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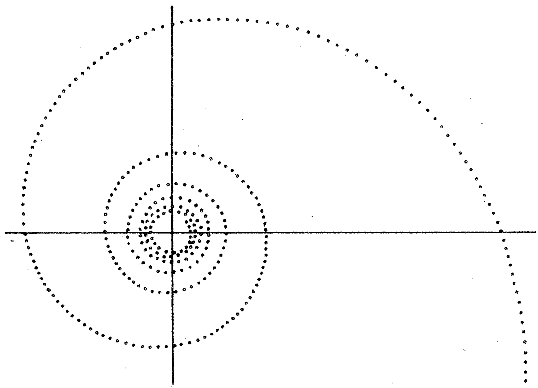

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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This issue brings to an end seven years of *Function*. In response to numerous requests, we have prepared an index. This gives the main articles printed over that period, but does not go into all the minor stories, jokes, problems, letters and "fillers". A larger omission is the (sometimes substantial) cover stories - these we hope to index later.

We take this opportunity to thank all those who have helped with this and earlier issues. The splendid cartoon on the back cover is by Colin Davies, and was sent to us by George Karaolis.

We also direct attention to the important correction on p.14 and thank those readers listed on p.19 who wrote to us about our dreadful mistake.

THE FRONT COVER

Our front cover shows the so-called *Sici spiral*. At any point of a curve, it may be approximated by a circle, whose radius R is termed the *radius of curvature*. The arc-length s of a curve is the distance, measured along the curve, from some suitable starting point. The Sici spiral has the property $R = e^{-s}$. Arcs from such spirals are pieced together to form the shapes of the French curves used by draughtsmen.

CONTENTS

THE BINOMIAL THEOREM

Bamford Gordon,
7 Burnside Ave., Hamilton

The origins of the binomial theorem are lost in antiquity. Certainly expressions for $(a + b)^2$ and $(a + b)^3$ were known by the Arabs and Indians at an early stage while Pascal's triangle (see below) appeared in a Chinese tract of 1303, written by Chu Shih-Chieh. The triangle of Pascal (1623-1662)

$$
\begin{array}{ccccccccccc}
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 \\
. & & . & & . & & . & & . & & .
\end{array}
$$

calculates successively the powers $a + b$, $(a + b)^2$, $(a + b)^3$, etc., as we see if we compare the triangle with

$$
\begin{aligned}
a + b &= a + b \\
(a + b)^2 &= a^2 + 2ab + b^2 \\
(a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\
(a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
(a + b)^5 &= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \\
\ldots . \quad & . \quad . \quad . \quad . \quad . \quad .
\end{aligned}
$$

Have you already worked out - if you have not seen it before - how Pascal's triangle is constructed? Each row is calculated from the preceding line as follows: each entry is the sum of the two numbers in the line above immediately to its left and to its right. This rule covers the first entry and last entry in each row if we imagine an entry 0 placed at the beginning and end of each row, for the purposes of the calculation.

The entries in each row are called *binomial coefficients*. We have a row of binomial coefficients corresponding to each positive integral power n of $a + b$: thus for $n = 4$, we have the power $(a + b)^4$ with corresponding coefficients 1,4,6,4,1, as read off from Pascal's triangle.

Let us introduce a notation for binomial coefficients. We write $(a+b)^n = C_0^n a^n + C_1^n a^{n-1}b + C_2^n a^{n-2}b^2 + \ldots + C_r^n a^{n-r}b^r + \ldots + C_n^n b^n$. The binomial coefficients† $C_0^n, C_1^n, \ldots, C_n^n$ are then given by the n-th row of Pascal's triangle.

It is easy to show that this is so. For suppose we have shown that the $(n-1)$-th row of Pascal's triangle gives the coefficients for $(a+b)^{n-1}$, i.e. we have shown that

$$(a+b)^{n-1} = T_0 a^{n-1} + T_1 a^{n-2}b + \ldots + T_r a^{n-r-1}b^r + \ldots + T_{n-1}b^{n-1}$$

where

$$T_0 \quad \ldots \quad T_1 \quad \ldots \quad T_r \quad \ldots \quad T_{n-1}$$

is the $(n-1)$-th row of Pascal's triangle. Then, it follows that

$$\begin{aligned}(a+b)^n &= (a+b)(a+b)^{n-1} = a(a+b)^{n-1} + b(a+b)^{n-1}\\ &= a(T_0 a^{n-1} + T_1 a^{n-2}b + \ldots + T_r a^{n-r-1}b^r + \ldots\\ &\qquad + T_{n-1}b^{n-1})\\ &\qquad + b(T_0 a^{n-1} + \ldots + T_{r-1}a^{n-r}b^{r-1} + \ldots\\ &\qquad + T_{n-2}ab^{n-2} + T_{n-1}b^{n-1})\end{aligned}$$

$$\begin{aligned}&= (T_0 + 0)a^n + (T_1 + T_0)a^{n-1}b + \ldots + (T_r + T_{r-1})a^{n-r}b^r + \ldots\\ &\qquad + (T_{n-1} + T_{n-2})ab^{n-1} + (0 + T_{n-1})b^n .\end{aligned}$$

In other words, the coefficients of $(a+b)^n$ are obtained from the $(n-1)$-th row of the Pascal triangle precisely by the rule of calculation that we gave for constructing the triangle:

$$C_r^n = T_{r-1} + T_r \,. \qquad (*)$$

Thus, if the $(n-1)$-th row of Pascal's triangle gives the binomial coefficients for $(a+b)^{n-1}$, then so also the n-th row of Pascal's triangle gives the coefficients for $(a+b)^n$. It is clear that the first row of Pascal's triangle gives the coefficients for $(a+b)^1 = a+b$. Hence, by the above argument, the second row gives the binomial coefficients for $(a+b)^2$; applying the argument again shows that then the third row gives the coefficients for $(a+b)^3$; and so on, without end. Pascal's triangle gives the binomial coefficients for all n.

---

†Perhaps a more common notation for C_r^n is $\binom{n}{r}$.

Formula (*) above, which is the rule for constructing Pascal's triangle, may be written

$$C_r^n = C_{r-1}^{n-1} + C_r^{n-1}$$

and is sometimes called Van der Monde's theorem.

Pascal's triangle is ideally set up for calculating binomial coefficients, say on the computer. It gives an algorithm, i.e. a step-by-step procedure, that generates the binomial coefficients one by one. Van der Monde's theorem may also to used to find a formula for C_r^n, as we now show.

Let us write $n(n - 1)(n - 2)\ldots 2.1$ as $n!$. The number $n!$ is the product of the factors n and all positive integers less than n, and is called *factorial* n. Thus $3! = 6$, $2! = 2$, $1! = 1$. It will be convenient to agree that $0!$ equals 1. The formula that we want to show is that

$$C_r^n = \frac{n!}{r!(n - r)!} .$$

Let us check the 4th row of Pascal's triangle.

$$C_0^4 = \frac{4!}{0!\,4!} = 1$$

$$C_1^4 = \frac{4!}{1!\,3!} = 4$$

$$C_2^4 = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1} = 6$$

$$C_3^4 = \frac{4!}{3!\,1!} = 4$$

$$C_4^4 = \frac{4!}{4!\,0!} = 1 .$$

Thus the formula applies for the coefficients in the 4th row. It is easily checked similarly that the formula holds for the first, second and third rows of Pascal's triangle.

