifying lengths or distances in terms of whole numbers of *unit* lengths (as in a "span of five cubits" or a "journey of twenty leagues"). Their idea, which we shall refer to as the "paving stone technique", was to let the simple objects stand in place of the units, to treat these objects as "paving stones", and, in effect, to "pave" the given region as exactly as possible with variously chosen stones, thus:



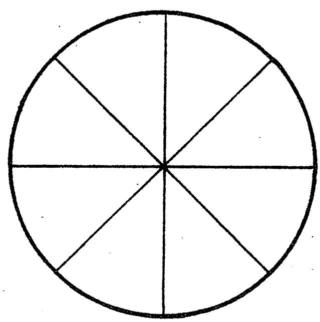
The unknown area is then *approximately* equal to the paved area, i.e. the area which is actually covered by the stones. It was (and is) considered to be part of the intrinsic nature of area that the whole should be equal to the sum of the parts, and so the paved area, in turn, is equal to the sum of the *known* areas of the individual (nonoverlapping) stones.

Such approximations certainly could be made precise enough to satisfy the practical requirements of the time. The greater sophistication of Greek mathmatics came about because of their desire for *perfect* exactitude. Consequently the triangle gained favour over the rectangle as a paving stone because any region bounded by straight lines could always be paved *exactly* by finitely many triangles, thus:

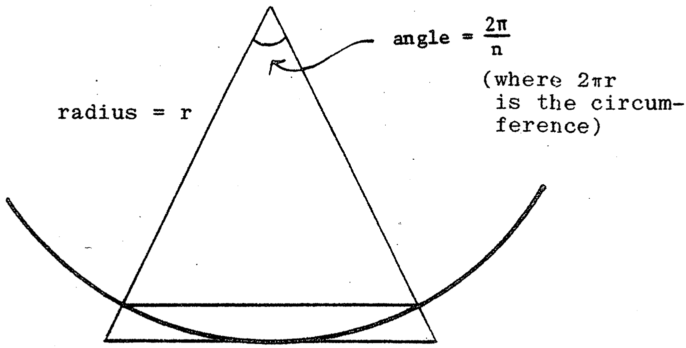


The real ingenuity of the Greeks lay in their derivation of exact information about the areas of regions bounded by *curved* lines. Such regions could not be exactly paved by stones which had uncurved edges. So the Greeks devised a "method of exhaustion" whereby the exact area is almost literally *squeezed* out of a great many inexact pavings. This method was their version of today's "limiting argument". It is used in Book 12 of Euclid to reveal the fact that the area of a circle lies in constant ratio to the area of the square on its radius. This famous ratio (the *number* π, as we know it) was later found by Archimedes to be less than $\frac{22}{7}$, but greater than $\frac{223}{71}$. We now know that π cannot be expressed as the *exact* ratio of two whole numbers.

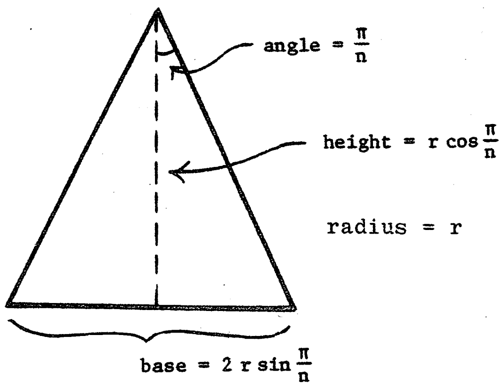
To illustrate how exact area measurements may be squeezed out of inexact pavings, we shall present a hybrid derivation of the area A of a circle in terms of its radius r. The triangulations (that is, the pavings with triangular stones) will be precisely those which Euclid considered, but the notation and limiting argument will be strictly modern. The basic idea is to subdivide the circle, in the manner of cutting up a pie, into a certain number, call it n, of equal slices, thus:



With each slice (or sector) of the circle, we associate two triangles, one lying entirely within the slice (the *inscribed* triangle), and the second completely covering the slice (the *superscribed* triangle), in the manner pictured below:



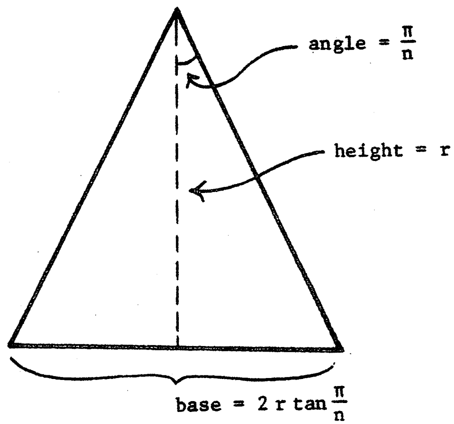
Notice that each triangle is isosceles and that the angle determined by the equal sides has radian measure $\frac{2\pi}{n}$. A closer look at the inscribed triangle, thus:



shows that its area is $r^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$. Let us now consider the n inscribed triangles as a paving (an inexact triangulation of the circle). The total paved area is then $\ell_n = n\, r^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$. And since the paved region lies *entirely within it* follows that $A \geqslant \ell_n$. But now observe that we really may use *any* number $n \geqslant 3$ in this argument. So it must also be true that

$$A \geqslant \lim_{n\to\infty} \ell_n = \lim_{n\to\infty} \pi\, r^2 \left(\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \right) \cos \frac{\pi}{n} = \pi r^2$$

because $\frac{\sin \theta}{\theta} \to 1$ if $\theta \to 0$, and $\cos \theta \to 1$ if $\theta \to 0$. (This is the "limiting argument".) A closer look at the superscribed triangle, thus:

