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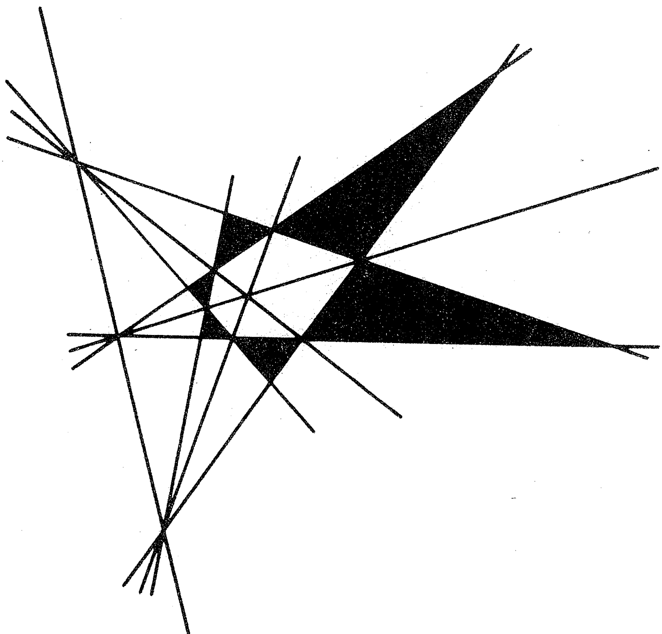

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Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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We hope readers will find an interesting variety in this issue of *Function*. First of all there are the letters and solutions to problems. This indicates a response to our last issue that the editors find most gratifying.

Then, of course, there are the articles. Two look at familiar things in new ways. Students learning matrix algebra will enjoy Charles and Wendler's perceptive article that makes mathematical mileage from an old chestnut; Peter Higgins presents an alternative derivation of the formula for $1 + 2 + \ldots + n$. And much more besides.

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THE FRONT COVER

M.A.B. Deakin, Monash University

Our front cover derives from *projective geometry*, which arose from the theory of perspective drawing.

It shows a regular hexagon with its main diagonals drawn. This same picture occurs at the right (Figure 1). By symmetry, the diagonals all intersect at a point *U*, while the sides are arranged as 3 pairs of parallel lines (*AY* with *BX*, *CY* with *BZ* and *AZ* with *CX*).

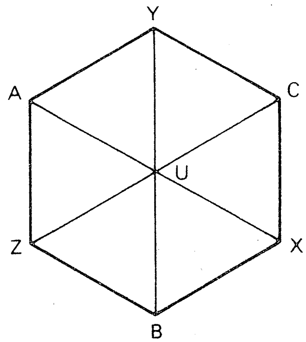

Figure 1

The diagram on the cover (and also reproduced as Figure 2 opposite) shows this picture from a tilted perspective, however. In this case, the parallelism disappears and is replaced by other properties.

A regular hexagon may be inscribed in a circle. In the new tilted perspective, this circle (not drawn) becomes an ellipse, one of the conic sections. (For more on these, see John Mack's article in this issue.)

We now have *Pascal's Theorem* which, in our case, states that *if the points A, Z, B and Y, C, X lie on a conic section, then the intersection of AY and BX (say P), the intersection of AZ and CX (say Q), and the intersection of BZ and CY (say R) are collinear,* that is to say *PQR* in Figure 2 is a straight line.

Note that in the square-on perspective of Figure 1, *P*, *Q*, *R* all lie at infinity. *PQR* is sometimes referred to as the *line at infinity*.

Pascal's Theorem is named after its discoverer, the 17th Century mathematician *Blaise Pascal*, who first formulated it in 1639. It generalises a special case in which the conic consists

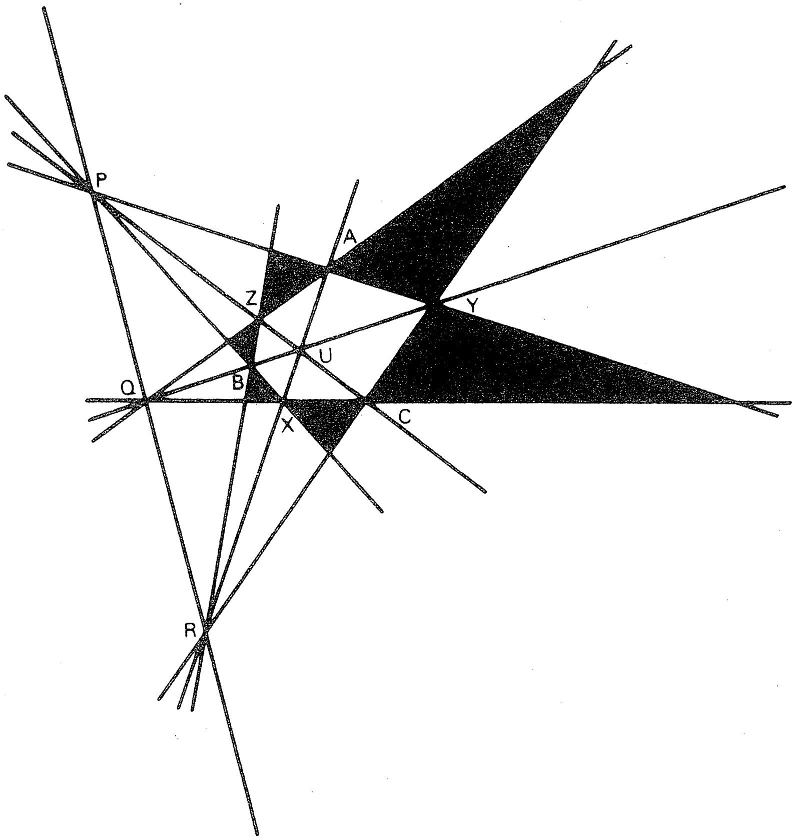

Figure 2

of two straight lines. The special case is referred to as *Pappus' Theorem*. Pascal's Theorem is basic to the branch of mathematics known as *Projective Geometry*, which generalises the theory of perspective.

We still have not exhausted the properties of Figure 1 and Figure 2, for note that the main diagonals of Figure 1 parallel the sides. E.g. $CZ \parallel YA$, etc. In the perspective of Figure 2, this means that CZ passes through P, YB passes through Q and AX passes through R. These diagonals still all pass through a single point of mutual intersection *U*.

This diagram formed the basis for much of the mathematical art of the late H.P. Nightingale, whose son Charles, a telecommunications scientist, wrote describing the construction involved in the journal *Leonardo* (Vol.6, 1973, pp.213-217). *Leonardo* is an international journal devoted to contemporary visual arts. A selection of articles from it, including this one, was published as a book: *Visual Arts, Mathematics and Computers* (Pergamon, 1979), which we heartily recommend.

We quote from Charles Nightingale's account of a diagram equivalent to Figure 2.

> "First draw a straight line. Then select three points on it arbitrarily, say *P*, *Q*, *R*. Further select an arbitrary point *U* [not on the line] and draw line *QU*. Select a point anywhere on line *QU*, say *Y*. Draw lines *PY*, *RY*. Let the intersection of *RU* with *PY* be *A* and that of *PU* with *RY* be *C*. Let the intersection of *QA* with *PC* be *Z* and that of *QC* with *RU* be *X*. Then the following holds true:
>
> The intersection of *PX* and *RZ* (say *B*) lies on the line *QY*.
>
> The resulting figure *AYCXBZ* is then a regular hexagon in the projective sense."

This construction uses the properties stated above in reverse to construct a perspective picture of the regular hexagon.

Nightingale's art extended considerably our notion of what is meant by "perspective", using the full range of projective geometry. In place of the ellipse mentioned above, we could use another conic section, say a hyperbola. Such "perspective" lies outside the visual experience that we normally bring to the world, where a circle seen in perspective is necessarily projected as an ellipse.

In the art that Nightingale based on the cover picture, the construction shown is merely one element, a tool. Three examples are reproduced (unfortunately only in black and white) in the course of the *Leonardo* article referred to above. In these, not merely a single regular hexagon appears, but we see representations of arrays of inscribed regular hexagons and the like.

As Charles Nightingale remarks:

"Perspective is a special case of projective geometry, just as a snake is a kind of reptile. Not all reptiles are snakes and similarly not all projective geometry is perspective. It is reasonable to inquire whether the geometry could sustain an artistic life independent of perspective. The striking perspectives of Renaissance painting catch and fascinate the eye with their infinite depth of focus, which is unattainable in photography. Yet, nature still rules. Remove the figures, the windows and all detail, leaving only the geometric structure and one is still firmly trapped in the observed world. Nightingale stepped away from the observed world, away from perspective and into a more general conception."

