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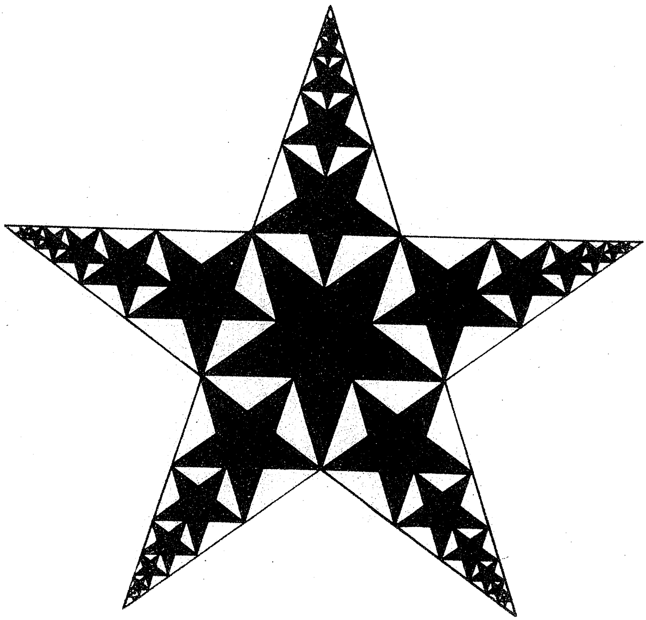

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

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This issue contains some interesting articles by some interesting people. We mention here some biographical details of two of them.

Captain John Noble manages his marine supply business "Great Circle Services". Previously, he was in command of New Zealand ships, and was, for 20 years, a Port Phillip Sea Pilot. He is also the author of a number of books concerning ships and the sea.

Professor John Howie is Regius Professor of Mathematics at St Andrews University, Scotland, and was a visitor to Monash in 1979. The ancient British universities (Oxford, Cambridge, St Andrews, Glasgow, Aberdeen, Edinburgh) have a number of professors appointed by the crown and designated "Regius". Mathematics was taught at St Andrews from its beginning (in 1411), but the first Regius Professor (James Gregory) was appointed to the chair endowed by Charles II in 1668. John Howie is the fifteenth such professor.

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THE FRONT COVER

J.N. Crossley, Monash University

Then þay schewed hym þe schelde, þat was of schyr goulez
Wyth þe pentangel depaynt of pure gold hwez.

The pentangle or pentagram featured on the cover has been known since at least 3000 B.C. in Babylon. The Pythagoreans of sixth century B.C. southern Italy (Magna Graccia) used it, possibly as a symbol for identification of the members of the sect. It was also used as a sign for Health and in this sense the pentagram was used frequently in the sixteenth century A.D.

Some people believe that the pentagram or pentangle was central in the discovery of irrational numbers (i.e. numbers which cannot be written as fractions). The reason for this is that the ratio of the side of a pentagon to its diagonal is irrational, that is $CD:CF$ in the figure or, equivalently, $BD:BE$. (There are lots of similar triangles here and the angles of any of the triangles in figure 1 are all multiples of 36° or $\pi/5$ so verifying this remark is not hard.) The ratio is the *golden ratio* treated by Proclus, in his commentaries on Euclid (see also *Function*, Vol.3, Part 4, pp.29,30), and ubiquitous in the world of art and architecture. Formally the ratio is defined as follows. Consider the line

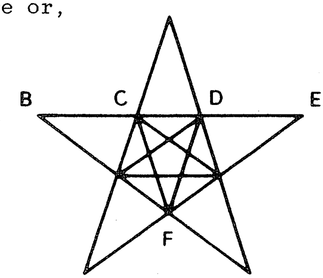

Figure 1.

A ———————— B ———— C then $AB:AC$ is the golden ratio if

$AB:AC = BC:AB$. Numerically $(-1 + \sqrt{5})/2$. We can see the ratio is irrational but prove it directly from the figure on the the cover. The fact that the figure keeps on repeating itself while getting smaller and smaller shows that the ratio of side to diagonal can never be measured by a common unit of length to give a ratio of integers (for otherwise the process of drawing the inner pentagram would stop - which it doesn't).

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†The letter þ (thorn) has the sound *th* and was used in Middle English. It survives, in bowdlerised form, as a *y* in "*Ye Olde* ...", and such phrases.