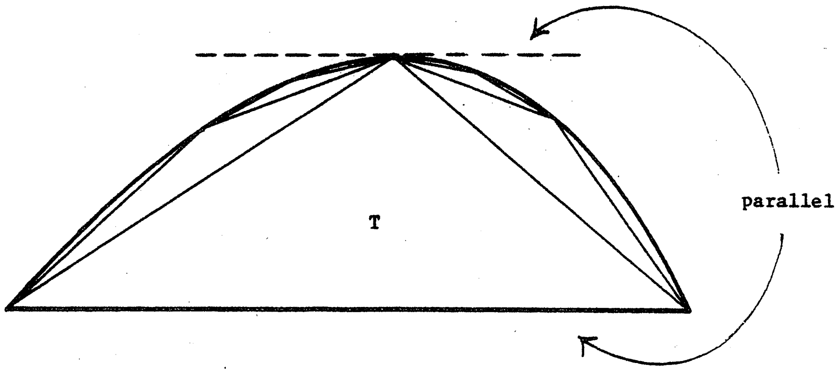


shows that its area is $r^2 \tan \frac{\pi}{n}$. The n superscribed triangles pave a region of total area $u_n = n\, r^2 \tan \frac{\pi}{n}$. This region *completely covers* the region inside the circle, and so $u_n \geqslant A$. Hence, as before:

$$A \leqslant \lim_{n\to\infty} u_n = \lim_{n\to\infty} \pi\, r^2 \left(\frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \right) \frac{1}{\cos \frac{\pi}{n}} = \pi\, r^2 .$$

The demonstration is now complete. We have *disqualified* all possible candidates for the true value of A except for one and one only: $A = \pi\, r^2$. Notice that it was necessary to consider *infinitely* many triangulations in order to achieve this exact result.

In an even more ingenious demonstration, Archimedes found the area of a parabolic segment by considering triangulations of the sort pictured below:

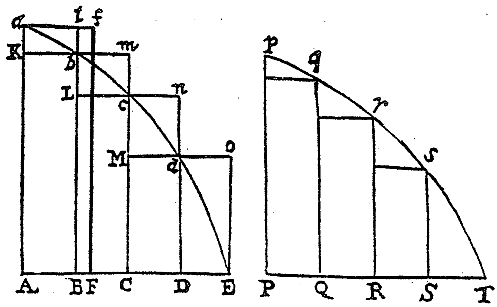


The uppermost vertex of the triangle labelled *T* is located at the precise point where the tangent to the parabola is parallel to the base of the segment. Archimedes showed that the area of the segment is exactly $\frac{4}{3}$ the area of *T*.

The tragic death of Archimedes at the hands of a Roman soldier in 214 B.C. marked the final stages of the Roman conquest of Greece. Nearly two millenia were to pass before very many exact results came to light which had not already been known to the Greeks, although a good deal of effort was devoted to approximations. Two things, perhaps, were needed before further progress became possible: (1) A better developed notion of *number*. Number is to measurement as money is to commerce. It provides a "common currency" for the description and comparison of areas and other types of measurement, whereas the Greek "barter system" only permitted one area to be directly compared to (or exchanged with) another. (2) The abandonment of the triangular paving stone. It made the calculations too difficult, and each particular case required too much ingenuity in order to ferret out workable triangulations. Undoubtedly such a step was psychologically very difficult. To abandon the triangle was to abandon Greek tradition.

3. *From Newton to Riemann: The Fundamental Theorem of Calculus.*

Near the beginning of Isaac Newton's *Principia Mathematica* (1687) we find the following figures:



These figures depict a spectacular breakthrough in the use of pavings to obtain exact numerical measurements of area. Triangular paving stones are nowhere to be seen. Newton had replaced them by rectangles and, what is more, by long, thin rectangles which cover the given region in the manner of wallpaper (or floorboards). If a more accurate paving was desired, the rectangles were made thinner, but not appreciably shorter. The curved part of the boundary of the region was precisely described by a mathematical *function* f, the area of the region was explicitly realized as a mathematical *limit* of paved areas, and this limit was called the *integral* of f over an *interval* $[a,b]$, or $\int_a^b f(x)dx$ for short. Most important of all, the Fundamental Theorem of Calculus made it possible (most of the time) to *calculate* this integral exactly, and to do so, moreover, without so much as a sideways glance at pavings. One simply needed to produce an *antiderivative* F of f, and the unknown area became just $F(b) - F(a)$. It is necessary to consider pavings (and so to perform "numerical integration") only on the occasions when F is unavailable. In this way the ad hoc methods of the past were supplanted by a systematic method of great power and scope.

All of these ideas appear in Newton's work, but they were not expressed with precision until the work of Cauchy in the 1820's. It was actually Cauchy who clarified the notions of derivative and integral by basing them upon the mathematical idea of limit. Cauchy's work was extended by Riemann in the 1850's, and the resulting integral bears Riemann's name. This is the integral which appears in all modern calculus textbooks.