HONESTY, CAUTION AND COMMON SENSE

N.S. Barnett,
Footscray Institute of Technology

The growth in the use of statistical methods has followed the general expansion of technology and biological sciences. Biology was a 'boom' subject in the late seventies, brought about, in part, by an effort to focus public attention on the environment and its vulnerability to man's interference. Many disciplines, formerly regarded as qualitative rather than quantitative, have undergone change in recent decades, presenting now a more 'scientific image'. An effort to make disciplines more quantitative in many cases has led to an increased use of statistical methods. Market Researchers, on a broad front, use statistical techniques in trying to understand and explain market trends. In fact, one of the very first recorded opinion polls cum sample surveys was a type of marketing exercise.

It was conducted by a young journalist named Armstrong whilst on an 'idea man' trip for the Des Moines, Iowa, Merchants Trade Journal in 1914. A full report of the survey appeared in the April 1915 edition of the Journal. The survey was conducted in Richmond, Kentucky, reputedly, at the time, the 'deadest town in all America'. Following an argument with a local furniture merchant in Richmond regarding the lackadaisical approach of local business, the young Armstrong set himself three days to find out more about local trade than was known by resident merchants. By a shrewd piece of detective work at the local freight office he was able to obtain the names and addresses of local citizens receiving merchandise from out of town together with a description of items being purchased. The local banks and post office provided him with information on bank drafts and money orders made by local depositors over the previous month to mail order houses and stores out of town. For three days he conducted house-to-house and farm-to-farm interviews, obtaining first-hand comments on the deficiencies of local stores. Discontented housewives were more than eager to give voice to their grievances; issues of poor shopping conditions and indifferent attitudes of assistants were amongst those highlighted. The whole enterprise was conducted and the report, tabulations and recommendations typed and presented in four days. Eye-opening revelations were made public and, to a degree, business attitudes changed.

In the 1980's, business in general is becoming more and more aware of the role that statistical arguments can play in helping make business decisions. Traditional applications of statistical methods in industry are also being revitalised, for example their use in quality control procedures.

Thousands of companies in the United Kingdom participated in the activities of Quality and Reliability Year (1966-67), which was organised by the National Council for Quality and Reliability, to encourage companies to reduce costs by adopting 'good' quality practices. One such practice quoted was the use of statistical quality control. At the International Academy for Quality conference in 1978 the superior performance of certain Japanese manufactured products was, in part, explained by emphasis on quality and, in particular, it was said that the application of statistical methods is one of the pillars of Japanese Quality Control.

This growth in interest and use of statistical argument has been fuelled by the growth in computer technology and the ready availability of computing facilities.

Reading the statistical work of the 20's, 30's and 40's, one is immediately struck by the regard the authors had to the feasibility of performing actual calculations - since these had to be done by tedious, time-consuming methods. With a computer, time-consuming methodology is no longer a barrier. Thus modern statistical techniques tend to lack the 'beauty' of much of the earlier work but from a pragmatic point of view the justification of more modern techniques must lie in their application to practical problems.

It is important to realise that many of the fundamental ideas used in statistical inference are a product of the twentieth century, developed by people who were themselves scientifically oriented, for instance, R.A. Fisher and K. Pearson. Australia has certainly not been without its own distinguished statisticians, for example, E.J. Pitman, C.E. Weatherburn and M. Belz, all of whom were mathematically and/or scientifically oriented. In fact, the development and perpetuation of the fundamental statistical ideas alluded to above occurred in the scientific environment - in the realm of quantitative measurement and observation.

In recent years a fundamental change has occurred in that many, if not the majority, of users of statistical techniques are no longer working in the scientific realm. Further, because of the ready availability of computer packages, many users of statistical techniques are not statistically trained. These and other factors have created a climate in which statistical methods can be and are being mis-applied and in which certain fundamental statistical notions are overlooked. One such notion is the fact that a statistical inference does not constitute a proof - it is at best a 'speculation' based on evidence - it is a quantitative appraisal of the evidence pointing towards certain conclusions. It is fundamentally dangerous to make unreserved conclusions based on a statistical analysis alone. A second fundamental point, often overlooked, pertains to the environment in which most statistical techniques have been developed - put simply: the statistical methods applied must have regard to the nature of the data available. For example, the type of data obtained from a scientific experiment is substantially different from the type of data obtained from sample surveys in which data may pertain to opinions and strengths of feelings towards certain products - even though these 'feelings' may have been quantified in some manner. A third phenomenon which is constantly evidenced is the disproportionate cost, time and effort expended on a statistical analysis whose results must be at best suspect and at worst downright misleading. Most statisticians can cite examples of action taken by government departments or private industry based on spurious and costly statistical analyses. There is a need to see a statistical analysis of a decision-making tool to be used, where possible (and it often is), in conjunction with other decision-making tools; this is particularly relevant in applications to business, industry and commerce. The fact that business decisions frequently have to be made quickly has prompted the need for 'short-cut' statistical analyses to obtain a hurried statistical appraisal of available figures. There have thus evolved a number of quick and easy techniques. With these there is even more danger in basing action entirely on the statistical analysis alone - but it has been done! Such techniques are appealing in that they, by their very nature, are easy to apply. This must be compared with the large body of theoretical justification and assumption which underpin many of the standard statistical methods. Despite the computational complexity however, the aim of many of these standard techniques is very simple but it is easy to lose sight of this fact and get bogged down in the mathematical and statistical manipulation and justification.

In summary then, remember that a statistical analysis in its classical setting does not constitute a proof - it is, rather, a quantitative appraisal of the evidence. The nature of the data is important when applying techniques because these techniques rely heavily on certain assumptions. When at all possible never hang the whole decision-making process or draw unreserved conclusions on the strength of a statistical analysis alone. Finally, the time, cost and effort expended on a statistical analysis should be proportional to its usefulness. One needs to proceed with honesty, a good deal of caution and, dare I say it, with a healthy degree of common sense. The survey example cited in the introduction is an example of this latter point bearing in mind that the voluminous amount of mathematical and statistical justification of sample survey procedures was not available in 1914!

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THAT DAY IS NOW

Statistical thinking will one day be as necessary for efficient citizenship as the ability to read and write.

H.G. Wells

IMPOSSIBLE ?†

Look carefully at Figure 1, the logo of the Netherlands Mathematical Olympiad. It was designed, some years ago, by Marc van Leeuwen, a participant and prize-winner not only in our national but also in the International Mathematics Olympiad. With the logo Marc wanted to demonstrate, in two ways, something impossible: an equilateral triangle with three right angles!

This is impossible because, as you know, the sum of the angles of every triangle is 180°, whereas with three right angles you get 270°, which is 90° too many.

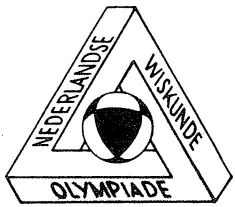

Figure 1. Two equilateral triangles each with three right angles.

However, in spherical geometry†† it is a different matter and you can see that in the centre of the logo. This contains a sphere which consists of white and black sections.

Using a little imagination you could take it to represent a picture of a globe divided into eight sections by three great circles, say the equator, and two circles through the North pole and South pole. These eight pieces are coloured alternately white and black. In the logo you see *one* such black "octant" as well as the points of the three adjacent black octants. Each octant is bounded by three great circles intersecting each other at right angles. Thus you have a "triangle" with equal "sides" and three right angles.

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†This article is a translation of one appearing in the Netherlands Journal *Pythagoras*. It appears here under an exchange agreement between *Pythagoras* and *Function*. We thank A.-M. Vandenberg for the translation.

††See *Function* Vol.6, Parts 4 and 5.

THE TRIANGLE OF BARS

In the second "impossible" triangle are the words "Nederlandse Wiskunde Olympiade" (meaning "Netherlands Mathematical Olympiad"). It is a nice example of an artist's ability to create the illusion of spatial figures by means of flat pictures. It looks just as if bars have been drawn - ordinary straight bars joined together at right angles. But the drawing is deceptive, for three such bars can never form a triangle in this manner. But why can you still draw it like this? In Figure 2 we have made things even worse. It looks as if you can simply put together the triangle in this picture! And there is nothing wrong with the bars. A carpenter would not have any problems with them. Each connection *separately* is perfectly correct, too. You can see that if you just lift one such connection out of the drawing (Figure 3). But the construction as a whole is incorrect. Two connections together would succeed but adding the third one is impossible (Figure 2).