Does the formula hold in general? Suppose we have already checked its truth for the first row, second row, ..., up to the $(n-1)$-th row. Then, we may check for the n-th row by Van der Monde's theorem. For, if we have already shown that

$$C_{r-1}^{n-1} = \frac{(n - 1)!}{(r - 1)!\,(n - r)!}$$

and

$$C_r^{n-1} = \frac{(n - 1)!}{r!(n - 1 - r)!} ,$$

then, by Van der Monde's theorem

$$C_r^n = \frac{(n - 1)!}{(r - 1)!\,(n - r)!} + \frac{(n - 1)!}{r!(n - 1 - r)!} .$$

The right hand side of this equation may be rearranged to give

$$(n-1)!\left[\frac{1}{(r-1)!(n-r)[(n-1-r)!]}+\frac{1}{r[(r-1)!]((n-1-r)!)}\right]$$
$$=\frac{(n-1)!}{(r-1)!(n-1-r)!}\left[\frac{1}{n-r}+\frac{1}{r}\right]$$
$$=\frac{(n-1)!}{(r-1)!(n-1-r)!}\;\frac{r+(n-r)}{(n-r)r}$$
$$=\frac{(n-1)!\times n}{r\times(r-1)!\times(n-r)\times(n-r-1)!}$$
$$=\frac{n!}{r!(n-r)!}\,.$$

Thus, by Van der Monde's theorem,

$$C^n_r=\frac{n!}{r!(n-r)!} \qquad (**)$$

which is the result we were trying to verify: repeated application of Van der Monde's theorem shows that the formula (**) holds for all binomial coefficients.

There is another approach to showing that formula (**) holds which helps to explain the importance of the binomial theorem in statistics. This approach involves counting the number of different permutations and combinations that can be made with a set of objects. Consider permutations first. Suppose we have a set of n distinct objects, and choose r of them, in turn. Taking into account the order in which the objects are chosen, how many different ways of making such a choice are there? This number is called the *number of permutations of n objects taken r at a time.*

Observe that there are n ways of choosing the first object; when that is chosen there are $n-1$ objects left, and so $n-1$ ways of choosing the second object; when the first two objects are chosen there are $n-2$ ways of choosing the third object; and so on. Thus there are $n\times(n-1)\times(n-2)\times\ldots\times(n-r+1)$ ways of choosing a sequence of r objects.

Now suppose that the order in which we choose the r objects does not matter: we are not interested in choosing a *sequence* of r objects, but a *set* of r objects. Consider any particular sequence of r objects, selected as we have just described. The set of r objects involved in this sequence will be chosen when the objects are selected in any other order. The number of orders they could be selected in is just $r\times(r-1)\times\ldots\times 2\times 1$, $= r!$, by the argument just given. Thus the number of permutations of n objects taken r at a time will include $r!$ permutations corresponding to each possible set of r objects.

Hence the number of different sets of r objects that can be selected from a set of n objects, called the *number of combinations of n objects taken r at a time*, is $[n\times(n-1)\times\ldots\times(n-r+1)]/(r!)$. Observe that

$$n\times(n-1)\times\ldots\times(n-r+1)=\frac{n!}{(n-r)!}$$

and we have that the number of combinations of n objects taken r at a time is

$$C_r^n = \frac{n!}{r!(n-r)!}$$

So this provides us with another way of looking at binomial coefficients as numbers of combinations; and this new way provides us with an alternative way of deriving the formula (**) for a binomial coefficient.

For

$$(a+b)^n = (a+b)(a+b)\ \dots\ (a+b)\,,$$

a product with n factors, and when the expression on the right is expanded out the coefficient of $a^{n-r}b^r$ is what we have called C_r^n. But the coefficient of $a^{n-r}b^r$ just measures the number of ways we can get the term $a^{n-r}b^r$ when multiplying out n factors $a+b$. In fact we get $a^{n-r}b^r$ when we choose b from r of the factors $(a+b)$ and choose a from the rest. So the coefficient of $a^{n-r}b^r$ is the number of ways that r factors can be chosen from n factors, i.e. the number of combinations of n objects taken r at a time, i.e. is $\dfrac{n!}{r!(n-r)!}$. Thus the binomial coefficient $C_r^n = \dfrac{n!}{r!(n-r)!}$.

We made a comment on the history of the discovery of the binomial theorem at the beginning of this article. It is perhaps worth commenting that Newton made the extremely important discovery that the formulae we have developed could be extended from positive integral n to negative and fractional n, for appropriate choices of a and b. When n is negative or fractional the factorial symbol $n!$ will no longer make sense as we have defined it. So now we must write

$$C_r^n = \frac{n \times (n-1)(n-2)\dots(n-r+1)}{r!}\ ;$$

note that we are keeping r as a positive integer.

Taking $a = 1$, the binomial theorem would then look like,

$$(1+b)^n = 1 + C_1^n b + C_2^n b^2 + \dots + C_r^n b^r + \dots\,, \qquad (***)$$

and, when n is not a positive integer, this series on the right does not terminate. In fact, this always works when $-1 < b < 1$. For example, let $b = -\frac{1}{2}$ and $n = -1$. Then the formula (***) becomes

$$(1-\tfrac{1}{2})^{-1} = 1 + C_1^{-1}(-\tfrac{1}{2}) + C_2^{-1}(-\tfrac{1}{2})^2 + \dots + C_r^{-1}(-\tfrac{1}{2})^r + \dots\,.$$

A calculation gives

$$\begin{aligned} C_r^{-1}(-\tfrac{1}{2})^r &= \frac{-1(-1-1)(-1-2)\dots(-1-r+1)}{r!}(-\tfrac{1}{2})^r \\ &= (-1)^r \times \frac{1 \times 2 \times 3 \times \dots \times r}{r!}(-1)^r(\tfrac{1}{2})^r \\ &= (-1)^{2r}\,\frac{r!}{r!}(\tfrac{1}{2})^r \\ &= (\tfrac{1}{2})^r\,. \end{aligned}$$

Hence the formula (***) reduces to

$$(1 - \tfrac{1}{2})^{-1} = 1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + \ldots + (\tfrac{1}{2})^r + \ldots .$$

Hence the left hand side equals

$$\frac{1}{1 - \frac{1}{2}} = 1/(\tfrac{1}{2}) = 2 ;$$

while the right hand side is a geometric series. Let s_k denote the sum of the first k terms of this geometric series:

$$s_k = 1 + \tfrac{1}{2} + \ldots + (\tfrac{1}{2})^{k-1} .$$

Then $$\tfrac{1}{2}s_k = \tfrac{1}{2} + \ldots + (\tfrac{1}{2})^{k-1} + (\tfrac{1}{2})^k ;$$

hence, subtracting,

$$s_k - \tfrac{1}{2}s_k = 1 - (\tfrac{1}{2})^k$$

whence

$$s_k = 2(1 - (\tfrac{1}{2})^k) ,$$

i.e. $$s_k = 2 - \frac{1}{2^{k-1}} .$$

Thus s_k differs from 2 by $\frac{1}{2^{k-1}}$. For k large, $\frac{1}{2^{k-1}}$ is small, and diminishes steadily to 0 as k increases. Check (on a hand calculator, for example) that $\frac{1}{2^{10}} < \frac{1}{1000} = \frac{1}{10^3}$, whence $\frac{1}{2^{20}} < \frac{1}{10^6}$, ..., $\frac{1}{2^{10m}} < \frac{1}{10^{3m}}$. We say that s_k has limit 2 as n tends to infinity and that the infinite series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^r} + \ldots$$

has "sum" 2.