Although the integral was originally conceived as a device for calculating areas, it became in the hands of Cauchy and Riemann more of an *abstract* technique of calculation. (Students of calculus will be aware that some of the rectangular "areas" are counted negatively in the "Riemann sums" which approximate the integral.) This abstractness enables it to be used in more than one way. It may be used to calculate areas, volumes, arc lengths, and many other physical quantities. Velocity is the integral of acceleration with respect to time, work is the integral of force with respect to distance, and so on. However, the spirit behind all of these applications is the paving stone technique. Indeed, the ancient Greek idea of using progressively smaller units of measurement (alias paving stones, alias gradations on some measuring device) to measure a physical quantity to a specified degree of exactitude is basic to virtually all numerical measurement in modern science.

Riemann himself employed integration as an averaging technique in the mathematical analysis of trigonometric series. His was perhaps the first voice of modern mathematics. The traditional mathematician said: "I see an attractive object. I shall study its properties." For Cauchy the attractive object was a continuous function. Its integral was one of its properties. The modern mathematician says: "I see an attractive property. I shall study the totality of objects which possess this property." Thus, a "Riemann-integrable function" is simply any function, however bizarre, for which *Cauchy's* definition of integral makes sense. The feelings of the traditionalists toward the modernists were summed up by the mathematician Hermite when he wrote: "I recoil in fright and horror from this lamentable plague of functions which do not have derivatives!"

The modern view has spawned a kind of mathematical heroism, a climbing-of-the-mountain-because-it-is-there. The result has been an abstract mathematics of great generality, depth, beauty, and unenvisioned application. The modern view has also spawned a cancerous overgrowth of abstraction, a kind of art-for-art's-sake. This disease, popularly known as abstractionitis, is frequently marked by willful obscurantism, fueled by the desperation to publish, and sustained by cult worship. For better or for worse, the evolution of the traditional mathematical study of measurement into modern mathematical measure theory will be governed by the modern view.

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STATISTICS IN GOVERNMENT

Each month, the Bureau of Labor Statistics (BLS) of the U.S. Department of Labor announces the Consumer Price Index and the unemployment rate - the two most politically sensitive statistical indices in the country. These indicators are so critical because they directly concern anyone who worries about his or her pocketbook. About half of us, for instance, have some part of our income escalated by the CPI. Calculating these figures as accurately as possible - and presenting them as objectively as possible - involves some unique scientific challenges for the BLS, as well as some formidable political pressures.

The ability of the BLS to provide accurate statistical information on the economy has grown by leaps and bounds in recent years. It is striking to think that up until the late 1960s, the BLS calculated its figures by hand. Advances in computation have enabled the Bureau to change its approach fundamentally, since it is now possible to process so much more data. According to BLS Commissioner Janet Norwood, "there has (recently) been an enormous emphasis on statistical validity and on mathematical approaches, because the state of the science has developed so far that we find there are a lot of things we can do."

From SIAM News (newsletter of the (US) Society for Industrial and Applied Mathematics), March, 1985.

MATHEMATICS IN TENNIS

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In recent years there has been an increasing interest in the application of mathematics to problems in sport. Students interested in a range of mathematical applications in sports such as baseball, American football, basketball, hockey, cricket, tennis, golf, athletics and rowing should read Ladany and Machol, (1977), *Optimal Strategies in Sports*, North Holland Publ. Co. As an example, we will look at applying some probability to investigate the effects of different scoring systems in tennis.

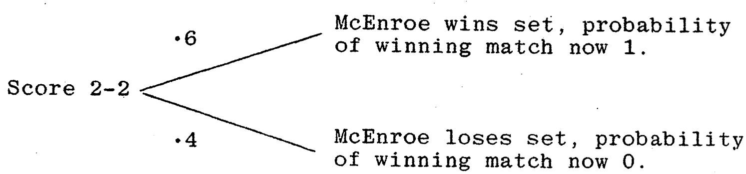
Consider a 5-set tennis match between McEnroe and Cash. Suppose McEnroe has a 60% chance of winning any set. The following matrix can be set up, where the body of the matrix shows the probability of McEnroe winning the match when the score is as shown.

		Cash's Score			
		3	2	1	0
McEnroe's Score	3	×	1	1	1
	2	0	*		
	1	0			
	0	0			

Thus the cell marked * would contain McEnroe's chance of winning when the score is 2 all.

The first row is all 1's since McEnroe wins if he reaches 3 sets, and the first column is all 0's since he loses if Cash reaches 3.

A tree diagram now allows us to calculate the number in the cell marked *.



Hence the number in cell at 2-2 is $0{\cdot}6 \times 1 + 0{\cdot}4 \times 0 = 0.6$. In a similar way the number in any cell is made up of 60% of the number above it plus 40% of the number to its left. We can progressively fill in the table using a calculator.

The results are shown below.

		Cash's score			
		3	2	1	0
McEnroe's Score	3	×	1	1	1
	2	0	·60	·84	·94
	1	0	·36	·65	·82
	0	0	·22	·48	·68

Thus we find McEnroe's chance of winning from any position, and in particular that he has a 68% chance of winning a 5-set match, but only a 65% chance of winning a 3-set match.