Whether the Pythagoreans did use this argument we shall probably never know. Certainly the pentagram has captured many people's imagination. The cover design was suggested by Mr P. Greetham of Boronia Technical School, who also writes on Magic Squares in this issue of *Function*. Our quotation at the beginning comes from a famous British poem: Sir Gawain and the Green Knight which dates from at least 1400 A.D. It reads "Then they showed him the shield that was of shining red with the pentangle coloured pure gold". The author goes on to point out the significance of five, but not that of the "triple intersecting triangle" as the pentagram is called by Lucian the ancient Greek.

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PURE MATHEMATICS CAN BE USEFUL

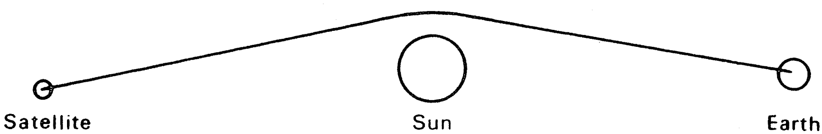
Rudolf Lidl, University of Tasmania

Abstract mathematical ideas have an enormous (unfortunately often unnoticed) impact on science and society. They helped to make possible the revolution in electronics that transformed the way we communicate; neither television, satellites, calculators, nor computers would be possible were it not for numerous results of "pure" mathematics.

Biologists, astronomers, chemists and physicists have less difficulty in communicating the excitement and essence of their fields to the general public. Mathematicians, however, face the problem of the abstract, "otherworldly" vocabulary of their subject, when they try to talk to nonmathematicians. Molecules, DNA and even black holes refer to things with a material sense, providing the chemist, biologist and physicist with an effective communication link based on physical reality. In contrast not even analogy and metaphor are capable of bringing the mathematical vocabulary within range of human experience. Unfortunately, in addition many educated people are oblivious to the existence or significance of mathematics. Their concerns only centre on the traditional feeling that "I never was any good at maths". To display some of the essential features of problems which can be solved by modern mathematics, here are a few typical questions to indicate problem areas: How large should a telephone exchange be to reduce queues? When playing poker, does bluffing pay? What is the shortest way for a commercial traveller to visit 30 given towns in Australia? How can we decode a cryptograph? How can we protect number sequences against errors in telegraphing? What is the cheapest, safest and fastest coding of messages that have to be sent over a noisy communication channel? Mathematical answers to these questions show that a subject as pure and passionless as mathematics can have a lot to say about the world in which we live.

To be a little bit more specific, modern algebra is basic in radar and communication systems, particularly for long-range radars, such as satellite tracking radars and radars which make maps of the Moon or measure ranges of Venus. The techniques used consist of transmitting a long sequence of signals of electromagnetic energy, coded in such a way that one can match the transmitted and received energy in exactly one way. So called periodic sequences (an analogy is $\frac{1}{7} = \cdot 142857142857\ldots$) with a very long period are used to generate the transmitted radar signals and algebraic coding techniques make it possible to encode signals and also to decode, match or correlate returned signals to which errors have been added through "noisy" communication channels. Such periodic sequences can be constructed by using certain ("primitive") binary polynomials, e.g. $x^{20} + x^3 + 1$ to give a sequence of period $2^{20} - 1$, which is about right for radar observation of the Moon. For polynomials of degree 30, periods of length $2^{30} - 1$ are obtained, suitable for observations of Venus; if polynomials of degree 50 are used, then satellite communication systems can be obtained which generate sequences transmitted at intervals of 10^{-6} sec, but such a length that they repeat only once a year. The algebraic concepts used thereby are properties of fields with finitely many elements, (called "Galois fields" or finite fields), polynomials with coefficients in these fields and formal power series $\sum_{k=-\infty}^{\infty} c_k x^k$. Finite fields are also used in the modelling of finite state machines; for decades they have been basic in experimental designs and many other types of combinatorial designs. A very recent application with important practical impact is to "fast Fourier transforms", where the use of elements from a finite field eliminates the problem of computer generated round-off errors.

The techniques involving periodic sequences also lead to the development of a high-precision interplanetary ranging system and to test the General Theory of Relativity far more accurately than by previous methods.



According to Einstein, we should observe two non-Newtonian phenomena as a radio signal travels from Earth to Probe, and back. First, there is the "bending" of the ray by the sun's gravitational field. Second, the photons gain energy from the sun's field, and since they cannot speed up faster than the speed of light the extra energy shortens the wave length (increases the frequency) by Planck's formula $E = h\nu$. Fortunately, both of these effects are in the same direction. Each has the effect of increasing the number of cycles above what we would expect from the Newtonian model.