With this interpretation the left hand side and right hand side of (***) are equal.

This observation of Newton was the starting point for the use of infinite series for calculations. Nowadays most mathematical calculations made on a computer are in fact done by using rapidly converging infinite series.

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BEGGING THE QUESTION

One [computer program] designed to teach the alphabet opens with the on-screen message: "Hello! Please type your name".

Newsweek, 17.1.83.

SCIENTIFIC LAWS II

G.B. Preston, Monash University

How does one discover a law that will describe the results of a set of observations? In practice scientific observations are subject to inaccuracy due to a variety of causes. Some inaccuracies may be inherent in the situation, but are within known limits; some may be due to a lack of adequate knowledge of all the factors influencing the observation. In this article we ignore such inaccuracies and their causes and instead concentrate simply on the problem of finding laws to interpret a given set of, presumed accurate, observations. We shall also restrict the discussion to the simple case of a sequence of observations, each giving a single number, thus giving a sequence of observations $u_1, u_2, \ldots, u_n$. We wish to find a function $u(t)$, such that $u(1) = u_1, u(2) = u_2, \ldots, u(n) = u_n$. We shall call such a function $u(t)$ a *law connecting* or *generating* the members $u_1, u_2, \ldots, u_n$ of the sequence we are trying to interpret.

For example, the law $u(t) = t^3$ generates the sequence $u(1) = 1$, $u(2) = 8$, $u(3) = 27$, ...

When we have found a law $u(t)$, generating the terms $u_1, \ldots, u_n$ that we started with, then it is possible to calculate $u(t)$ for other values of t, e.g. $u(n + 1)$, etc. However we only had n observations $u_1, u_2, \ldots, u_n$; whether the calculated value of $u(n + 1)$ might correspond to another observation will not concern us in this article.

There are many kinds of functions, or known formulae, that could be tried as possibilities for a law $u(t)$. For example you could try $u(t) = 2^t$ or $u(t) = \log_{10}(t)$. However we shall restrict ourselves here to polynomials, i.e. to $u(t)$ of the form

$$u(t) = a_k t^k + a_{k-1} t^{k-1} + \ldots + a_1 t + a_0 ,$$

in which $a_0, a_1, \ldots, a_k$ are constant numbers. Such an expression is called a *polynomial* of *degree* k, and $a_0, a_1, \ldots, a_k$ are called *coefficients*, a_i being the coefficient of t^i.

This is a decided restriction because, for example, if the real explanation of the observations made was given by the formula $u(t) = 2^t$, then no polynomial whatsoever would do to cover all possible values of t; though, as we shall see, a polynomial can be found to fit any specified number of observations.

The fitting of polynomials to data has turned out to be most useful in practice. We consider two approaches, one due to Lagrange, the other to Newton. But first a general comment.

Suppose we are trying to find a polynomial of agreed degree n to fit our data. Then, if there are $n + 1$ or more distinct observations the polynomial, if one exists, will be unique.

This uniqueness depends on the fact that for any polynomial

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 ,$$

whose coefficients are not all zero, there are at most n distinct values of x for which its value is 0. Such values are called the *roots* of $p(x)$. So, if we have two polynomials $p_1(x)$ and $p_2(x)$ that take the same values for $n + 1$ or more values of x, then $p_1(x) - p_2(x)$ must be zero at each of these values of x, i.e. $p_1(x) - p_2(x)$ has at least $n + 1$ roots. This is impossible, as has just been stated, unless all the coefficients of $p_1(x) - p_2(x)$ are zero. Thus

$$p_1(x) - p_2(x) = 0, \text{ identically,}$$

i.e.

$$p_1(x) = p_2(x) .$$

There does in fact exist a polynomial $p(x)$ of degree n that takes specified values for $n + 1$ different values of x; and as already shown this polynomial is unique. There are two standard prescriptions for this polynomial that we now give.

We first give Lagrange's formula[†]. This formula is the one used by J.A. Deakin in his letter in *Function*, Volume 7, Part 1, pp.24-5. Lagrange's polynomial is

$$\begin{aligned} L(x) = {} & u_0 \frac{(x - c_1)(x - c_2)\ldots(x - c_n)}{(c_0 - c_1)(c_0 - c_2)\ldots(c_0 - c_n)} \\ & + u_1 \frac{(x - c_0)(x - c_2)\ldots(x - c_n)}{(c_1 - c_0)(c_1 - c_2)\ldots(c_1 - c_n)} \\ & + u_2 \frac{(x - c_0)(x - c_1)(x - c_3)\ldots(x - c_n)}{(c_2 - c_0)(c_2 - c_1)(c_2 - c_3)\ldots(c_2 - c_n)} \\ & + \ldots \\ & + u_n \frac{(x - c_0)(x - c_1)\ldots(x - c_{n-1})}{(c_n - c_0)(c_n - c_1)\ldots(c_n - c_{n-1})} . \end{aligned}$$

[†]Jean-Louis Lagrange (1736-1813), perhaps the greatest eighteenth century mathematician, began his mathematical career as a professor at Turin, appointed to this post in his late 'teens. He then spent twenty years at the court of Frederick the Great, in Berlin. On Frederick's death he accepted an invitation to move to Paris where he spent the last twenty-six years of his life.

Here $c_0, c_1, \ldots, c_n$ are distinct numbers, and $u_0, u_1, \ldots, u_n$ are constants.

Notice that
$$L(c_0) = u_0 ,$$
$$L(c_1) = u_1 ,$$
$$L(c_2) = u_2 ,$$
$$\vdots$$
$$L(c_n) = u_n .$$

Thus this polynomial takes the specified values $u_0, u_1, \ldots, u_n$, for $x = c_0, c_1, \ldots, c_n$, respectively. [Check for yourself: see how cleverly Lagrange arranged that the expression multiplying u_i becomes zero, when $x = c_0, c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_n$, but becomes 1 when $x = c_i$.]

Newton (1642 - 1727) had a different formula; but of course there is only one polynomial so his formula when evaluated gives the same polynomial as Lagrange's formula. Newton's polynomial is

$$N(x) = \lambda_0 + \lambda_1(x - c_0) + \lambda_2(x - c_0)(x - c_1) + \ldots$$
$$+ \lambda_n(x - c_0)(x - c_1)\ldots(x - c_{n-1}) ,$$

and where $\lambda_0, \lambda_1, \ldots, \lambda_n$ are evaluated one after the other, as follows. First, set $x = c_0$. Note that then the expressions multiplying $\lambda_1, \lambda_2, \ldots, \lambda_n$ each become zero. Hence we have

$$N(c_0) = \lambda_0 .$$

But $N(c_0)$ is the value we wish the polynomial to take when $x = c_0$; hence $N(c_0) = u_0$. Thus

$$\lambda_0 = u_0 .$$

Now, having found λ_0 we find λ_1 by substituting $x = c_1$ in the formula:

$$N(c_1) = \lambda_0 + \lambda_1(c_1 - c_0) ,$$

since the expressions multiplying $\lambda_2, \lambda_3, \ldots, \lambda_n$ each become zero when $x = c_1$. Hence, since $N(c_1) = u_1$, the prescribed value at $x = c_1$, and $\lambda_0 = u_0$, as already calculated, we have

$$u_1 = u_0 + \lambda_1(c_1 - c_0) ,$$

whence

$$\lambda_1 = \frac{u_1 - u_0}{c_1 - c_0} .$$

We continue like this, calculating in turn each of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$, and thus determining the polynomial $N(x)$. Although in fact identical with $L(x)$, $N(x)$ does not look much like $L(x)$ at first sight.