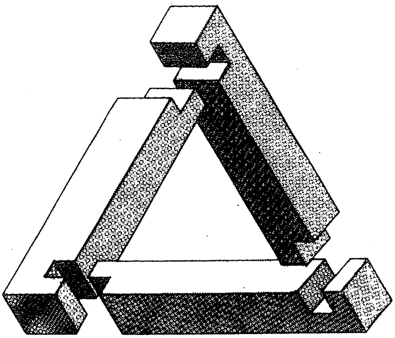

Figure 2. An impossible triangle of bars.

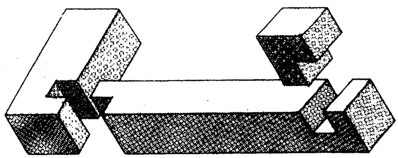

Figure 3: It is still possible with two connections.

SOME MORE IMPOSSIBILITIES

Figure 4 shows another 3 variations: an impossible square and two impossible pentagons - impossible, at least, if we want to regard them as spatial figures consisting of straight bars. The plaited pentagonal star we find the nicest. You could imagine that each vertex separately forms a right-angled joint, but looking further at the drawing as a whole, you notice the entire thing is wrong. An artist's trick *par excellence*!

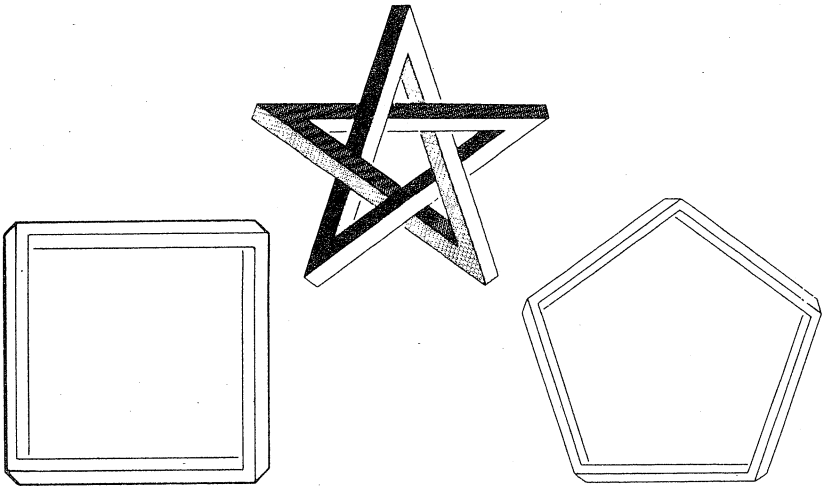

Figure 4. An impossible square and two impossible pentagons.

REAL HEXAGONS

There is nothing wrong with the hexagon of Figure 5. It can really be constructed. There are indeed "regular" hexagons with six right angles and six equal sides. Not flat, of course, but three-dimensional. There are even two essentially different types. One is shown in Figure 5, the other in Figure 6. Yet we find the one in Figure 5 somewhat "more regular" than that in Figure 6. Do you see why?

We have also made those hexagons with pieces of plastic electrical conduit and right-angled "elbows". Figure 7 shows the result. You can extend the "not so regular" hexagon to an octagon, a decagon, etc.

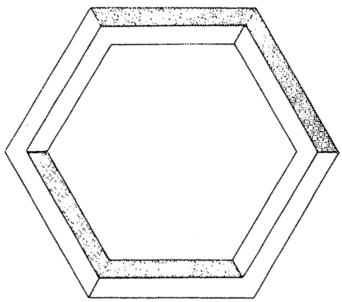

Figure 5. A regular hexagon with right angles.

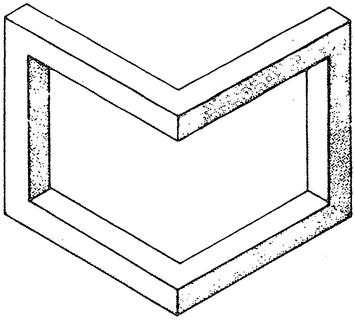

Figure 6. Another right-angled hexagon.

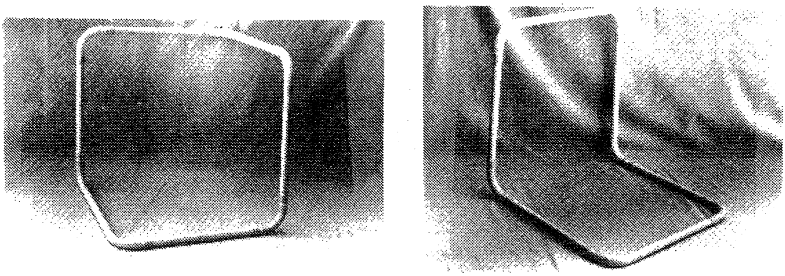

Figure 7. Two equilateral hexagons with right-angled corners made with lengths of electrical conduit.

THE RIDDLE OF THE HEPTAGON

Now a question. Is there a similar type of heptagon? That is to say, a closed figure made of *seven* equally long straight tubes and seven right-angled joints?

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A PREDICTION WE WISH HAD COME TRUE

Since of late mathematicians have set themselves to the study of it [medicine] men do already begin to talk so intelligibly and comprehensibly even about abstruse matters that it may be hoped in a short time that those who are designed for this profession are early, while their minds and bodies are patient of labour and toil, initiated in the knowledge of numbers and geometry, that mathematical learning will be the distinguishing mark of a physician from a quack.

Richard Mead, Physician to Queen Anne.

YET ANOTHER PANDORA'S BOX!

D. Charles and W. Wendler,

Pascoe Vale Girls High School

Mathematics displays its variety with its plethora of integrated structures. Its cohesiveness is shown when a discovery in one branch flows into and modifies, highlights, or simplifies another branch.

Often, in trying to solve a problem, the path to the solution opens other byways leading to seemingly unconnected topics. If the mathematician is single-minded enough, these are left until the problem at hand is solved. The old adage of one answer giving rise to a hundred new questions appears to hold true.

Of particular interest is the solution to the problem of two litre vessels, A and B. A is filled with water and B is filled with milk. A spoonful of water is transferred from vessel A to vessel B, the contents of B are allowed to mix thoroughly and a spoonful of the now homogeneous mixture is transferred from B back to A. The above process is repeated an indefinite number of times and the relevant question is: "How many times must the process be repeated until the mixtures in each vessel are of equal consistency?" The answer usually given is that they are never equal because the amount transferred is always less than half the difference between the respective proportions; but let us examine the problem more carefully.

We have two choices, either we can look at the amount of a particular liquid in a given container after n cycles, say the amount of water in container A, or we can look at the system of both containers as a whole and observe the change in the "state" of the system. Let us choose the latter as it seems more systematic (pardon the pun).

Since there are two liquids, each in two vessels let us construct a "state" matrix S_n, which gives the amount of liquid of each type in each container after n operations.

Initially the amounts may be represented as follows:

$$\begin{array}{cc} \text{Vessel } A & \text{Vessel } B \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \text{water} \\ \text{milk} \end{array}$$

Let us assume the spoon holds x litres $(0 < x < 1)$, and try to calculate S_1.

$$\begin{array}{ll} & \begin{matrix} A & B \end{matrix} \\ \begin{matrix} \text{water} \\ \text{milk} \end{matrix} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1-x & x \\ 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1-x+\frac{x^2}{x+1} & x-\frac{x^2}{x+1} \\ \frac{x}{x+1} & 1-\frac{x}{x+1} \end{bmatrix} \end{array}$$

$$S_0 \qquad\qquad S_{\frac{1}{2}} \qquad\qquad S_1$$

i.e. $$S_1 = \frac{1}{x+1}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

We now have:

$$S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad S_1 = \frac{1}{x+1}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

We moved from S_0 to S_1 by performing an operation on S_0. This operation is in fact matrix multiplication so we have:

$MS_0 = S_1$ where M is the matrix operator corresponding to the liquid transfer.

Since S_0 is the identity matrix we have:

$$MI = S_1$$
$$\therefore M = S_1$$
$$\therefore M = \frac{1}{x+1}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

n successive applications of M therefore should give us S_n, i.e.

$$S_n = M^n.$$

In calculating S_2 from first principles for example we obtain:

$$S_2 = \frac{1}{(x+1)^2}\begin{bmatrix} x^2+1 & 2x \\ 2x & x^2+1 \end{bmatrix}$$

Check this for yourself.