This is easily extended to any series of matches, sets, games, or points for any head to head contest in any sport.

In general, if $P(i,j)$ is the probability that player A wins a match up to n points when A's score is i, and B's is j, and p is the probability that A wins any point, a tree diagram gives

Score (i,j) $\xrightarrow{p}$ A wins, score goes to $(i + 1,j)$

Score (i,j) $\xrightarrow{1 - p}$ A loses, score goes to $(i,j + 1)$

Then $P(i,j) = p.P(i + 1,j) + (1 - p).P(i,j + 1)$.

Player A wins when A's score is n and loses when B's score is n, so

$$\begin{array}{ll} P(n,j) = 1 & 0 \leqslant j < n \\ P(i,n) = 0 & 0 \leqslant i < n \end{array} .$$

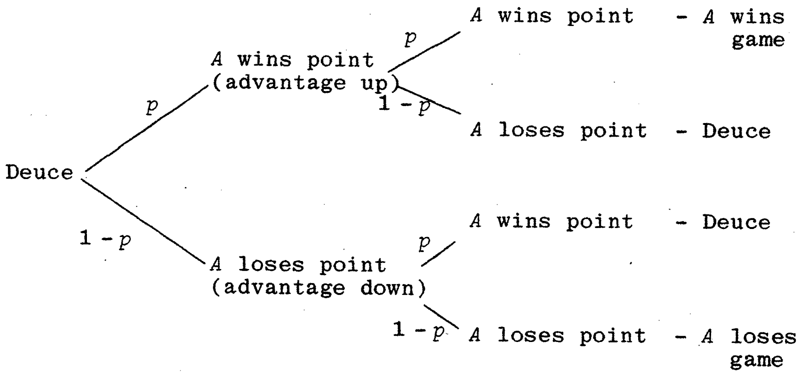
The short computer program below, written in Microsoft Basic, solves these equations progressively, for any match up to 25 points.

```
10 DIM P(25,25)
20 INPUT"number of points in match",N
30 INPUT"probability of player A winning point",P
40 FOR I=0 TO N:P(I,N)=0:P(N,I)=1:NEXT I
50 FOR I=N-1 TO 0 STEP -1
60 FOR J=N-1 TO 0 STEP -1
70 P(I,J)=P*P(I+1,J)+(1-P)*P(I,J+1)
80 PRINT P(I,J),
90 NEXT J
100 PRINT
110 NEXT I
120 END
```

In general we would be interested in $P(0,0)$, the chance of winning at the beginning of the game, but the program gives us the chance of winning from any position. The previous results were obtained by running the program with $n = 3$ and $p = 0.6$.

When applying these equations to a set of tennis, where p is now the probability of winning a game, p may alter depending on whether A is serving or not. We would then need to input the two values of p at line 30, and incorporate a line at 65 which chooses between them depending on whether $i + j$ is even or odd.

Situations such as deuce cause a slight problem since at advantage the score 'reverts' to deuce. Thus we need to know the answer for the deuce cell before calculating the advantage cell, but we need the advantage cell before calculating the deuce cell. This can be handled either by iteration - letting your computer program repeatedly calculate each one until its answers are not changing, or by more simply considering 2 points ahead. Again a tree diagram helps.



So if X is the probability that player A wins game at deuce, we have

$$X = p^2.1 + (1 - p)^2.0 + 2p(1 - p)X \ .$$

Solving for X gives

$$X = \frac{p^2}{1 - 2p + 2p^2} = \frac{p^2}{(1 - p)^2 + p^2} \ .$$

Also the chance of winning from advantage up is $p.1 + (1 - p)X$ and from advantage down is pX.

Since deuce usually occurs one point before usual winning score (e.g. at 40 all in a game of tennis, 20 all in table tennis etc.) we can incorporate these equations by adding to our program:

```
45 X = P*P/(P*P + (1-P)*(1-P))
46 P(N,N)=X: P(N,N-1)=P+(1-P)*X: P(N-1,N)=P*X.
```

The scoring systems of most sports are nested - e.g. in tennis a certain number of points wins a game, a certain number of games wins a set, a certain number of sets wins a match.

This is most easily incorporated by working through a series of *recurrence*† *relations*. Starting with the probability of winning a point, we can calculate $P(0,0)$ the probability of winning the game. This then becomes P in the next set to calculate the probability of winning the set etc. It is quite surprising how a small advantage in winning a point gives a large advantage in winning the match.

For example, suppose McEnroe wins 70% of points that he serves. Running our program with $n = 4$, $p = \cdot 7$ show that McEnroe will win 90% of his service games. Suppose Cash wins 65% of points that he serves. The program shows, with $n = 4$ and $p = \cdot 35$, that McEnroe will win 17% of Cash's service games. Using a slight approximation, (we could avoid this with some extra work) the program with $n = 7$ and $p = (\cdot 70 + \cdot 35)/2 = \cdot 525$ tells us that McEnroe will win approximately 58% of tiebreaker games. By inserting this in as lines 45,46 and making other adjustments so that when $i + j$ is even, $p = \cdot 17$ but when $i + j$ is odd, $p = \cdot 90$ we find that McEnroe will win 65% of sets.