There is a simple formula for calculating $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$. It looks a little simpler if we take the special case when $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, ..., $c_n = n$; and we now derive this formula for this special case. We then have

$$N(x) = \lambda_0 + \lambda_1 x + \lambda_2 x(x - 1) + \ldots + \lambda_n x(x - 1)\ldots(x - (n - 1)),$$

and, as before,

$$\lambda_0 = u_0$$
$$\lambda_1 = u_1 - u_0 .$$

To obtain the general formula, introduce the notation

$$\Delta u_k = u_{k+1} - u_k ;$$

so that, in the same notation,

$$\Delta(\Delta u_k) = \Delta u_{k+1} - \Delta u_k .$$

Write $\Delta(\Delta u_k) = \Delta^2 u_k$; and, in general, let

$$\Delta^{r+1} u_k = \Delta^r u_{k+1} - \Delta^r u_k .$$

Then, in this notation,

$$\lambda_0 = u_0 ,$$
$$\lambda_1 = \Delta u_0 ,$$

and, as we now show,

$$\lambda_k = \frac{\Delta^k u_0}{k!} . \qquad (*)$$

Recall that $k!$, called 'factorial k', denotes the product $k(k - 1)\ldots2.1$. In particular $1! = 1$, so the formula (*) holds for $k = 1$. Suppose that we have proved that (*) holds for $k = 1$, $k = 2$, ..., up to $k = m - 1$; let us show that, on this assumption it holds for $k = m$.

We thus assume that

$$N(x) = u_0 + (\Delta u_0)x + \frac{\Delta^2 u_0}{2!}x(x-1) + \ldots$$
$$+ \frac{\Delta^{m-1}u_0}{(m-1)!}x(x-1)\ldots(x-m+2) + \lambda_m x(x-1)\ldots(x-m+1)$$
$$+ \ldots + \lambda_n x(x-1)(x-2)\ldots(x-n+1)\,.$$

Setting $x = m$, and remembering that $N(m) = u_m$, we therefore have

$$u_m = u_0 + (\Delta u_0)m + \frac{\Delta^2 u_0}{2!}m(m-1) + \ldots$$
$$+ \frac{\Delta^{m-1}u_0}{(m-1)!}m(m-1)\ldots 2 + \lambda_m.m!\ .$$

But, this is just

$$u_m = u_0 + \binom{m}{1}\Delta u_0 + \binom{m}{2}\Delta^2 u_0 + \ldots + \binom{m}{m-1}\Delta^{m-1}u_0 + \lambda_m m!\ ,(**)$$

where the $\binom{m}{r}$ are binomial coefficients (see the article on Pascal's triangle in this issue).

On the other hand, by the definition of the symbol Δ,

$$\begin{aligned} u_m &= u_{m-1} + \Delta u_{m-1} = (1+\Delta)u_{m-1} \\ &= (1+\Delta)((1+\Delta)u_{m-2}) \\ &= (1+\Delta)^2 u_{m-2} \\ &= \ldots \\ &= (1+\Delta)^m u_0\,. \end{aligned}$$

Thus, from the binomial expansion of $(1+\Delta)^m$ (see again the article by Gordon on Pascal's triangle),

$$\begin{aligned} u_m &= (1+\Delta)^m u_0 \\ &= \{1 + \binom{m}{1}\Delta + \binom{m}{2}\Delta^2 + \ldots + \binom{m}{m}\Delta^m\}u_0 \\ &= u_0 + \binom{m}{1}\Delta u_0 + \binom{m}{2}\Delta^2 u_0 + \ldots + \binom{m}{m}\Delta^m u_0\,. \qquad (***) \end{aligned}$$

Subtracting (**) from (***) we have

$$0 = \binom{m}{m}\Delta^m u_0 - \lambda_m m!\ ,$$

whence

$$\lambda_m = \frac{\Delta^m u_0}{m!} \qquad (*)$$

as required. It follows that (*) holds for all m.

A small modification of the above argument leads to a similar formula for the coefficients λ_k when we choose $c_0, c_1, \ldots, c_n$ arbitrarily, instead of giving them the special values $c_0 = 0$, $c_1 = 1$, ... , $c_n = n$, as we did above.

These formulae of Lagrange and Newton are called *interpolation formulae*, because they enable us to "interpolate" beyond the values, or observations, we have been given. Such a procedure would be appropriate if we had reasons to believe, or hope, that the polynomial being fitted to the data should also fit further observations, i.e. that we had in fact found a law underlying the situation being observed. A test of this possibility could be made by making further observations and checking whether or not the calculated polynomial was consistent with them.

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MORDELL'S CONJECTURE PROVED

The Australian (27/7) and *Newsweek* (1/8) both carried stories on the proof by the West German mathematician, Gerd Faltings, of a result in number theory. This result, known as Mordell's conjecture, had become one of the outstanding problems of number theory, like Fermat's last theorem, to which it is related.

The press announcements came too late to be noticed in our last issue, and we do not yet have full details of Prof. Faltings' proof. However, we do hope to give a fuller account of the matter sometime next year.

Consider Fermat's last theorem: namely, if $n > 2$, there are no positive integers a, b, c, for which $a^n + b^n = c^n$. Divide both sides of this equation by c and write $x = a/c$, $y = b/c$. We then have

$$x^n + y^n = 1 . \qquad (*)$$

Now in the case when $n = 2$, Equation (*) represents a circle of radius one and from each of the known Pythagorean triples (like 3,4,5; 5,12,13 etc.) we get a point on this circle, whose coordinates are rational. As there are infinitely many Pythagorean triples, the circle passes through infinitely many such *rational points*.

However, if $n > 2$, Equation (*) represents (for $x \geqslant 0$, $y \geqslant 0$) a quadrant of one of a family of supercircles. These curves are more complicated than circles. The complexity of curves is measured by mathematicians on a technical scale we shall not describe here. These more complicated curves are said to have a *genus* greater than one.

Mordell's conjecture states that *no curve whose genus exceeds one can pass through more than a finite number of rational points.*

It follows from the, now known, truth of this statement that for any *particular* n (>2), because supercircles have genus greater than one, there can be no more than a finite number of exceptions to Fermat's last theorem. This clearly represents some progress in the proof of this long outstanding result, but equally clearly, it leaves a lot of room for counter-examples.

CORRECTION

The article on series in our last issue contained (on pp.8, 9) some serious errors. In the first place, the first, easy, example is incorrect as stated. We remove not those terms with even denominators, which leads as several correspondents have pointed out (see p.19) to a divergent series, but all those terms whose denominators *contain an even digit*. The series is thus:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}+\frac{1}{17}+\frac{1}{19}+\frac{1}{31}+\frac{1}{33}+\cdots .$$

[What is the next term in the sequence 1,3,5,7,9,11,13,15,17,19 ?]