Since the ranging device counts cycles, we can observe an effect of the order of an extra hundred or so cycles. This experiment has been performed in 1969, with Mariner's 6 and 7 space probes. They also sent back over 200 photographs of Mars, each of them divided into 658 240 dots and each dot given a brightness level ranging from 1 to 2^8. Therefore, each photograph required about five million bits of information. These bits were encoded, (using an error-correcting code, based on properties of modern algebra) and transmitted back to earth at a rate of 16 200 bits per second, where they were received and decoded into photographs.

History of mathematics clearly shows that mathematical theories developed as part of pure mathematics appeared to be of importance or sometimes even indispensable as a tool in other branches of science. One of the best-known examples of a surprising application is the theory of conics, which was not applied until 1800 years after their discovery by the Greek geometer Appolonius of Perga, when Johannes Keppler described the planetary motions by ellipses. Most important advances in pure mathematics rarely wait 1800 years for application. With the increasing momentum of mathematical and scientific activity, the speed of application has also increased. It took only 60 years from the development of matrix theory as a part of pure mathematics in 1860 by A. Cayley to its application in physics by Werner Heisenberg. Subsequently, it took 30 odd years from the development of tensor calculus in the 1870's to its application as the basic mathematical tool of relativity theory by A. Einstein in the 1910's.

Mathematics has an incredibly long and sophisticated history; it can measure its history in millenia. In comparison, the physical sciences build on concepts and theories which are at most a few hundred years old, much of the social and even life sciences is of even more recent vintage. Although the contributions of pure mathematicians may often seem to be small, there is always a certain character of permanence about them; and to produce and study anything of permanent interest is to do something utterly beyond the powers of the vast majority of scientific disciplines.

Here are some suggestions for further reading on the topic of the uses of mathematics and its relevance today:

G. BIRKHOFF and T.C. BARTEE: *Modern Applied Algebra.* McGraw-Hill. 1970.

F.E. BROWDER: *The Relevance of Mathematics.* Amer. Math. Monthly 83 (1976), 249-254.

Mathematics in the Modern World. Freeman, San Francisco 1968.

L.A. STEEN (ed.): *Mathematics Today, Twelve Informal Essays.* Springer Verlag, New York-Heidelberg-Berlin, 1978.

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PI THROUGH THE AGES

J.M. Howie, St Andrews University

A little-known verse from the Bible reads

> And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.
>
> (I Kings 7, 23)

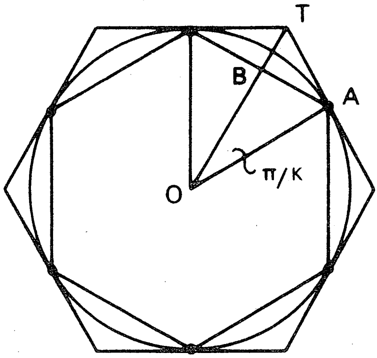
The same verse can be found in II Chron. 4, 2. It occurs in a list of specifications for the great temple of Solomon, built around 950 B.C., and its interest for us here is that it gives $\pi = 3$. Not a very accurate value, of course, and not even very accurate in its day, for Egyptian and Mesopotamian values of $3\frac{1}{8} = 3{\cdot}125$ and $\sqrt{10} = 3{\cdot}162$ have been traced to much earlier dates; though in defence of Solomon's craftsmen it should be noted that the item being described seems to have been a very large brass casting, where a high degree of geometrical precision is neither possible nor necessary.

The fact that the ratio of the circumference to the diameter of a circle is constant has been known for so long that it is quite untraceable. The earliest values of π, including the 'biblical' value of 3, were almost certainly found by measurement. The first theoretical calculation seems to have been carried out by Archimedes of Syracuse (287-212 B.C.), perhaps best known for his 'displacement principle' in hydrostatics, but arguably one of the greatest mathematicians of all time. Let me attempt, using all the resources of modern mathematical notation, to describe the essence of the argument by which he showed that

$$3\tfrac{10}{71} < \pi < 3\tfrac{1}{7}.$$

Before I do that, however, let me draw attention to the very considerable sophistication involved in the use of inequalities here. Archimedes knew, what so many people to this day do not, that π does not *equal* 22/7, and made no claim to have discovered an exact value. If we take his best estimate as the average of his two bounds we arrive at $3{\cdot}1418$, an error of about $\cdot 0002$.