Now running the original program (without lines 45,46) with $n = 3$ and $p = \cdot 65$ we get that McEnroe wins 76% of matches. Thus a very slight advantage in each point (winning 70% of serves against opponent winning 65%) means the better player will win over 3/4 of best of 5-set matches.

In many cases we may be interested in the length of a match. If $\mu(i,j)$ is the mean number of points in the rest of the match when the score is (i,j), then it can be shown that the basic recurrence relations become

$$\mu(i,j) = 1 + p.\mu(i + 1,j) + (1 - p)\mu(i,j + 1)$$

$$\mu(n,j) = \mu(i,n) = 0, \quad 0 \leqslant i,j < n \quad .$$

Some of the problems which can be investigated using the above techniques are:

* The effect of 3-set, 5-set, 6, 8 or 10-game advantage, no advantage, tie breaker sets on chances of better player and length of game.

* Probabilities of winning from any position in tennis, table tennis, etc.

* The effect of giving players starts, e.g. if you beat a player 21 - 15 at table tennis, is it fair that you give 6 start next game?

* The most important points in games, i.e. at which scores does the probability of winning/losing alter the most?

* The efficiency of scoring systems - which system gives the better player the most chance of winning in as few points as possible?

In a later article, we may look at squash and badminton, where you only score points on your own serve.

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†See *Function Vol.2, Part* 5 and *Vol.3, Part* 2.

GURSON'S CONSTANT

George J. Strugnell,

106 Bell St., Coburg

The death of J.G. Gurson in Adelaide late last year passed almost unnoticed: he was a strange old fellow, a supporter of lost and unpopular causes, who constantly advocated reforms based, as he claimed, on logical conclusions. When decimal currency was introduced, he proposed a ten-month year with alternating 36- and 37-day divisions. The abolition of death duties, he claimed, would widen the gap between the rich and the poor and he urged not only their reintroduction, but also their more stringent imposition based upon an exponential equation heavily loaded against the wealthy. "What they (the heirs) have never had they will never miss." Moreover, he recommended that all taxes, duties and imposts should be calculated in a similar fashion. The Upper House of Parliament, he argued, should consist of technocrats and its membership determined not by election, but by qualification in the various fields of science, art and literature, with their academic merit tested by oral and written examination and their admission decided by the drawing of lots. Not surprisingly, he was generally regarded as an eccentric mathematical dilettante and on the few occasions newspaper editors saw fit to publish his numerous letters, they made them appear as hoaxes.

My introduction to the affairs of the late Mr Gurson occurred in the following circumstances. His executor, a solicitor practising at a seaside suburb of Adelaide, wrote to me seeking advice and enclosing a copy of the Will and a voluminous note-book found amongst the deceased's papers. The estate was substantial and Gurson, who had never married, lived frugally and died the only child and orphan of immensely wealthy parents. He bequeathed a generous portion of it to form a perpetual trust fund, from which the interest should be paid "annually as an honorarium for the welfare and support of such retired mathematician living in indigent circumstances as my trustees in their absolute discretion consider to have made during the previous year the greatest contribution to the furtherance of my research into the innate commensurability of nature."

The executor's dilemma was that he did not know what research the deceased had been pursuing and he could not make head or tail of the contents of the note-book: this was understandable, for apart from masses of abstruse calculations with strange signs and symbols apparently of the author's own concoction, the commentary is entirely and atrociously hand-written in classical Greek, with one exception, the following enigmatic quatrain at the beginning:

> My name is Jethro Glenelg Gurson,
> I am a truly Jovial person:
> Whilst all the earth around me slumbers,
> I moonrake with Ionic numbers.

Throughout the note-book, almost on every page, appear the expressions, ι + 2λ = 3ɜ and ʞ; and towards the end, Eύρνκα! But what had he found?

Many fruitless hours I spent studying Gurson's notes: one of his devices I discovered was that to express a reciprocal he simply turned the symbol upside down, e.g., ɯ $= \frac{1}{\pi}$. Suddenly one night there occurred to me an inspiration, which led me immediately to telephone the executor. Yes, the deceased had subscribed to the Nautical Almanac: yes, he owned a telescope and numerous sets of astronomical ephemerides. It all fell into place: Gurson had been investigating celestial mechanics! By 'Jovial' and 'Ionic' he was referring to the planet Jupiter and to Io, one of its moons. Consulting the Encyclopaedia Britannica, I ascertained that Jupiter has four principal satellites (commonly called the Galilean moons after their discoverer) named in order of their distance from their primary: Io, Europa, Ganymede and Callisto. The second is very nearly as large as our moon, the others exceed it in size and they all orbit at astonishingly high speeds. Pierre Simon Laplace discovered a remarkable relationship between the longitudes of the three inner ones, namely, that that of Io plus twice that of Ganymede minus three times that of Europa is 180°, so that they cannot all come into conjunction with one another or into opposition or conjunction with the sun at the same time.