The second example also contains errors and we give below an amended version to replace p.9. However, see also the letter from Bamford Gordon on p.17.

We put

$$a_1 = 1+\frac{1}{2}+\cdots+\frac{1}{8}$$
$$a_2 = \frac{1}{10}+\cdots+\frac{1}{88}$$
$$a_3 = \frac{1}{100}+\cdots+\frac{1}{888} .$$

Now let b_n be the number of terms in the sum a_n. Clearly $b_1 = 8$. Now consider b_2. The first digit in the denominator may be 1,2,3,...,8 and the second may be 0,1,2,...,8. There are thus 8×9 possibilities in all and thus $b_2 = 8 \times 9$. Now b_3 contains terms whose denominators begin with one of the digits 1,2,3,...,8, continues with one of the digits 0,1,2,...,8 and concludes with one of the digits 0,1,2,...,8. There are thus $8 \times 9 \times 9$ or 8×9^2 possibilities in all.

Continuing in this way, we see that $b_n = 8 \times 9^{n-1}$.

Now the largest term in the sum a_n is the first, and this is exactly 10^{-n+1}. Thus

$$a_n < 10^{-n+1}.b_n$$
$$= 8 \times \left(\frac{9}{10}\right)^{n-1} .$$

But the entire series has sum $a_1 + a_2 + a_3 + \ldots$, and this sum is less than

$$8 + 8 \times \frac{9}{10} + 8 \times \left(\frac{9}{10}\right)^2 + \cdots ,$$

a geometric series with first term 8 and common ratio $\frac{9}{10}$. So our series sums to a value less than $8/(1 - 9/10)$, or 80.

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LETTERS TO THE EDITOR

GREATLY IMPROVING A RESULT

We would like to comment on Bruce Henry's contribution "Improving a Result" on page 26 of the August issue.

The note in the previous issue stated that 100! + 2,... 100! + 100 are all composite.

Wilson's Theorem states that if p is a prime number, then $(p - 1)! + 1$ is a multiple of p. Since 101 is prime, 100! + 1 is composite (a multiple of 101). So 100! - 1 is followed by 101 composite numbers. If 100! - 1 proved to be composite, then there would be 201 composite numbers from 100! - 100 to 100! + 100 inclusive.

100! - 1 is composed of 159 digits and a search for a factor less than 10000 proved unsuccessful, which is probably understandable.

However, a further examination of some n factorials where n is 1 less than a prime number was productive with 60!.

60! + 1 is a multiple of 61

and 60! - 1 is a multiple of 733 .

So, there are 121 composite numbers between 60! - 60 and 60! + 60 inclusive.

Indeed we can do better than this, for 52! -1 is composite, being a multiple of 61. Since, by Wilson's theorem 52! + 1 is a multiple of 53, there are 105 composite numbers between 52! - 52 and 52! + 52 inclusive.

David Shaw and Year 11 students
Geelong West Technical School.

[*A great improvement, indeed.* 100! $\simeq 10^{158}$, *while* 52! $\simeq 10^{68}$, *which is, of course, a very large number, but minute compared with the former. Mr Shaw also sent copies of the BASIC programmes used, but space prevented their being printed here. Eds.*]

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DIVISIBILITY AND CONGRUENCES

Bruce Henry's paper on tests for divisibility (*Function, Vol.7, Part 4*) avoids the use of congruences and I assume this is deliberate. Using congruences, one can argue as follows:

1. Since $1000 \equiv -1 \pmod{1001}$, $(10^3)^n \equiv (-1)^n \bmod 1001$.

Hence given an integer N, write it in blocks of 3, starting from the right. E.g. if

$$N = 1234567893 ,$$
$$N = 1\ 234\ 567\ 893 .$$

So $$N = 1 \times (10^3)^3 + 234 \times (10^3)^2 + 567 \times 10^3 + 893$$

and hence

$$N \equiv 1(-1)^3 + 234(-1)^2 + 567(-1) + 893 \pmod{1001}$$
$$N \equiv 559 \pmod{1001}.$$

If we put $N_0 = 559$, then N is divisible by 7, 11, 13 iff N_0 is, since $1001 = 7 \times 11 \times 13$. Since $7 \nmid 559$, $11 \nmid 559$, $13 | 559$, $13 | N$.

2. For the prime divisor 17, we note that $17 \times 59 = 1003$, and $1000 \equiv -3 \pmod{1003}$. So we may use the same method as above, taking congruences mod 1003, replacing $(10^3)^n$ by $(-3)^n$. Further, since $(-3)^3 = -27 \equiv -10 \pmod{17}$, then if we are interested only in divisibility by 17, we may replace $(-3)^3$ by -10. Thus for

$$N = 1\ 234\ 567\ 893 ,$$
$$\begin{aligned} N &\equiv 1(-3)^3 + 234(-3)^2 + 567(-3) + 893 \pmod{1003} \\ &\equiv -10 + 234 \times 9 - 567 \times 3 + 893 \pmod{17} \\ &\equiv 1\ 288 \pmod{17} \\ &\equiv -3 + 288 \pmod{17} \text{ (doing same thing again)} \\ &\equiv 285 \pmod{17}. \end{aligned}$$

Since $17 \nmid 285$, $17 \nmid N$.

On the example given,

$$\begin{aligned} N &= 1\ 606\ 463\ 297 \\ &\equiv 1(-3)^3 + 606(-3)^2 + 463(-3) + 297 \pmod{1003} \\ &\equiv 4335 \pmod{1003} \\ &\equiv 4(-3) + 335 \\ &\equiv 323 \pmod{1003} . \end{aligned}$$

Since $17 | 323$, $59 \nmid 323$, $17 | N$.

3. For the case $p = 19$, observe $19 \times 53 = 1007$ and $1000 \equiv -7 \pmod{1007}$. So we reduce mod 1007, replacing $(10^3)^n$ by $(-7)^3$. Further, since $(-7)^3 \equiv -1 \pmod{19}$, we may replace $(-7)^3$ by -1 if we reduce mod 19.

E.g. $$\begin{aligned} N &= 1\ 606\ 463\ 297 \\ &\equiv 1(-7)^3 + 606(-7)^2 + 463(-7) + 297 \pmod{1007} \\ &\equiv -1 + 606 \times 49 - 463 \times 7 + 297 \pmod{19} \\ &\equiv 26749 \pmod{19} \\ &\equiv 26(-7) + 749 \pmod{19} \\ &\equiv 567 \pmod{19} \end{aligned}$$

and $19 \nmid 567$. So $19 \nmid N$.

4. For the case $p = 23$, observe $23 \times 43 = 989$, $1000 \equiv 11 \pmod{980}$ and also that $11^2 \equiv 6 \pmod{23}$, $11^3 \equiv -3 \pmod{23}$, so it is very easy to reduce N first mod 989 and then mod 23 .

John Mack
University of Sydney

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CORRECTING AN ARGUMENT

There is a couple of misprints on page 8 of Part 4 of *Function* Volume 7: the two occurrences of "even denominators" should be replaced by "an even digit in the denominator".