From our formula we obtain:

$$S_2 = MS_1 = M^2 = \frac{1}{(x+1)^2}\begin{bmatrix} x^2+1 & 2x \\ 2x & x^2+1 \end{bmatrix}$$

Similarly:

$$S_3 = MS_2 = \frac{1}{(x+1)^3}\begin{bmatrix} 3x^2 + 1 & x^3 + 3x \\ x^3 + 3x & 3x^2 + 1 \end{bmatrix}$$

We are now in a position to calculate the various concentrations after n operations. Iteration, unfortunately, is always a tedious method but happily for us a shortcut can be seen by examining the state matrices.

$$S_0 \qquad \frac{1}{1}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad\qquad S_1 \qquad \frac{1}{(x+1)}\begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix}$$

$$S_2 \qquad \frac{1}{(x+1)^2}\begin{bmatrix} x^2 + 1 & 2x \\ 2x & x^2 + 1 \end{bmatrix} \qquad\qquad S_3 \qquad \frac{1}{(x+1)^3}\begin{bmatrix} 3x^2 + 1 & x^3 + 3x \\ x^3 + 3x & 3x^2 + 1 \end{bmatrix}$$

Not only is S_n diagonally symmetric but the sum of any two adjacent terms is 1 (after division by the common factor $\frac{1}{(x+1)^n}$). This of course is to be expected since "conservation of liquid" must hold.

Looking more closely we see that the top left and bottom right have, as numerators, the sum of the terms with *even powers of* x in the expansion of $(1 + x)^n$ whereas the other positions consist of a numerator in odd powers of x in the expansion of $(1 + x)^n$.

The recipe for finding S_n is now greatly simplified. e.g. What is the value of the state matrix after 8 transfers (S_8) ?

$$(1 + x)^8 = \underline{1} + 8x + \underline{28x^2} + 56x^3 + \underline{70x^4} + 56x^5 + \underline{28x^6} + 8x^7 + \underline{x^8}$$

$$\therefore S_8 = \frac{1}{(x+1)^8}\begin{bmatrix} 1+28x^2+70x^4+28x^6+x^8 & 8x+56x^3+56x^5+2x^7 \\ 8x+56x^3+56x^5+8x^7 & 1+28x^2+70x^4+28x^6+x^8 \end{bmatrix}$$

We now suppose that the elements of S_n each approach a limit as $n \to \infty$. Common sense tells us that the liquids will mix almost evenly after a large number of transfers so we expect that:

$$\lim_{n\to\infty} M^n = \begin{bmatrix} 0\cdot5 & 0\cdot5 \\ 0\cdot5 & 0\cdot5 \end{bmatrix} .$$

We demonstrate that the elements do indeed converge to 0·5 as follows.

Suppose that at some stage the state matrix is given by:

$$S_n = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad \text{where } a > b \text{ and } a + b = 1.$$

Then

$$S_{n+1} = \frac{1}{x+1} \begin{bmatrix} 1 & x \\ x & 1 \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$= \frac{1}{x+1} \begin{bmatrix} a + bx & b + ax \\ b + ax & a + bx \end{bmatrix} .$$

Write this as

$$\begin{bmatrix} a - \delta & b + \delta \\ b + \delta & a - \delta \end{bmatrix} ,$$

where $\delta = (a - b)\left(\frac{x}{x+1}\right)$ and note that $\delta > 0$.

We see that δ is proportional to the difference between a and b. Then, as $\delta \to 0$, a and b both approach 0·5.

Application of M to the matrix $\begin{bmatrix} 0\cdot5 & 0\cdot5 \\ 0\cdot5 & 0\cdot5 \end{bmatrix}$ leaves it unaltered.†

The following theorem is then seen to hold:

$$\lim_{n\to\infty} \frac{1 + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots}{(1+x)^n}$$

$$= \lim_{n\to\infty} \frac{nx + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots}{(1+x)^n}$$

$$= 0\cdot5.$$

---

†Note: Readers familiar with The Markov Chain model outlined in the H.S.C. option "matrices" may like to devise an alternative proof.

Let us examine the Binomial Approximation. It is usually quoted in textbooks as:

$$(1 + x)^n \cong 1 + nx \quad \text{for } x \text{ small and all values of } n.$$

Were this true in all cases, then

$$\frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \frac{n(n-1)(n-2)(n-3)}{4!}x^4 + \ldots$$

would be small in relation to $(1 + x)^n$. In consequence, the numerators in the box above could be approximated by, in the top line, 1, and in the second line, nx. This would then give

$$\frac{1}{(1 + x)^n} \cong \frac{nx}{(1 + x)^n} \cong \frac{1}{2}, \text{ when } n \text{ is large.}$$

But this is patently untrue, even when n is large and x is small. The Binomial Approximation is more correctly stated as:

$$(1 + x)^n \cong 1 + nx, \text{ when } x \ll 1 \text{ and } nx < 1.$$

One can easily see that, as n becomes large, third order and higher order terms in x do not become negligible. We leave it to the reader to examine other aspects of this peculiar separation of terms in the expansion of $(1 + x)^n$.

Another impossible object (see pp.8-11) is shown at right. It was drawn on a stamp to commemorate the 10th Austrian International Mathematical Congress, held in Innsbruck in September, 1981. Each four years the Austrian Mathematical Society hosts such a congress, widely attended by delegates from Germany and the Eastern European countries.

A GEOMETRICAL PROOF THAT $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n + 1)$

Peter Higgins, Monash University

The story goes that a teacher, wanting to keep his nine-year-olds quiet for an hour or so, told them to add up all the numbers from one to one hundred. His plan was thwarted by the immediate announcement of the correct answer, 5050, by one of his pupils.

The boy with the answer was Carl Gauss and no doubt he knew of the above formula. Someone may have told him, but it seems likely as not that he discovered it himself for he went on to become one of the very greatest of mathematicians.

One way to prove this formula is as a standard exercise in a method of proof called mathematical induction. However here is a geometric approach.

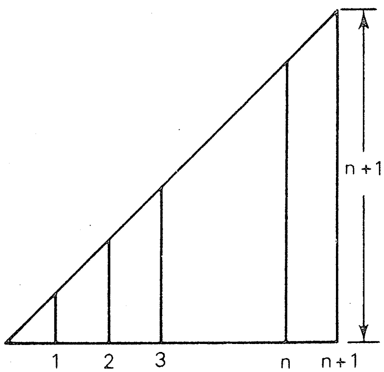

Figure 1

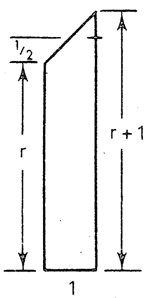

Figure 2

Figure 1 shows an isosceles right-angled triangle whose equal sides are both of length $n + 1$. Clearly its area is $\frac{1}{2} \times$ base $\times$ height, i.e. $\frac{1}{2}(n + 1)^2$.

However, the area may be calculated in another way. Divide the base into intervals each one unit long and so divide the triangle into trapezoidal strips as shown. The area of a typical strip (Figure 2) of base 1 and with height varying between r and $r + 1$ is $1 \times (r + \frac{1}{2})$ as the construction drawn in Figure 2 demonstrates. Thus, as r goes from 0 to n, the strips have areas $0 + \frac{1}{2}$, $1 + \frac{1}{2}$, $2 + \frac{1}{2}$, ..., $(n + \frac{1}{2})$ respectively.

Hence the total area is

$$(0 + \tfrac{1}{2}) + (1 + \tfrac{1}{2}) + (2 + \tfrac{1}{2}) + \ldots + (n + \tfrac{1}{2}).$$

But this equals

$$1 + 2 + 3 + \ldots + n + \tfrac{1}{2}(n + 1).$$

Therefore

$$1 + 2 + 3 + \ldots + n + \tfrac{1}{2}(n + 1) = \tfrac{1}{2}(n + 1)^2$$

and so

$$1 + 2 + 3 + \ldots + n = \tfrac{1}{2}(n + 1)^2 - \tfrac{1}{2}(n + 1) = \tfrac{1}{2}n(n + 1).$$

This derivation is related to concepts used in integral calculus.