Now for the Archimedes argument. Consider a circle of radius 1, in which we *inscribe* a regular polygon of $3 \times 2^{n-1}$ sides, with semiperimeter q_n ; and *escribe* a regular polygon of $3 \times 2^{n-1}$ sides, with semiperimeter p_n. In the diagram this has been done for $n = 2$.



$OA = 1,$

$AB = \sin \frac{\pi}{K},$

$AT = \tan \frac{\pi}{K},$

where $K = 3 \times 2^{n-1}$.

The effect of this procedure is to define an increasing sequence

$$q_1, q_2, q_3, \ldots$$

and a decreasing sequence

$$p_1, p_2, p_3, \ldots$$

such that

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = \pi .$$

From the diagram, and using trigonometrical notation, we see that the two semiperimeters are given by

$$p_n = K \tan \frac{\pi}{K}, \quad q_n = K \sin \frac{\pi}{K},$$

where $K = 3 \times 2^{n-1}$. Equally, we have

$$p_{n+1} = 2K \tan \frac{\pi}{2K}, \quad q_{n+1} = 2K \sin \frac{\pi}{2K},$$

and it is not at all a difficult exercise in trigonometry to show that

$$\tfrac{1}{2}\left(\frac{1}{p_n} + \frac{1}{q_n}\right) = \frac{1}{p_{n+1}}, \qquad (1)$$

$$p_{n+1}\, q_n = q_{n+1}^2 . \qquad (2)$$

Archimedes, starting from $p_1 = 3 \tan \frac{\pi}{3} = 3\sqrt{3}$ and $q_1 = 3 \sin \frac{\pi}{3} = \frac{3\sqrt{3}}{2}$, calculated p_2 using (1), then q_2 using (2), then p_3 using (1), then q_3 using (2); and so on until he had calculated p_6 and q_6. His conclusion† was that

$$q_6 < \pi < p_6.$$

Let me emphasise that my use of trigonometrical techniques here is totally unhistorical: Archimedes did not have the huge advantage of an algebraic and trigonometrical notation and had to derive (1) and (2) by entirely geometrical means. Moreover, he did not even have the advantage of our decimal position notation for numbers, so that the calculation of p_6 and q_6 from (1) and (2) was by no means a trivial task. So it was a pretty stupendous feat both of imagination and of calculation and the wonder is not that he stopped with polygons of 96 sides, but that he went so far.

For of course there is no reason in principle why one should not go on. Various people did, including

Ptolemy (c. 150 A.D.)	3·1416
Tsu Ch'ung Chi (430 - 501 A.D.)	355/113
Al-Khowarizmi (c. 800)	3·1416
Al-Kashi (c. 1430)	14 places
Viète (1540 - 1603)	9 places
Van Coolen (c. 1600)	35 places

Except for Tsu Ch'ung Chi, about whom next to nothing is known and who is very unlikely to have known of Archimedes' work, there was no theoretical progress involved in these improvements, only greater stamina in calculation. Notice how the lead, in this as in all scientific matters, passed from Europe to the East for the millenium from 400 to 1400 A.D. Al-Khowarizmi lived in Baghdad, and incidentally gave his name to 'algorithm'; while the words 'al jabr' in the title of one of his books gave us the word 'algebra'. Al-Kashi lived still further east, in Samarkand, while Tsu Ch'ung Chi, one need hardly add, lived in China.

The European Renaissance brought about in due course a whole new mathematical world. Among the first effects of this reawakening was the emergence of arithmetical formulae for π. One of the earliest was that of Wallis (1616 - 1703):

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† My colleague Dr G.M. Phillips has pointed out that the sequence of weighted means $\frac{1}{3}(p_n + 2q_n)$ converges to π much more rapidly than either of the sequences (p_n) or (q_n). (This can be shown quite easily by considering the Maclaurin expansions of $K\tan(\pi/K)$ and $K\sin(\pi/K)$.) Neither he nor I can think of any way in which Archimedes could have hit on this improvement by a geometrical argument; but probably we underestimate him.

$$\frac{2}{\pi} = \frac{3.3.5.5.7.7...}{2.2.4.4.6.6...}$$

and one of the best-known is

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots .$$

This is sometimes attributed to Leibniz (1646 - 1716) but in St Andrews (with some justification - see Boyer [1], p.422) we always attribute it to James Gregory (1638 - 1675), who was appointed our first Regius Professor of Mathematics in 1668.

These are both dramatic and astonishing formulae, for the expressions on the right are completely arithmetical in character, while π arises in the first instance from geometry. They show the surprising results that infinite processes can achieve and point the way to the wonderful richness of modern mathematics.