The "Second, More Surprising, Example" also needs some correction in the proof offered. First, trivially, a_1 has 8 terms, not 9 terms, as stated on lines 7 and 8, page 9. Second, the inductive argument is at fault. There are at least four errors in the argument presented: the following is the major one.

The statement (lines 14-17): "the other eight each contain exactly as many terms as there are in the interval

$$0 < d < 10 \quad ,"$$

is false. I am not here quibbling with the fact that this statement is badly worded and should perhaps read "there are denominators in the" instead of "there are in the". However, giving it this amended reading, it is false. For example, consider a_2, the denominators of whose terms lie between 10 and 10^2, a range that can be divided into nine intervals

$$\alpha \,.\, 10 \leqq d < (\alpha + 1) \,.\, 10 \quad , \quad \alpha = 1,2,\ldots 9 \quad ,$$

as in the article.

Take $\alpha = 1$. The article claims that this interval

$$10 \leqq d < 20 \qquad \ldots \qquad \text{(a)}$$

contains "exactly as many terms as there are in the interval

$$0 < d < 10 \text{ "}. \qquad \ldots \qquad \text{(b)}$$

Now, there are 8 terms in the interval (b); but there are 9 terms with denominators in the interval (a).

The statement is false for all n and inspection will show that this slip invalidates the inductive argument as presented.

I managed to get hold of the original version in *Math-Jeunes*, of which this article is a translation and found that, although, in the original, a slightly different argument was presented to justify the claim that the sum of the terms of the harmonic series that have no digit 9 in the denominator, is convergent, again the argument is incorrect. I offer the following in the hope that it is a correct argument.

We show that the number of terms in each a_1 is less than 9^i. The argument is just a small modification of that in the original *Math-Jeunes* and also a small modification of the translation offered in *Function*.

First note that a_1 contains exactly 8 terms and so the statement "the number of terms in a_i is less than 9^i" holds for $i = 1$.

Suppose that this statement has been proved for $i = 1, 2, \ldots, n$; we now show that it then follows that the statement is true for $i = n + 1$.

a_{n+1} contains all the terms of the series whose denominators lie between 10^n and 10^{n+1}. This interval can be divided into nine subintervals

$$\alpha \, . \, 10^n \leqq d < (\alpha + 1) \cdot 10^n \quad , \quad \alpha = 1, 2, \ldots, 9 \, .$$

Every d in the last of these subintervals begins with the digit 9, so this subinterval contains no denominators of the given series. Let us examine the remaining 8 subintervals. Each contains the same number β, say, of integers d that contain no 9's, namely the number of such integers d that satisfy

$$0 \leqq d < 10^n \, . \tag{1}$$

Denote by $|a_i|$ the number of terms in a_i. Then the number of integers d satisfying (1) is

$$\beta = 1 + |a_1| + |a_2| + \ldots + |a_n| \, .$$

We have already proved that $|a_i| < 9^i$, if $i \leqq n$; thus $|a_i| \leqq 9^i - 1$, if $i \leqq n$. Hence

$$\begin{aligned} \beta &\leqq 1 + (9 - 1) + (9^2 - 1) + \ldots + (9^n - 1) \\ &= 1 - n + 9 + 9^2 + \ldots + 9^n \\ &= 1 - n + (9^{n+1} - 1)/8 \, . \end{aligned}$$

Hence

$$\begin{aligned} |a_{n+1}| = 8\beta &\leqq 8 - 8n + 9^{n+1} - 1 \\ &< 9^{n+1} \, , \end{aligned}$$

as required.

You will notice that in the final stage of this argument, if we had tried to use just $|a_i| < 9^i$ (instead of $|a_i| \leqq 9^i - 1$) and obtained (as is true)

$$\beta < 1 + 9 + 9^2 + \ldots + 9^n \, ,$$

then we can deduce only that

$$|a_n| = 8\beta < 8 + 9^{n+1} - 1 = 9^{n+1} + 7 \, ,$$

an inequality not strong enough to establish the required result.

You will notice also that it is necessary to assume that the result $|a_i| < 9^i$ has been proved for $i = 1, 2, \ldots, n$ for us to be able to give the proof offered that then $|a_{n+1}| < 9^{n+1}$: it does not suffice to assume only that it has been proved for $i = n$ (as attempted both in the *Function* article and the *Math-Jeunes* article). An argument by induction of this kind is sometimes called "a proof by full induction".

It is not too difficult to calculate the precise number of terms in $|a_n|$: in fact, $|a_n| = 8 \times 9^{n-1}$.

Bamford Gordon
7 Burnside Avenue
Hamilton.

[*We acknowledge receipt of three more letters on the same subject. W.J. Ewens and T.C. Brown, both of Monash University, and J.C. Barton of Drummond St., North Carlton, pointed out errors in the article. See the correction on p.14. We printed Mr Gordon's letter in preference to these others because of the mathematical points of interest that it raises. Eds.*]

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OLYMPIAD REPORT

The 1983 International Mathematical Olympiad was held in Paris from July 4th to July 12th. Australia sent six candidates, two from Adelaide, and one each from Sydney, Perth, Hobart and Rockhampton. (Where are all the Victorians?)

The results were our best ever. Overall Australia finished 19th out of 32 teams, well behind the winning teams of West Germany, U.S.A., Hungary, Russia, Romania and Vietnam (in that order), but well ahead of Sweden, Austria, Italy and others.

Dick Vertigan of Hobart won a Silver Medal with 31 points out of a possible 42 and two others, Richard Moore from Adelaide and Andrew Kepert from Perth, won Bronze Medals.

Good on ya, team!

NEW "LARGEST PRIME"?

From the U.S. comes the news that David Slowinski of Cray Research Inc. claims the number

$$2^{132\,049} - 1$$

to be prime. If this claim is verified, this new prime will be the largest known. Last year, Slowinski found

$$2^{86\,243} - 1$$

to be prime, and this is the previous record.

PROBLEM SECTION

A number of problems remained outstanding when our previous issue went to press. We give solutions of these and others.

SOLUTION TO PROBLEM 7.3.1

Of the $n!$ possible permutations of n different symbols $1,2,3,\ldots,n$, $f(n)$ are such that *no* symbol is undisturbed and $g(n)$ are such that *exactly one* remains where it was. Prove that

$$f(n) = g(n) + (-1)^n \ .$$

This problem was first posed in 1708 by Montmort, and the formula quoted supplied by Euler in 1779. We say that, for the n symbols, there are $f(n)$ *complete derangements*, which may be of two types:

(i) Those in which the first symbol in the new ordering, a_1 say, has changed places with 1, i.e. 1 has moved to the a_1th place, and a_1 has moved to the first place;

(ii) All the others.

Consider (i) first. There are $n - 1$ possible values of a_1. Once this is fixed, the remaining $n - 2$ *places can be filled by any complete derangement of the remaining* $n - 2$ *symbols*, i.e. in $f(n - 2)$ ways. So there are $(n - 1)f(n - 2)$ complete derangements of the first kind.

Now for (ii). a_1 may be chosen in $n - 1$ ways. The remaining symbols are $2,3,\ldots,$ but with 1 replacing a_1. However, 1 will not be in the a_1th position after the derangement, so the new set is completely deranged. This can occur in $f(n - 1)$ ways, and as a_1 may be chosen in $n - 1$ ways, there are $(n - 1)f(n - 1)$ complete derangements of Type (ii).