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

CONES AND CONIC SECTIONS I †

John Mack, University of Sydney

Introduction

The *conic sections* are the curves obtained when a plane intersects a circular cone and include curves such as ellipses and hyperbolas. The study of these curves in the H.S.C. 4 unit mathematics course confines attention to their focus-directrix properties, standard equations $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ and standard para-

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† The first part of a talk given to talented students in Sydney, 1982. The conclusion will appear in our next issue. Dr Mack writes "Since giving this talk, I have had opportunity to see the Open University film on conics. This develops their properties exactly as I do and uses some magnificent illustrative models, so that one does not have to rely on one's imagination. Australian distributors are Holt-Saunders, but the films are fiendishly expensive."

metrisations. My purpose is to relate conic sections to the cone and to give some insight into the geometrical significance of some of their properties, including their foci and directrices.

I begin by describing some of the ways that a cone can be obtained. (By 'cone' I mean a right circular cone.) It is a surface of revolution obtained by rotating a given line about a fixed line which meets the given line in V. V becomes the vertex of the cone, the fixed line is its axis, and the given line is a generator of the cone. The surface is symmetrical in the point V and consists of two pieces called *nappes*, although it is common to refer to each of these (i.e., half the whole cone) also as a cone.

Imagine our cone with axis vertical. If we drop a ball into the upper half, it will stop when it fits snugly into the cone. By symmetry it will touch the cone in a horizontal circle, which will be a "small circle" on the surface of the ball. The generators of the cone are precisely the lines joining V to points on this circle and are also the tangent lines to the ball from V.

Just as the two tangents from an external point in the plane of a circle to the circle have the same length, so all the tangents from an external point to a ball have the same length. This is an important simple fact, which we shall use several times.

Intersection of a plane and a cone

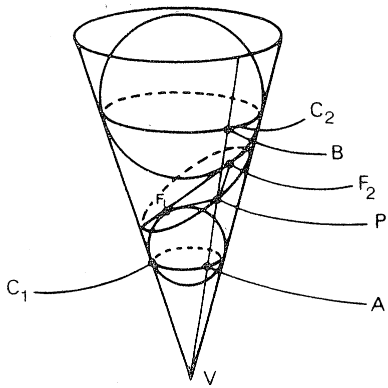
A plane perpendicular to the axis of a cone meets it in a *circle*, unless the plane passes through the vertex V, when the intersection degenerates to a *single point*. If we slowly tilt the plane, we obtain first nearly circular ellipses which grow larger and become less round as we tilt. When the plane has tilted so that it becomes parallel to a generator of the cone, the ellipses change into a parabola. As soon as the tilt increases again, the parabola changes immediately into the two-branched hyperbola. Further tilting produced no new curves. For special planes we will obtain degenerate curves such as a single line through V or a pair of lines through V.

Alternatively, take a fixed plane and consider the family of parallel planes and their curves of intersection with a given cone. We may obtain a family of circles, or of ellipses, or of hyperbolas in this way. Can we obtain a family of parabolas?

The ellipse

There is time to discuss one curve only and we shall investigate the ellipse. Again, I suppose we have the axis of the cone vertical and a plane cutting the upper half of the cone in an ellipse. This time, we imagine what happens as we drop balls into it, beginning with a very small one and increasing their size steadily.

Small spheres will sit very close to the vertex. As they grow larger, they sit steadily higher in the cone and their surfaces get closer to the plane of the ellipse, until, for a sphere S_1 of one size only, the sphere touches both the cone in a circle C_1 and the underside of the plane in a single point F_1. As the spheres increase further in size, they rise higher inside the cone, as does the circle of contact with it. These spheres will meet the plane of the ellipse in circles, growing at first in size (from the single point contact at F_1) and then diminishing as the spheres rise higher, until, at a second size S_2, the sphere will just touch the upper side of the plane at a single point F_2, while touching the cone in a circle C_2.



Choose any point P on the ellipse, let VP be the generator of the cone passing through P and let it meet C_1 and C_2 in A and B respectively. Join PF_1 and PF_2.

Consider the sphere S_1 and the point P. PA and PF_1 are tangents to S_1 from P and hence are of equal length.

Similarly, PB and PF_2 are tangents to the sphere S_2 and hence are of equal length. Thus

$$PF_1 + PF_2 = PA + PB = AB.$$

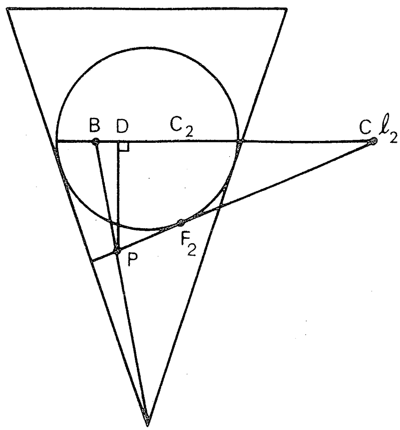
But the distance AB is the distance along a generator between the circles C_1 and C_2 and is fixed, independently of P. So the points F_1 and F_2 have the special property that *the sum of their distances from any point* P *on the ellipse is a constant*. Thus the ellipse may be constructed by tying a thread of length AB to pins at F_1 and F_2, pulling it taut with a pencil and tracing the path of the pencil point P in the given plane as the pencil encircles F_1 and F_2.

Before proceeding further, let us apply the same argument to the case of an inclined plane cutting a right circular cylinder with axis vertical. Spheres of one size only now fit exactly into the cylinder and exactly two will touch the plane, one from above and one from below. Applying the argument used above, this time to the generators of the cylinder, shows that the same result is true and the curve of intersection is an ellipse. Further, by the symmetry of the cylinder about any point on its axis, we see that the ellipse is symmetrical about the point of intersection of the plane and the axis of the cylinder. (Note however that the centre of an ellipse obtained by intersecting a plane with a cone does not lie on the axis of the cone.)

As another aside, imagine that we have a strong point source of light at V (the vertex of our cone), that a horizontal circular cross-section of the cone is opaque, and that we are interested in the shape of the shadow cast by this circle on a plane surface lying above it. Obviously, this shadow is a section of the cone and will be an elliptical, parabolic, hyperbolic or circular shape depending on the inclination of the plane. Similarly, the shadow cast by a circular disc illuminated by a beam of parallel light rays perpendicular to its plane is shown by our cylinder model to be of elliptical or circular shape.

The focus-directrix property of the ellipse

The two points F_1 and F_2 constructed above are called the *foci* of the ellipse. With each focus is associated a special line called a *directrix*, obtained as follows.



Position ourselves so we are looking "side-on" at the plane of the ellipse, so that it looks like a line, as does the horizontal plane of the circle C_2. These two planes then meet in a line ℓ_2 which is in the direction of our line of sight and so appears as a point.

Let PC be the perpendicular from P to ℓ_2, meeting ℓ_2 in C. Let PD be the perpendicular from P to the plane of C_2, meeting this plane in D. (In the plane of C_2, D will lie on the line joining B to the centre of the circle C_2.) The triangle PDC is right-angled at D because CD lies in the plane of C_2. Hence

$$\cos C\hat{P}D = \frac{PD}{PC} .$$

Similarly, the triangle PDB is right-angled at D and hence

$$\cos B\hat{P}D = \frac{PD}{PB}$$

$$= \frac{PD}{PF_2}$$

because $PB = PF_2$. Thus, by division,

$$\frac{PF_2}{PC} = \frac{PF_2}{PD} \cdot \frac{PD}{PC} = \frac{\cos C\hat{P}D}{\cos B\hat{P}D} .$$

But $C\hat{P}D = 90° - B\hat{C}P$ and hence is independent of P since $B\hat{C}P$ is the angle between the plane of C_2 and the plane of the ellipse. Also, $B\hat{P}D$ equals the angle between the generator VB and the axis of the cone and is also independent of P, as it is one half the 'vertex angle' or 'angle of aperture' of the cone. Hence

$$\frac{\cos C\hat{P}D}{\cos B\hat{P}D} = \text{a constant, } e \text{ say,}$$

where (because $C\hat{P}D > B\hat{P}D$, since the plane of the ellipse is not tilted enough to be parallel to a generator of the cone) $0 < e < 1$. Thus

$$\frac{PF_2}{PC} = e ,$$

i.e., the distance of P from the focus F_2 is e times the distance from the corresponding directrix ℓ_2.