From the point of view of the calculation of π, however, neither is of any use at all. In Gregory's series, for example, to get 4 decimal places correct we require the error to be less than 0·00005 = 1/20 000, and so we need about 10 000 terms of the series. However, Gregory also showed the more general result

$$\tan^{-1}x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \ldots \quad (-1 \leqslant x \leqslant 1) \quad {}^{\dagger} \qquad (3)$$

(from which the first series results if we put $x = 1$). So using the fact that $\tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$ we get

$$\frac{\pi}{6} = \frac{1}{\sqrt{3}}\left(1 - \frac{1}{3.3} + \frac{1}{5.3^2} - \frac{1}{7.3^3} + \ldots\right),$$

which converges much more quickly. The 10th term is $1/19.3^9\sqrt{3}$, which is less than ·00005, and so we have at least 4 places correct after just 9 terms.

An even better idea is to take the formula (beloved by sixth form examiners the world over)

$$\frac{\pi}{4} = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\frac{1}{3} \qquad (4)$$

and then calculate the two series obtained by feeding first $\frac{1}{2}$ and then $\frac{1}{3}$ into (3):

$$\frac{\pi}{4} = [\tfrac{1}{2} - \tfrac{1}{3}(\tfrac{1}{2})^3 + \tfrac{1}{5}(\tfrac{1}{2})^5 - \ldots] + [\tfrac{1}{3} - \tfrac{1}{3}(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 - \ldots].$$

Clearly we shall get very rapid convergence indeed if we can find a formula saying something like

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{a} + \tan^{-1}\frac{1}{b},$$

with a and b large. In 1706 Machin found such a formula:

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}. \qquad (5)$$

---

† Remember the notations $\tan^{-1}x \equiv \text{artan}\, x$.

(Actually this is not at all hard to prove; if you know how to do (4) then there is no real extra difficulty about (5), except that the arithmetic is worse. Thinking it up in the first place is, of course, quite another matter.)

With a formula like this available the only difficulty in computing π is the sheer boredom of continuing the calculation. Needless to say, a few people were silly enough to devote vast amounts of time and effort to this tedious and wholly useless pursuit. One of them, an Englishman named Shanks, used Machin's formula to calculate π to 707 places, publishing the results of many years of labour in 1873. Shanks has achieved immortality for a very curious reason, which I shall explain shortly.

I have said that the calculation of π to ever more places is wholly useless. If Shanks had found that the expansion of π *terminates*, or that it becomes a *recurring decimal*, that would have been of great importance, but in fact he knew very well that such a thing could not happen, since as early as 1761 it had been proved by Lambert that π is *irrational*. One way of expressing this is to say that there cannot exist integers a_0, a_1 for which π is a solution of the equation $a_0 + a_1x = 0$. Shortly after Shanks's calculation it was shown by Lindemann that π has the much worse property of being *transcendental*: that is to say, π cannot be a root of a polynomial equation

$$a_0 + a_1x + \ldots + a_nx^n = 0$$

for any n *or for any choice of integers* $a_0, a_1, \ldots, a_n$. This *was* an important result, since for reasons I cannot go into here it laid to rest the classical Greek problem of 'squaring the circle'. It is a consequence of the transcendentality of π that no ruler and compass construction can exist for constructing a square equal in area to a given circle.

Very soon after Shanks's calculation a curious statistical freak was noticed by De Morgan, who found that in the list of 707 digits there was a suspicious shortage of 7's. He mentioned this as a curiosity in his 'Budget of Paradoxes' of 1872 (reprinted in [4]) and a curiosity it remained until 1945, when Ferguson discovered that Shanks had made an error in the 528th place, after which all his digits were wrong. In 1949 a computer was used to calculate π to 2000 places. In this expansion and in all subsequent computer expansions (the most recent I have heard of being to 500 000 places!) the number of 7's does not differ significantly from its expectation; and indeed the sequence of digits has so far passed all statistical tests for randomness.