By taking ι, ε, γ and κ as representing the orbital periods of those four satellites and transliterating Gurson's often repeated expressions into more conventional mathematical notation, I arrived at $\frac{1}{\iota} + \frac{2}{\gamma} = \frac{3}{\varepsilon}$ and $\frac{1}{\kappa}$? (a semicolon being the Greek equivalent of a question mark). What had clearly perturbed Gurson was the absence of Callisto, the largest of them, from the Laplacian relationship and he set to to rectify the omission. First he rearranged the equation to

$$\left(\frac{1}{\iota} - \frac{1}{\varepsilon}\right) = \left(\frac{2}{\varepsilon} - \frac{2}{\gamma}\right),$$

then to

$$\left(\frac{1}{\iota} - \frac{2}{\varepsilon}\right) = \left(\frac{1}{\varepsilon} - \frac{2}{\gamma}\right),$$

and finally to

$$\left(\frac{1}{\iota} - \frac{3 - \upsilon}{\varepsilon}\right) = \left(\frac{\upsilon}{\varepsilon} - \frac{2}{\gamma}\right),$$

where υ equals any number. Adverting to the fact that on each side of the last equation the second numerator exceeds the first by $2 - \upsilon$, and that each of the first and second numerators on the right exceeds its respective counterpart on the left by $\upsilon - 1$, by arithmetic progression he projected a third component, thus: $\left(\frac{1}{\iota} - \frac{3 - v}{\varepsilon}\right) = \left(\frac{v}{\varepsilon} - \frac{2}{\gamma}\right) = \left(\frac{2v - 1}{\gamma} - \frac{v + 1}{\kappa}\right)$. By solving the equation constituted by the second and third components algebraically, he finally ascertained the value of v, now a constant, to be 1·288 471 708 091, which by approximating fractions may be reduced to $\frac{4873}{3782}$. Here was the means whereby the orbital period of one Galilean moon could be expressed in simple terms of those of any two others, for there were now available not only $\frac{1}{\iota} + \frac{2}{\gamma} = \frac{3}{\varepsilon}$ but also

$$\frac{v + 1}{\kappa} = \frac{2v + 1}{\gamma} - \frac{v}{\varepsilon} = \frac{2v + 1\frac{1}{2}}{\varepsilon} - \frac{v + \frac{1}{2}}{\iota} = \frac{1\frac{1}{3}v + 1}{\gamma} - \frac{\frac{1}{3}v}{\iota} .$$

For the convenience of sceptics particulars are now given of the relevant orbital periods.

	Sidereal Period in mean solar days
Io	1·769 137 864 95
Europa	3·551 181 106 65
Ganymede	7·154 552 718 16
Callisto	16·689 018 606 10

You will see how accurately these fit Gurson's formula. The result works equally well with the synodic periods which are slightly different.

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PROBLEM SECTION

We begin by commenting on the incomplete solution to Problem 8.4.4 in our last issue. We had to show that $uv(u^2 - v^2)$ could not be a perfect square and got around to showing that u, v (and, as a result, $u^2 - v^2$) have no divisors in common. There remains to be disproved one further possibility, namely that each of $u, v, u^2 - v^2$ is itself already a perfect square. However, this cannot be.

SOLUTION TO PROBLEM 8.4.4 (CONTINUED).

If $u = x^2$, $v = y^2$, $u^2 - v^2 = z^2$ (say), then

$$x^4 - y^4 = z^2 .$$

It is known that this equation has no integral solutions (see Problem 9.2.2). Hence the problem is solved.

We now give the solutions to the other problems set in Volume 8.

SOLUTION TO PROBLEM 8.5.1.

We asked:

Let r, x, y and z be real or complex variables. Show that:

(i) $y \propto x$ if y changes by a factor of r whenever x changes by a factor r. [Assume y is known to be a function of x.]

(ii) $z \propto xy$ if $z \propto x$ for each fixed y and $z \propto y$ for each fixed x. [Assume z is known to be a function of both x and y, i.e., to each suitable x and y, there corresponds just one z.]

Colin Wratten, who submitted the problem, sent this solution.

(i) Suppose $y = f(x)$, then we must prove $f(x) = kx$ for some constant k, independent of x, given that

$$f(rx) = rf(x).$$

For any non-zero x_1, x_2, put $r = x_1/x_2$ and so find

$$f(x_1) = f\left(\frac{x_1}{x_2}\, x_2\right) = \frac{x_1}{x_2}\, f(x_2)$$

and so

$$\frac{f(x_1)}{x_1} = \frac{f(x_2)}{x_2}.$$

Hence $f(x)/x = k$ is a constant unless perhaps $x = 0$. But $f(0) = f(2 \times 0) = 2f(0)$ and so $f(0) = 0$. We conclude that $f(x) = kx$, as required.

(ii) Suppose $z = f(x,y)$. We must prove that $f(x,y) = kxy$, for some constant k independent of both x, y. By Part (i) above, we have

$$f(x,y) = k_1(y)x = k_2(x)y,$$

where $k_1(y)$, $k_2(x)$ denote, respectively, functions of y,x alone. Then

$$\frac{f(x,y)}{xy} = \frac{k_1(y)}{y} = \frac{k_2(x)}{x}$$

for non-zero x,y. And because

$$\frac{k_1(y)}{y} = \frac{k_2(x)}{x}$$

$k_1(y)/y$ is independent of y, and $k_2(x)/x$ is independent of x. Thus these both equal some constant k. So $f(x,y) = kxy$, unless perhaps x or y is zero. But this last case can be dealt with as before, and so the result holds true generally.

[This last result can be extended by induction to functions of n variables.]