A similar argument applied to F_1 and the circle C_1 produces a directrix ℓ_1 with the property

$$\frac{\text{distance of } P \text{ from } F_1}{\text{distance of } P \text{ from } \ell_1} = \text{the same constant } e.$$

In the plane of the ellipse, take F_1F_2 as the x-axis, with the origin O midway between F_1 and F_2. If we let $(\pm a, 0)$ be the points of intersection of the ellipse with the x-axis then we shall find that F_1, F_2 have coordinates $(\pm ae, 0)$ and ℓ_1, ℓ_2 have equations $y = \pm \frac{a}{e}$. If P has coordinates (x, y), the condition $PF_2 = ePC$ gives the equation of the ellipse as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

LETTERS TO THE EDITOR

Two articles from our last issue attracted comment. In the first place, Sir Richard Eggleston addresses some of the legal issues raised by the article on the rodeo problem. Dr Olbrich's generalization of Pythagoras' Theorem attracted an alternative proof from Garnet J. Greenbury, one of our most faithful correspondents.

We have received two full solutions to the "1983" problem, and finally Kim Dean reports on a disturbing discovery.

THE RODEO

I read with interest Dr Watterson's article, 'The Rodeo' (Vol.7, Part 1, p.19). I do not repudiate the proposition for which I am quoted in it, but I would point out that my objection to L. Jonathan Cohen's treatment of the rodeo example is mainly based on another ground, namely, that it ignores the possibility of the defendant giving evidence.

At the best, statistical reasoning can only raise a *prima facie* case against the defendant, enabling the Court to give judgment against him if the defendant declines to give evidence. But in the case put by Dr Watterson, if J. Smith, instead of merely asserting from the Bar Table, "I did pay", had been prepared to enter the witness box and give sworn evidence to that effect, the statistical probability in favour of the plaintiff could hardly prevail against the sworn testimony of the defendant, unless, of course, the plaintiff, by cross-examining the defendant, had been able to show that on the day in question J. Smith did not have $4 to his name, or otherwise to persuade the Court that Smith was not telling the truth.

What is more difficult is the case where the plaintiff has a statistical probability in his favour and, for one reason or another, the defendant is unable to rebut the prima facie case. This could happen, for example, if John Smith had died before the case came on, and the action was being continued against his executor, or if the case were of a kind in which in the nature of things the statistical probability could not be rebutted. Such a case is discussed by Professor Kaye in his article, 'The Limits of the Preponderance of the Evidence Standard: Justifiably Naked Statistical Evidence and Multiple Causation' (American Bar Foundation Research Journal, 1982, Spring No.2, 487 at p.492). Kaye gives the following example: Suppose a man suffers from a particular type of cancer and the statistics show that it is more probable than not that the cancer is caused by a particular chemical agent to which the plaintiff has been exposed in the course of his work. The evidence, however, shows that not all people exposed to the chemical contract that type of cancer, and not all people who do contract it have been exposed to the chemical. Thus, while there is a statistical probability in favour of the chemical having caused the cancer, it is not certain; on the other hand, there is no way in which the defendant can rebut the plaintiff's case.

Does this mean that the defendant would be held liable for every case of cancer of this type occurring amongst his employees, even though the statistics show that some of them would have contracted the disease in any event?

It is difficulties of this kind which have led some judges to assert that a mere statistical probability is not enough. Thus in an American case, where a woman complained that an unidentified bus had forced her into a collision with another car, and proved that the defendant busline was the only one authorised to operate on the route in question, and that one of the defendant's buses was scheduled to run on that route at about the time of the accident, the judge said that although the mathematical probabilities favoured the plaintiff, this was not enough (*Smith* v. *Rapid Transit Inc.* 58 N.E. 2d 754 (1945)). The judge in that case insisted that the plaintiff must engender in the mind of the judge an 'actual belief' in the truth of his assertion. Since the plaintiff could not exclude the possibility of a chartered bus having been responsible, the judge was not satisfied.

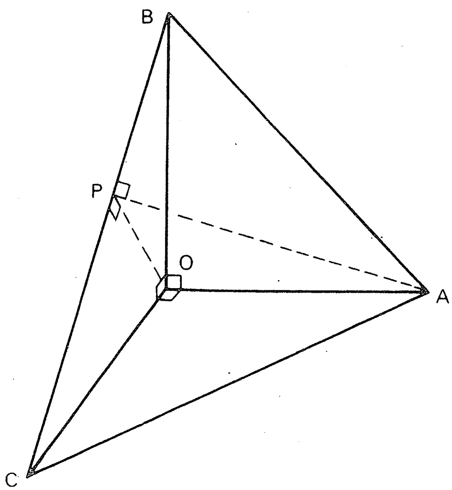
The authorities in this country are conflicting. When the matter is finally clarified, one would like to think that the judges would take into account the kind of comparison of utilities and disutilities to which Dr Watterson draws attention.

R.M. Eggleston,
Faculty of Law, Monash University

THE PYTHAGOREAN THEOREM IN THREE DIMENSIONS

Here is an alternative proof of Olbrich's generalized Pythagorean Theorem.

Let $OABC$ be a right-angled tetrahedron in which OA, OB and OC are mutually perpendicular.



The result to be proved is:

The square of the area of the hypotenuse face is equal to the sum of the squares of the areas of the other three faces, i.e. $ABC^2 = OAB^2 + OBC^2 + OCA^2$.

Let the lengths of OA, OB and OC be a, b and c. Draw AP perpendicular to BC.

$$\begin{aligned} CP &= OC \cos OCB \\ &= OC \frac{OC}{BC} \\ &= \frac{c^2}{\sqrt{b^2 + c^2}} . \end{aligned}$$

Thus

$$\begin{aligned} ABC &= \tfrac{1}{2}\sqrt{b^2 + c^2} \,.\, \sqrt{a^2 + c^2 - \frac{c^4}{b^2 + c^2}} \\ &= \tfrac{1}{2}\sqrt{(b^2 + c^2).(c^2 + a^2) - c^4} \\ &= \tfrac{1}{2}\sqrt{(bc)^2 + (ca)^2 + (ab)^2} . \end{aligned}$$

It follows that

$$\begin{aligned} ABC^2 &= (\tfrac{1}{2}bc)^2 + (\tfrac{1}{2}ca)^2 + (\tfrac{1}{2}ab)^2 \\ &= OAB^2 + OBC^2 + OCA^2 . \end{aligned}$$

For a related result, see my article "Cosine rule in 3-space", published in *The Mathematics Teacher*, Vol.75, Part 1 (Jan. 1982).

Garnet J. Greenbury,
Brisbane.

1983 SOLVED

In relation to Problem 6.1.1, you raise the problem of expressing the integers 1 to 100 in terms of the digits 1, 9, 8, 3 in that order. I have solved this problem.

The task for 1983 is simpler than that for 1982, as $3! \neq 3$ and so, in effect, an extra digit is available.

The table below gives the solution.

$1 = -1 - 9 + 9 + 3$	$11 = 1^9 \times (8 + 3)$
$2 = (1 + 9) \div (8 - 3)$	$12 = (1 + 9 - 8) \times 3!$
$3 = -1 + 9 - 8 + 3$	$13 = -1 + 9 + 8 - 3$
$4 = 1 + (9 - 8) \times 3$	$14 = -1 - 9 + 8 \times 3$
$5 = 1 + 9 - 8 + 3$	$15 = 1 + 9 + 8 - 3$
$6 = (1 + 9 - 8) \times 3$	$16 = 1 + (9! \div 8!) + 3!$
$7 = 1 \times (9 - 8) + 3!$	$17 = 19 - 8 + 3!$
$8 = 19 - 8 - 3$	$18 = 1 \times 9 \times (8 - 3!)$
$9 = (19 + 8) \div 3$	$19 = -1 + 9 + 8 + 3$
$10 = \sqrt{1 + 9 \times (8 + 3)}$	$20 = 1 \times 9 + 8 + 3$