Adding, we now have,

$$f(n) = (n - 1)f(n - 2) + (n - 1)f(n - 1) \ .$$

Now $g(n)$ may be readily calculated, for keep one symbol fixed which may be done in n ways and completely derange the others, which may be done in $f(n - 1)$ ways.

Thus $$g(n) = nf(n - 1) \ .$$

Then

$$f(n) - g(n) = (n - 1)f(n - 1) + (n - 1)f(n - 2) - nf(n - 1)$$
$$= -\{f(n - 1) - (n - 1)f(n - 2)\} \ ,$$

that is to say

$$f(n) - nf(n - 1) = - \{f(n - 1) - (n - 1)f(n - 2)\} ,$$

and the bracketed term on the right is the same as the left hand side, except that $n - 1$ replaces n. We may thus repeat this reasoning to find

$$f(n - 1) - (n - 1)f(n - 2) = - \{f(n - 2) - (n - 2)f(n - 3)\}$$

and so on. Thus

$$f(n) - nf(n - 1) = (-1)^{n-2}\{f(2) - 2f(1)\} .$$

But clearly $f(2) = 1$, $f(1) = 0$. So

$$f(n) - g(n) = (-1)^{n-2} = (-1)^n$$

as required.

For more on this problem, see Problem 7.5.1 below.

SOLUTION TO PROBLEM 7.3.2

This asked for arithmetic expressions for the numbers 1,2,...,20 in terms of π. George Karaolis, 66 Cruikshank Street, Port Melbourne, sent us this list.

$1 = [\sqrt{\pi}]$	$11 = [(\pi \times \pi) + \sqrt{\pi}]$
$2 = [[\sqrt{\pi}] + \sqrt{\pi}]$	$12 = [(\pi \times [\pi]) + \pi]$
$3 = [\pi]$	$13 = [(\pi \times \pi) + \pi]$
$4 = [\pi + \sqrt{\pi}]$	$14 = [(\pi \times \pi \times \sqrt{\pi}) - \pi]$
$5 = [\pi \times \sqrt{\pi}]$	$15 = [(\pi + \sqrt{\pi}) \times \pi]$
$6 = [\pi + \pi]$	$16 = [\pi \times [\pi] \times \sqrt{\pi}]$
$7 = [(\pi \times \sqrt{\pi}) + \sqrt{\pi}]$	$17 = [\pi \times \pi \times \sqrt{\pi}]$
$8 = [\pi + \pi + \sqrt{\pi}]$	$18 = [(\pi + \pi) \times [\pi]]$
$9 = [\pi \times \pi]$	$19 = [\pi \times (\pi + \pi)]$
$10 = [\pi \times \pi \times \pi]/[\pi]$	$20 = [(\pi \times \pi \times \sqrt{\pi}) + \pi]$.

SOLUTION TO PROBLEM 7.3.3

Rather to our surprise, we received no solutions for this relatively straightforward problem.

A deck of 52 playing cards is shuffled and placed face down on a table. Cards are removed from the top of the pile until a black ace is encountered. In which position is this ace most likely to be found?

The first black ace is most likely to be in the top position. Its probability of being found there is $\frac{2}{52}$. The chance of its being in the second spot is $\frac{50}{52} \cdot \frac{2}{51}$, or $\frac{50}{51} \cdot \frac{2}{52}$, which is less than $\frac{2}{52}$. The probability that the first black ace is in the third spot is

$$\frac{50}{52} \cdot \frac{49}{51} \cdot \frac{2}{50} = \frac{2}{52} \cdot \frac{50}{51} \cdot \frac{49}{50} < \frac{2}{52},$$

etc.

SOLUTION TO PROBLEM 7.3.4

Let x, y, z be relatively prime, and satisfy $x^2 + y^2 = z^2$. Prove that there exist integers u, v such that $z = u^2 + v^2$. Find x, y in terms of u, v.

John Percival, Year 12, Elderslie H.S., N.S.W. solved this problem. He first noted that one of x, y must be odd, the other even, so that z is odd. Suppose now that x is even, and $x = 2w$, i.e. $x^2 = 4w^2$. Then

$$x^2 = 4w^2 = z^2 - y^2 = (z + y)(z - y) . \qquad (*)$$

But
$$(z + y) + (z - y) = 2z .$$

The greatest common divisor of $z + y$, $z - y$ must divide $2z$ and must therefore be 2 as y, z are relatively prime. By (*), we must then have $z + y = 2v^2$, $z - y = 2u^2$ for some integers u, v.

It then follows that $x = 2uv$, $y = v^2 - u^2$, $z = v^2 + u^2$. This result is readily seen to satisfy the given equation.

SOLUTION TO PROBLEM 7.3.5

We asked for the solution of

$$\sqrt{x - 1} - \sqrt{x - \sqrt{3x}} = \frac{1}{2}\sqrt{x}.$$

George Karaolis reordered the terms and squared to get $(2\sqrt{x - 1} - \sqrt{x})^2 = 4(x - \sqrt{3x})$ and, again reordering and squaring, reached $[(x - 4) + 4\sqrt{3x}]^2 = 16x(x - 1)$. This equation simplifies to

$$(x - 4)(15x - 8\sqrt{3x} + 4) = 0 .$$

He then showed that $15x - 8\sqrt{3x} + 4 = 0$ has no real solutions. It thus follows that $x = 4$.

John Percival set $u = \sqrt{x}$ and found a quartic equation for u:

$$15u^4 - 8\sqrt{3}u^3 - 56u^2 + 32\sqrt{3}u - 16 = 0 .$$

This gives

$$(u^2 - 4)(15u^2 - 8\sqrt{3}u + 4) = 0 .$$

The second factor has no real zeros and so $u = \pm 2$. But $\sqrt{x} \neq -2$ so $\sqrt{x} = 2$, i.e. $x = 4$.

We may also note that for $\sqrt{x - \sqrt{3x}}$ to exist, we need $x \geqslant \sqrt{3x}$ or $x \geqslant 3$. Relatively straightforward arguments based on sketch-graphs show that the left hand side is less than the right when $x = 3$, but greater when x is large and that the graphs cross only once. The answer $x = 4$ may then be found by trial.

SOLUTION TO PROBLEM 7.4.1

A *palindromic number* is one which reads the same backwards or forwards. It can happen that when a number and its reverse are added, the sum is palindrome. [E.g. 1030 + 0301 = 1331.] In other cases, it isn't. 812 + 218 = 1030, which is not palindromic. But in this case a further operation of reversal and addition produces, as we saw above, a palindromic result.

Does this hold true in general if the process is repeated often enough? The answer depends on the number base employed and is known only in the case of base 2.

In that base, 10110 can never produce a palindromic number in this way. Can you prove this?

We reprint a solution from *Recreation in Mathematics: Some Novel Problems* by Roland Sprague (Blackie and Co. 1963).

Applying the procedure four times to the binary number 10110 gives the number 10110100, with no palindrome arising intermediately. We shall write the latter number as $101_{(2)}010_{(2)}$, so that it becomes the case $n = 2$ of a general type of number $101_{(n)}010_{(n)}$.