SOLUTION TO PROBLEM 8.5.2.

This problem, also submitted by Colin Wratten, read as follows.

Let $A = \sqrt{5} + \sqrt{22 + 2\sqrt{5}}$ and

$B = \sqrt{11 + 2\sqrt{29}} + \sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}}$. Prove that $A = B$.

Hai Tan Tran of Plympton Park, S.A., writes:

Since

$$\begin{aligned}\sqrt{16 - 2\sqrt{29} + 2\sqrt{55 - 10\sqrt{29}}} &= \sqrt{5 + 2\sqrt{5}\sqrt{11 - 2\sqrt{29}} + 11 - 2\sqrt{29}}\\ &= \sqrt{(\sqrt{5} + \sqrt{11 - 2\sqrt{29}})^2}\\ &= \sqrt{5} + \sqrt{11 - 2\sqrt{29}}\ ,\end{aligned}$$

$$\begin{aligned}B &= \sqrt{11 + 2\sqrt{29}} + \sqrt{11 - 2\sqrt{29}} + \sqrt{5}\\ &= C + \sqrt{5}\quad \text{(say)}.\end{aligned}$$

Then

$$C^2 = 22 + 2\sqrt{11^2 - 4 \times 29} = 22 + 2\sqrt{5}\ .$$

So

$$B = \sqrt{22 + 2\sqrt{5}} + \sqrt{5} = A\ .$$

An essentially similar proof was also supplied by Colin Wratten.

SOLUTION TO PROBLEM 8.5.3.

A particle is projected vertically into the air; it ascends to a certain height and then descends to the point of projection, all in the same straight line. Taking air-resistance into account, show that the initial (projection) speed is greater than the final (impact) speed and that the ascent time (the time to reach maximum height) is less than the descent time.

Again the problem comes from Colin Wratten, who also supplied the following solution.

Let $V_+(S)$ = ascent speed at height S

$V_-(S)$ = descent speed at height S

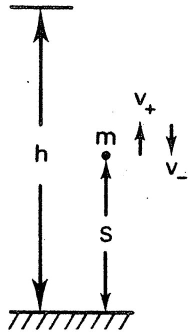
$G(S)$ = gravitational potential at height S

T_+ = ascent time

T_- = descent time

h = maximum height reached

m = mass of particle.



By conservation of total energy, $(\frac{1}{2}mV_+^2 + G) - (\frac{1}{2}mV_-^2 + G)$ is the work done against air-resistance between successive appearances at height S, and hence is positive for non-zero air-resistance. Thus $\frac{1}{2}mV_+^2 - \frac{1}{2}mV_-^2 > 0$, i.e., $V_+(S) > V_-(S) > 0$, for $0 \leqslant S < h$, $V_+(h) = V_-(h) = 0$. In particular, $V_+(0) > V_-(0)$, i.e. initial speed > final speed. Since the upward and downward journeys are of equal length and at any point on the flight path the upward speed exceeds the corresponding downward speed (except at maximum height where both are zero). It is physically evident that the ascent time is less than the descent time. [Formally, if t denotes time, then $V_+(S) = \frac{dS}{dt}$ and $V_-(S) = \frac{d}{dt}(h - S) = -\frac{dS}{dt}$ so

$$T_+ = \int_0^h \frac{dS}{V_+(S)} \quad \text{and} \quad T_- = -\int_h^0 \frac{dS}{V_-(S)} = \int_0^h \frac{dS}{V_-(S)}$$ and therefore

$T_+ < T_-$ since $V_+(S) > V_-(S) > 0$, i.e., $\frac{1}{V_+(S)} < \frac{1}{V_-(S)}$, for $0 \leqslant S < h$.]

We conclude with two new problems.

PROBLEM 9.2.1 (submitted by David Shaw, Geelong West T.S.).

This problem came from Hall and Knight's *Higher Algebra*, once a very widely used school text, first published in 1887, and occurs in Chapter XXVIII of that work. In the 1974 edition, the Dutchmen become Indians (Ram, Gopal, Amit; Jaya, Uma and Sujata), the hogs become sheep and the shillings rupees. One wonders why. The problem is credited to a 1743 *Miscellany of Mathematical Problems*.

"There are three Dutchmen of my acquaintance to see me, being lately married; they brought their wives with them. The men's names were Hendriek, Claas, and Cornelius; the women's Geertruij, Catriin, and Anna: but I forgot the name of each man's wife. They told me they had been at market to buy hogs; each person bought as many hogs as they give shillings for one hog; Hendriek bought 23 hogs more than Catriin; and Claas bought 11 more than Geertruij; likewise, each man laid out 3 guineas more than his wife. I desire to know the name of each man's wife."

PROBLEM 9.2.2 (carried over from Problem 8.4.4).

Show that no integers x, y, z exist such that $x^4 - y^4 = z^2$.

PERDIX

I continue with geometry problems. First please correct an error in my article in the last issue: on page 28, in the seventh line from the bottom, change '180' to '108'.

I shall continue the sequences of numbers for results, remarks, etc., started in issue 1.

PROBLEM 6. Let O be a point inside the polygon $A_1A_2 \ldots A_k$. Show that the sum of the lengths of $OA_1, OA_2, \ldots, OA_k$ is greater than half the perimeter of the polygon.