$21 = 1 + 9 + 8 + 3$
$22 = -1 + 9 + 8 + 3!$
$23 = -1 + 9 \times 8 \div 3$
$24 = (1 \times 9 \times 8) \div 3$
$25 = 1 + 9 \times 8 \div 3$
$26 = -1 + (9! \div 8!) \times 3$
$27 = (1^9 + 8) \times 3$
$28 = 1 + (9! \div 8!) \times 3$
$29 = -1 + (\sqrt{9})! + 8 \times 3$
$30 = 19 + 8 + 3$
$31 = 1 + (\sqrt{9})! + 8 \times 3$
$32 = -1 + 9 + 8 \times 3$
$33 = 1 \times 9 + 8 \times 3$
$34 = 1 + 9 + 8 \times 3$
$35 = (1 + \sqrt{9})!)(8 - 3)$
$36 = (1 + \sqrt{9} + 8) \times 3$
$37 = -19 + 8! \div (3!)!$
$38 = -1 - 9 + 8 \times 3!$
$39 = -1 \times 9 + 8 \times 3!$
$40 = (-1 + 9) \times (8 - 3)$
$41 = -1 + ((\sqrt{9})! + 8) \times 3$
$42 = (1 \times (\sqrt{9})! \times 8) - 3!$
$43 = 19 + 8 \times 3$
$44 = -1 + 9 \times (8 - 3)$
$45 = 1 \times 9 \times (8 - 3)$
$46 = 1 + 9 \times (8 - 3)$
$47 = -1 \times 9 + 8! \div (3!)!$
$48 = (-1 + 9! \div 8!) \times 3!$
$49 = -1 - (\sqrt{9})! + 8! \div (3!)!$
$50 = (1 + 9) \times (8 - 3)$
$51 = (1 \times 9 + 8) \times 3$
$52 = 1 + (9 + 8) \times 3$
$53 = (1 + (\sqrt{9})!) \times 8 - 3$
$54 = (1 + 9 + 8) \times 3$
$55 = 1 + (\sqrt{9})! + 8 \times 3!$
$56 = -1 + 9 + 8 \times 3!$
$57 = 1 \times 9 + 8 \times 3!$
$58 = 1 + 9 + 8 \times 3!$
$59 = (1 + (\sqrt{9})!) \times 8 + 3$
$60 = (1 + 9! \div 8!) \times 3!$
$61 = (-1 + 9) \times 8 - 3$
$62 = 1 \div (\sqrt{9})! \times 8! + 3!$
$63 = 1 + (\sqrt{9})! + 8! \div (3!)!$
$64 = -19 + 83$
$65 = -1 + (\sqrt{9})! \times (8 + 3)$
$66 = 198 \div 3$
$67 = (-1 + 9) \times 8 + 3$
$68 = -1 + 9 \times 8 - 3$
$69 = 1 \times 9 \times 8 - 3$
$70 = 1 + 9 \times 8 - 3$
$71 = -1 + \sqrt{9} \times 8 \times 3$
$72 = 1 \times \sqrt{9} \times 8 \times 3$
$73 = -1 - 9 + 83$
$74 = -1 + 9 \times 8 + 3$
$75 = 1 \times 9 \times 8 + 3$
$76 = 1 + 9 \times 8 + 3$
$77 = (1 + (\sqrt{9})!) \times (8 + 3)$
$78 = 1 - (\sqrt{9})! + 83$
$79 = 1 + (\sqrt{\sqrt{9}})^8 - 3$
$80 = -1 + 9^{(8-3!)}$
$81 = (19 + 8) \times 3$
$82 = 1 + 9^{(8-3!)}$
$83 = (1 + 9) \times 8 + 3$
$84 = 1 \times (\sqrt{9})! \times (8 + 3!)$
$85 = 1 + (\sqrt{9})! \times (8 + 3!)$
$86 = -1 + (\sqrt{\sqrt{9}})^8 + 3!$
$87 = 1 \times (\sqrt{\sqrt{9}})^8 + 3!$
$88 = (-1 + 9) \times (8 + 3)$
$89 = 1 \times (\sqrt{9})! + 83$
$90 = 1 + (\sqrt{9})! + 83$
$91 = -1 + 98 - 3!$
$92 = 1 \times (9 + 83)$
$93 = 1 + 9 + 83$
$94 = -1 + 98 - 3$
$95 = 1 \times (98 - 3)$
$96 = 1 + 98 - 3$
$97 = (\sqrt{\sqrt{1 + 9}})^8 - 3$
$98 = -1 + 9 \times (8 + 3)$
$99 = 1 \times 9 \times (8 + 3)$
$100 = 1 + 9 \times (8 + 3)$

Trevor Halsall, Student,
Pimlico H.S., Townsville, Queensland.

A similar letter reached us from Ken Ross, Science I, University of Melbourne. In some cases, Ken's answers are the same, in others different. Here are some of the more elegant.

$11 = -1 + (9! \div 8!) + 3$
$28 = (\sqrt{1 + \sqrt{9}}) \times (8 + 3!)$
$37 = (-1 + (\sqrt{9})!) \times 8 - 3$
$63 = 1 + (\sqrt{9})! + (8! \div (3!)!)$
$84 = (1 + 9)! \div 8! - 3!$

He also passed on to us these elegant expressions:

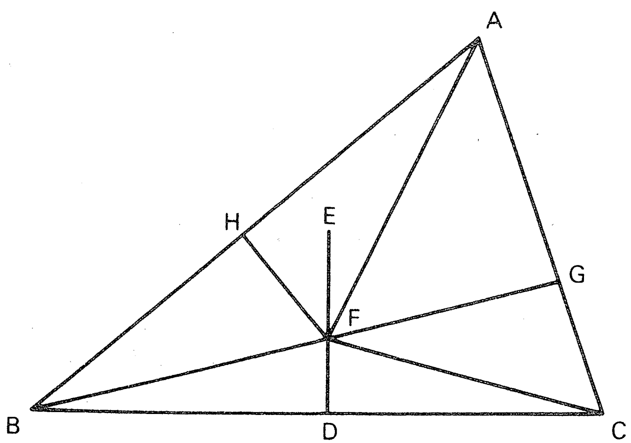
$82 = 1 + 9^{(8-3!)}$ (due to Adam Cutler)
$97 = 1 + (((\sqrt{9})!)! \div 8) + 3!$ (due to Ronnie Pila).
So the "1983 problem" seems to be well and truly disposed of.

THE INCONSISTENCY OF GEOMETRY

Each year I send to you an account of the work of the Welsh scientist Dai Fwls ap Rhyll. Dr Fwls does not always receive the credit that his original and far-reaching mind deserves. Last year you ran a story on Dr Fwls' discovery that arithmetic is in fact inconsistent, a result whose implications have still to be realised to the full.

Dr Fwls has now turned his attention to geometry with the same result. I give his proof that all triangles are equilateral.

Take any triangle ABC as shown in the diagram. Let D be the



mid-point of BC and let $ED \perp BC$. Bisect angle A and let the bisector meet ED in F. From F drop perpendiculars to AB and AC meeting these in H, G respectively.

Consider now triangles FDB, FDC.

$BD = CD$ as D is the mid-point of BC,
$ED = ED$ clearly,
$\measuredangle BDE = \measuredangle CDE$ since both are right angles.

Thus the two triangles are congruent and, in particular, $BF = CF$.

Consider now triangles AHF, AGF.

$AF = AF$ clearly,
$\measuredangle FAH = \measuredangle FAG$ as AF bisects $\measuredangle A$,

$$\measuredangle FHA = \measuredangle FGA \quad \text{both being right angles.}$$

Thus these two triangles are congruent, and we have

$$AH = AG \qquad (1)$$

and

$$FH = FG \; .$$

We now turn our attention to the triangles BFH, CFG.

$$\begin{aligned} BF &= CF && \text{proved above,} \\ FH &= FG && \text{proved above,} \\ \measuredangle FHB &= \measuredangle FGC && \text{and both are right angles.} \end{aligned}$$

Thus these triangles are congruent and, in particular,

$$BH = CG \quad . \qquad (2)$$

Now add Equations (1) and (2) to find $AB = AC$. Similarly, we may prove that $AC = BC$ and so the triangle is equilateral.

This proof by Dr Fwls demonstrates a manifest absurdity and thus establishes his contention that geometry is inconsistent.

Kim Dean,
Shoman University.

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THE COMPUTER REVOLUTION IN THE CLASSROOM

SIAM News, the newspaper of the Society for Industrial and Applied Mathematics (U.S.A.), for January 1983 headlines the fact that three colleges in America now require students to purchase a home computer, as part of their equipment.

Stevens Institute of Technology (Hoboken, New Jersey), Clarkson College (Potsdam, New York) and Drexel University (Philadelphia) all took this step.

A slightly different approach has been adopted by Carnegie-Mellon University (Pittsburg). It recently signed a three-year contract with IBM Corporation to lay the technological foundation, in equipment and programming, for computer workstations and communications services to be available to students and faculty, whether at home, in an office, or in a laboratory.

There are clear advantages in such developments. Nonetheless, cost is a big factor. Clarkson College seems to have done most to ease the financial burden, allowing the students to pay off their computers (issued by the college) over four years. But even though costs in this area are lower in the US than here and are continuing to fall, the burden is not inconsiderable.