We now show that four further steps applied to a number of this type will produce the numbers $101_{(n+1)}010_{(n+1)}$, with once again no palindrome arising intermediately. To make the addition more easy to perform, we write $101_{(n)}010_{(n)}$ in the equivalent form $101_{(n-2)}11010_{(n-2)}00$. The two first steps then give:

$$\begin{array}{r}
101_{(n-2)}11010_{(n-2)}00 \\
+\ 000_{(n-2)}10111_{(n-2)}01 \\
\hline
110_{(n-2)}10001_{(n-2)}01 \\
+\ 101_{(n-2)}00010_{(n-2)}11 \\
\hline
1011_{(n-2)}10100_{(n-2)}00
\end{array}$$

or, more conveniently for the next addition, $101_{(n-2)}110100_{(n-2)}00$. The third and fourth steps yield:

$$\begin{array}{r}
101_{(n-2)}110100_{(n-2)}00 \\
+\ 000_{(n-2)}010111_{(n-2)}01 \\
\hline
110_{(n-2)}001011_{(n-2)}01 \\
+\ 101_{(n-2)}101000_{(n-2)}11 \\
\hline
1011_{(n-2)}110100_{(n-2)}00
\end{array}$$

Thus, as asserted, we obtain $101_{(n+1)}010_{(n+1)}$ and no sum can be a palindrome, for all later numbers will have the forms 10...00 and 11...01, in strict alternation.

The problem is also discussed in Martin Gardiner's book *The Mathematical Circus*, pp.242-252. We received a partial solution from George Karaolis.

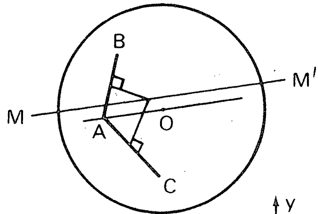
SOLUTION TO PROBLEM 7.4.2

Let S be a sphere with A, B, C three points interior to it. AB, AC are perpendicular to the diameter through A. There will be exactly two spheres passing through the points ABC and tangent to S. Prove that the sum of the radii of these spheres is equal to the radius of S.

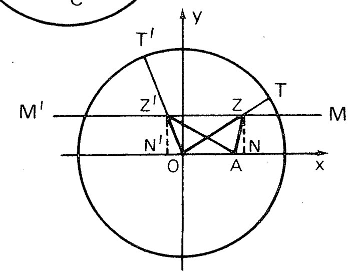
John Barton of 1008 Drummond St., North Carlton, solved this problem. We quote from his letter.

There are no such spheres, as specified, if the angle BAC is too great, or the points B and C are too far from the diameter through A. Otherwise, though, assuming we are dealing with the possible configurations, we can argue as follows.

A sphere through ABC has its centre on the line of intersection of the planes which bisect AB and AC respectively at right angles. This is the line MM' parallel to the diameter OA.



We now work in the plane of these two parallel lines.



To satisfy the geometrical condition that a sphere will pass through A (and hence also B and C) and also touch S, we have to find points (centres of spheres) Z, Z' on MM' such that

$$AZ = ZT \quad , \quad AZ' = Z'T' \ .$$

With axes as shown let the great circle on S be $x^2 + y^2 = b^2$ the line MM' be $y = m$, and the point A be $(a, 0)$. Denote unknowns Z, Z' by (ξ, m) and (ξ', m).

The line OT is $\{(x, y): y\xi = mx\}$, T is easily found to be $(b\xi/\sqrt{m^2 + \xi^2},\ bm/\sqrt{m^2 + \xi^2})$, and the condition $AZ = ZT$ easily gives the equation for the x-coordinates of Z, Z':

$$\xi^2 - a\xi = \frac{1}{4}(b^2 - a^2) - b^2m^2/(b^2 - a^2) \ . \qquad (1)$$

Provided $\sqrt{b^2 - a^2} > 2m$, which, geometrically, means that the line MM' is nearer the diameter OA than the (nearest point of the) small circle of intersection of S and the plane ABC, there are two (real) solutions for ξ, giving the x-coordinates of Z and Z'.

Further, from (1), $\xi + \xi' = a$,
that is, geometrically,
$$\vec{ON} + \vec{ON'} = \vec{OA}$$

and the figure $OAZZ'$ is an isosceles trapezium with $AZ = OZ'$, $AZ' = OZ$. It easily follows that

$$\begin{aligned} ZT + Z'T' &= ZT + AZ' \\ &= ZT + OZ = OT = b \ , \end{aligned}$$

as required.

SOLUTION TO PROBLEM 7.4.3

A snowplough starts ploughing snow at noon. By 1.00 p.m. it has travelled 10 km. By 2.00 p.m it has travelled 15 km. When did it start snowing? (Assume constant rates of snowfall and snow displacement.)

John Barton also solved this problem. Again we quote from his letter.

We assume that

(i) snow falls so that the rate of increase of snow-thickness on the ground, at all places, is constant ($=h$) length units per time unit

(ii) the snow plough has a constant rate of volume removal of snow equal to K cubic units per time unit

(iii) the path cleared by the plough has constant width equal to w length units.

Measuring time $t = 0$ from the beginning of snow-fall, the snow thickness at time t is ht. The area of section of the snow across the path is wht, and the rate of progress of the plough $K/(wht)$ length units per time unit.

If the plough has travelled x length units from its starting point and it starts when $t = t_0$, we have

$$\frac{dx}{dt} = \frac{K}{wht} \ ,$$

with solution
$$x = \frac{K}{wh} \log(t/t_0) \ .$$

We now use the hour as time unit. The length unit does not matter.

If $x = x_1$ at $t = t_0 + 1$ and $x = x_2$ at $t = t_0 + 2$, we have

$$\frac{x_1}{x_2} = \frac{\log(1 + 1/t_0)}{\log(1 + 2/t_0)} \text{ , and, since } \frac{x_1}{x_2} = \frac{2}{3} \text{ (data) ,}$$

we have
$$\log\{(1 + 1/t_0)^3\} = \log\{(1 + 2/t_0)^2\}$$
$$(1 + 1/t_0)^3 = (1 + 2/t_0)^2$$
$$(t_0 + 1)^3 = t_0(t_0 + 2)^2 \text{ ,}$$

which simplifies to $t_0^2 + t_0 - 1 = 0$, giving (positive) $t_0 = \frac{1}{2}(\sqrt{5} - 1) = 0.618$.

Now 0.618 hr = 37.08 min, so it started snowing 37.1 minutes before noon.

We close with some new problems for the summer break.

PROBLEM 7.5.1

The recurrence relation

$$f(n) - nf(n - 1) = (-1)^n$$

arose in connection with Problem 7.3.1. Derive the solution

$$f(n) = (n!)\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + \frac{(-1)^n}{n!}\right) .$$

PROBLEM 7.5.2

Is 1,000,343 prime or composite? Prove it.

PROBLEM 7.5.3 (From the 1982 South African Mathematical Olymniad)

Determine the smallest positive integer n such that, if the digit 7 is written after it and the digit 2 in front of it, the result is 91 times n.

PROBLEM 7.5.4 (From *Mathematical Digest*, July 1983)

A leaf is torn from a paperback novel. The sum of the remaining page numbers is 15000. Which pages were torn out?

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

HSC SCALING

Function Vol.7, Part 1 reported a new scaling procedure for HSC marks and criticised it severely. We are happy to report that VISE has rejected this and will now apply (at least in 1983) the 1982 procedures subject only to minor modifications.

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