Solution. In the triangle A_1A_2O the length of A_1A_2 is less than the sum of the lengths of OA_1 and OA_2, i.e.

$$A_1A_2 < OA_1 + OA_2.$$

Similarly

$$A_2A_3 < OA_2 + OA_3,$$

$$A_3A_4 < OA_3 + OA_4,$$

and so on. Adding up, we get

$$\text{Perimeter} = A_1A_2 + \ldots + A_kA_1 < 2(OA_1 + OA_2 + \ldots + OA_k).$$

DEFINITION 3.

Two triangles are said to be *similar* when the angles in one equal the angles in the other. [It of course suffices for similarity that 2 angles in one equal 2 angles in the other.]

RESULT 5. Let ΔABC be similar to ΔDEF, with $\angle A = \angle D$, $\angle B = \angle E$ (and hence $\angle C = \angle F$). Then

$$\frac{AB}{ED} = \frac{BC}{DF} = \frac{CA}{FE} \qquad (1)$$

i.e. the sides of one triangle are proportional to the sides of the other.

Conversely, if equations (1) hold for the sides of Δ's ABC and DEF, then the triangles are similar. [Show that if just one of the equations (1) is assumed to hold, then it does not follow that ABC and DEF are similar.]

PROBLEM 7. AB and CD are two parallel straight lines, E is mid-point of segment CD. Line AC meets BE at F and AE meets BD at G. Show that FG is parallel to AB.

PROBLEM 8. Let ABC be any triangle and draw parallel lines through the vertices A, B, and C to meet the opposite sides in D, E, and F, respectively. Show that the area of ΔDEF is twice the area of ΔABC.

Send me solutions please to Problems 7 and 8.

There are various ways of solving Problem 8. One method would use the result in the next problem.

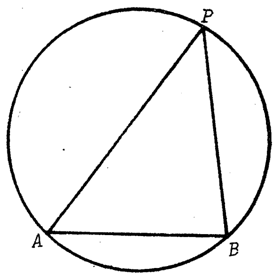
PROBLEM 9. Let ABC and DBC be two triangles such that AD is parallel to BC. Let BD and AC meet at E. Draw a line parallel to BC through E and let this line meet AB at F and CD at G. Show that $FE = EG$.

We now introduce circles into the problems and begin by stating some basic results that you should know. First some definitions.

DEFINITION 4.

Let A, B be two points on (the circumference of) a circle. Then the segment AB is called a *chord* of the circle.

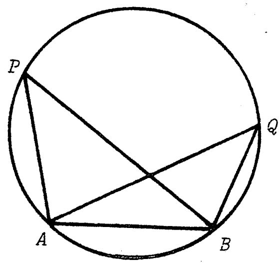
Let P be a further point on the circle. Then the angle $\underline{/P}$ of ΔAPB is called the *angle subtended by the chord AB at the circle.*



RESULT 6. If AB is a chord of a circle and P and Q are further points on the circle, both on the same side of AB, then the chord AB subtends the same angle at P as it subtends at Q.

Conversely, if R is any other point on the same side of AB as P and Q such that $\underline{/ARB} = \hat{P}$, then R lies on the circle ABQ.

[We use $\hat{P}$ as an alternative notation for $\underline{/P}$.]



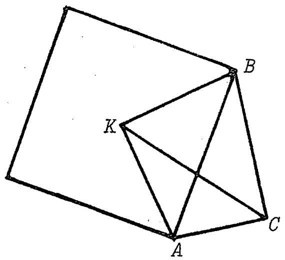
$\hat{P} = \hat{Q}$

RESULT 7. Any diameter of a circle subtends a right angle at the circle.

Conversely, if ABP is a triangle and $\underline{/P}$ is a right angle, then P lies on the circle with diameter AB.

PROBLEM 10. Let ABC be a right-angled triangle with hypotenuse AB and let K be the centre of the square on AB lying outside the triangle. Show that CK bisects $\underline{/C}$.

Solution. Clearly $\underline{/KAB} = \underline{/KBA} = 45°$ and $\underline{/K} = 90°$.



Hence (by Result 7) both K and C lie on the circle with diameter AB.

Hence $\underline{/KCB} = \underline{/KAB}$ (angles subtended by the same chord, KB, Result 6)

$= 45°$ (already shown).

Thus $\underline{/KCB} = 45°$ and, since $\underline{/ACB} = 90°$, KC bisects $\underline{/C}$.

Here are some further basic results about circles, that you will require to know. Check that you can prove them, and remember them.

DEFINITION 5. If 4 points lie on a circle then they are said to be *cyclic* and to form a *cyclic quadrilateral*.

RESULT 8. Let $PQRS$ be a cyclic quadrilateral. Then opposite angles are *supplementary* (i.e. add up to 180°).

Conversely, if $PQRS$ is a quadrilateral such that $\underline{/P} + \underline{/R} = 180°$, then P, Q, R and S lie on a circle.

PROBLEM 11. The point A is taken on a circle whose centre is O and X, S are the mid-points of the two chords AP, AQ. Prove that A, O, X, S lie on a circle.

PROBLEM 12. $ABCDEF$ is a regular hexagon with centre O. Show that $ABOF$ is a parallelogram. Show also that BDF is an equilateral triangle.