PROBLEM SECTION

YET MORE ON PROBLEM 6.1.1

Trevor Halsall (Pimlico H.S., Townsville) has found the following

$$94 = (-1 + ((\sqrt{9})!) \times 8) \times 2.$$

52 seems to elude fair representation. For Trevor's complete solution of the "1983" problem, see p.25.

SOLUTION TO PROBLEM 7.1.1

Augustus De Morgan, a 19th Century British mathematician claimed to have been x years old in the year x^2. Trevor Halsall writes:

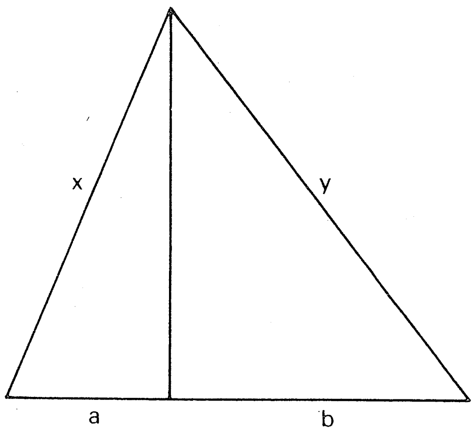
"The only perfect square in the 1800's is 1849 (= 43^2)."

Therefore De Morgan was born in 1849 - 43, i.e. 1806. (The only other, barely feasible, alternative would be 1722, but this would put De Morgan's productive period in the 1700's.) As a matter of historical record, the statement is accurate. De Morgan was born in 1806. The statement could also be made by someone born in 1805, if he referred to a period of the year 1849 prior to his ~~43rd~~ 44th birthday.

Also solved by J. Ennis, Year 11, M.C.E.G.S., and Mark Freeman, Year 12, Nepean H.S., Emu Plains.

SOLUTION TO PROBLEM 7.1.2

Trevor Halsall also solved this. It asked for the length of the altitude drawn to the 14-unit side in a (13,14, 15) triangle. We print Trevor's systematic solution below. Refer to the figure at right.



"By Pythagoras,

$$h^2 = y^2 - b^2 \quad (1)$$
$$h^2 = x^2 - a^2 \quad (2).$$

Hence, equating (1) and (2)

$$y^2 - b^2 = x^2 - a^2.$$

It now follows that

$$x^2 - y^2 = a^2 - b^2 = (a + b)(a - b).$$

Let
$$a + b = z \quad . \tag{3}$$

Then
$$a - b = \frac{x^2 - y^2}{z} \quad . \tag{4}$$

Add Equations (3) and (4) to find

$$2a = z + \frac{x^2 - y^2}{z} \quad ,$$

from which it follows that

$$a^2 = \left(\frac{x^2 - y^2 + z^2}{2z}\right)^2 \quad . \tag{5}$$

From Equations (2), (5)

$$h = \sqrt{h - \left(\frac{x^2 - y^2 + z^2}{2z}\right)^2} \quad .$$

This gives the length of the altitude in the general case. Let $x = p - 1$, $y = p + 1$, $z = p$. Then

$$h = \tfrac{1}{2}\sqrt{\{3(p^2 - 4)\}} \ .$$

When $p = 14$, $h = 12$."

There is actually a simpler method, not available in general, but applicable to this special case. A check of the diagram reveals two well-known Pythagorean Triples (5,12,13) and (9,12,15). (See *Function*, Vol.6, Part 3.) J. Ennis, Year 11, M.C.E.G.S. solved the problem in this way, and Mark Freeman, Year 12, Nepean H.S., by an ingenious variant of the same approach.

SOLUTION TO PROBLEM 7.1.3

A barrel contains n white marbles and m black. Marbles are drawn out one at a time. Prove that the probability that the black marbles are all withdrawn before the white is equal to the probability of obtaining a white marble on the first draw.

Mark Freeman, Year 12, Nepean H.S., Emu Plains, writes:

"Withdrawing all the black marbles before the white means the same as drawing a white marble last. All the marbles could be drawn out and placed in small containers. Clearly, whatever number of black and white marbles there are, the chance of having a white marble in the first container is exactly the same as the chance of finding one in the last container, so the statement is proved."

The same argument, expressed in rather different terms, was also supplied by J. Ennis.

SOLUTION TO PROBLEM 7.1.4

At a certain Australian casino, there is a game in which three dice are tossed. Before the tossing, you can bet on *one* of the numbers 1, 2, 3, 4, 5, or 6. If the three dice all show your number, you receive $11, if two show your number you receive $3 and if one dice shows your number you receive $2. Of course, you lose your dollar if no dice shows your number.

(i) What is the amount of money you expect to lose per game?

(ii) What is the probability that you would have x losing games in a row before your number came up? ($x = 0,1,2,\ldots$.)

This problem was also solved by J. Ennis.

(i) There are 216 possibilities for the result of the roll of three dice. Of these, 1 gives a profit of $10, 15 give you a profit of $2, and 75 give a profit of $1. Thus the expected return is

$$\$\ \frac{1\times 10 + 15\times 2 + 75\times 1 - 125\times 1}{216}$$

(as the profit from the remaining 125 possibilities is -$1). This results in a profit of -4.63¢, so your expected loss is about 5¢ a game.

(ii) On average, you lose $\frac{125}{216}$ of the games, so the probability of x consecutive losing games, followed by a win, is

$$\left(\frac{125}{216}\right)^{x}\left(\frac{91}{216}\right) = \frac{91}{216}\left(\frac{5}{6}\right)^{3x}.$$

We close with some new problems.

PROBLEM 7.2.1 (From *Mathematical Spectrum*.)

ABCDEF is a convex hexagon, all of whose angles are equal. Prove that

$$AB - DE = EF - BC = CD - FA.$$

PROBLEM 7.2.2 (From *Mathematical Digest*.)

Show that if $(a + b + c)^3 = a^3 + b^3 + c^3$,

then $(a + b + c)^5 = a^5 + b^5 + c^5$.

PROBLEM 7.2.3 (From the same source.)

A new planet has been discovered. Its shape is that of a right circular cone whose flat circular face is land. This continent has a diameter of 5000 km. The curved surface is entirely covered by water and the oceanic area is three times that of the continent. The inhabitants of the planet are all keen sailors and are planning an "around-the-cone" yacht race. What is the shortest route, starting and ending at the same point on the coastline, that circumnavigates the cone?

PROBLEM 7.2.4 (Submitted by J. Ennis.)

Prove, without using a calculator or tables, that

$$e + \ln 4 > 4.$$

PROBLEM 7.2.5

"I hear some youngsters playing in the garden", says Jones, a graduate student in mathematics. "Are they all yours?" "Heavens, no", exclaimed Professor Smith, the eminent number theorist. "My children are playing with friends from three other families in the neighbourhood, although our family happens to be the largest. The Browns have a smaller number of children, the Greens have a still smaller number, and the Blacks the smallest of all."

"How many children are there altogether?" asked Jones.

"Let me put it this way", said Smith. "There are fewer than 18 children, and the product of the numbers in the four families happens to be my house number which you saw when you arrived."

Jones took a notebook and pencil from his pocket and started scribbling. A moment later he looked up and said, "I need more information. Is there more than one child in the Black family?"

As soon as Smith replied, Jones smiled and correctly stated the number of children in each family.

Knowing the house number and whether the Blacks had more than one child, Jones found the problem trivial. It is a remarkable fact, however, that the number of children in each family can be determined solely on the basis of the information given above! How many children are in each family?

(A related problem appeared as Problem 1.2.6.)

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POLICY AND PREDICTABILITY

Mathematics has connections to the rest of science which are both predictable and unpredictable. Predictable connections include computing, statistics and modeling. These are major bread and butter activities of large subfields of mathematics.

The unpredictable applications of mathematics are harder to characterize. They spring from the universality and vitality of mathematics. Recent examples include the use of number theory in making and breaking codes, the use of algebra and topology to [study] the Yang-Mills equations and [new approaches to] defects in crystals.

From an address by James Glimm,
of Courant Institute of Mathematical
Sciences.

29.10.82.

NATIONAL MATHS WEEK

National Maths Week for 1983 will be held from August 7-13. Victorian readers can get more information from the Mathematical Association of Victoria. Interstate readers contact their local Mathematical Association.

At present, the organisers are still interested in ideas that could be implemented during the week in which the activities come to a focus.