The mention of statistics tempts me to include one last curiosity about the calculation of π, namely Buffon's needle experiment. If we have a uniform grid of parallel lines, unit distance apart and if we drop a needle of length $\ell < 1$ upon this grid, the probability that the needle falls across a line is (not obviously) $2\ell/\pi$. By now you will not be surprised to learn that various people have attempted to estimate π by throwing needles. The most remarkable result was that of Lazzerini (1901), who made 3408 tosses and got

$$\pi = \frac{355}{113} = 3{\cdot}1415929$$

(incidentally the value found by Tsu Ch'ung Chi). This outcome is suspiciously good, and the game is given away by the strange number 3408 tosses. Kendall and Moran [3] comment that a good value can be obtained by stopping the experiment at an optimal moment. If you do not know in advance how many throws there are to be then this is a *very* inaccurate way of computing π. Kendall and Moran indeed comment somewhat acidly that you would do better to cut out a large circle of wood and use a tape measure to find its circumference and diameter!

Still on the theme of phoney experiments, Gridgeman [2], in a paper which pours devastating scorn on Lazzerini and others, created some amusement by using a needle of carefully chosen length $\ell = 0{\cdot}7857$, throwing it *twice*, and hitting a line once. His estimate for π was thus given by

$$\frac{2 \times 0{\cdot}7857}{\pi} = \tfrac{1}{2},$$

from which he got the highly creditable value of $\pi = 3{\cdot}1418$. He was *not* being serious!

As a postscript, here is a mnemonic for the decimal expansion of π. Each successive digit is the number of letters in the corresponding word.

"How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics. All of thy geometry, Herr Planck, is fairly hard ..."

3·14159265358979323846264...

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1. Carl B. Boyer, *A history of mathematics*, Wiley, 1968.

2. N.T. Gridgeman, Geometrical probability and the number π, *Scripta Mathematica* 25 (1960) 183-195.

3. M.G. Kendall and P.A.P. Moran, *Geometrical probability*, Griffin, 1963.

4. James R. Newman, *The world of mathematics*, vol.IV, Allen and Unwin, 1956.

Editorial Note: We believe there is a book with π listed to 10^6 places. See also Function *Vol.3, Part 5, p.32 for a story about someone who memorized 15151 decimal places of π.*

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GAME THEORY AND NURSE ROSTERING

Peter G. Schulz,
Footscray Institute of Technology

Introduction

Throughout history man has been preoccupied with conflicts of interest such as games of chance, business competition, collective bargaining and international conflict. Games are mathematical models of conflict situations where the *payoffs* (outcomes) are determined by the *strategies* of the players. Because a player can choose his strategy, he has some control over the outcome of the game. But he is not in complete control, since the outcome also depends on the strategy of his opponents.

The modern mathematical approach to game theory is generally attributed to von Neumann who wrote several papers on the subject in the late 1920's and 1930's. The Second World War saw considerable work being done in the areas of logistics, submarine search, air defence etc. As a result, von Neumann and Morgenstern's book "Theory of Games and Economic Behaviour" became the first standard reference on Game Theory.

Games are classified by the number of players (m) and the number of moves or strategies (n). If one player's gain always equals the other player's loss (e.g. in poker) the game is called "zero-sum". The matrix of payoffs from one person to another for each strategy employed results in an $m \times n$ *playoff matrix.* If a player can select his strategy after he knows how his opponent has committed himself, his appropriate strategy is obvious and the game is trivial. The essence of game theory, however, is that each player must commit himself without knowing his opponent's decision; he only knows the payoff matrix and his opponent's past pattern of play.

In this article the following two games will be considered:

(i) strictly determined two person games, and

(ii) an application of game theory to the rostering of nurses at a hospital.

Strictly determined two-person games

Consider the following payoff matrix for a two-person game:

A \ B	1	2	3
1	40	-20	10
2	20	-1	5

B has three strategies and *A* two. *The matrix represents payments from B to A* e.g. if *A* selects strategy 2 and *B* selects strategy 1, *B* pays 20 units to *A*. If *A* selects strategy 1 and *B* selects strategy 2, *A* pays 20 units to *B*.

We can state the problem in the following way: we have to select *A*'s strategies (or combination of strategies) so that *A's expected gain* over a number of plays is a *maximum* <u>and</u> to select *B*'s strategies (or combination of strategies) so that *B's expected loss* over a number of plays is a *minimum.*

Consider *A*'s position: the continuous play of strategy 1 is unattractive. It does contain the largest payoff (40) but *B* will (if he finds *A* playing strategy 1 all the time) switch to strategy 2 thus forcing *A* to lose. *A* thus selects strategy 2. Mathematically, *A* calculates the minimum value in each row of the matrix (-20,-1) and selects the maximum (i.e. the "maximin") which occurs for strategy 2.

B wishes to keep the payoff small - he is ill-advised to play either strategy 1 or strategy 3. Mathematically *B* calculates the maximum value for each column (40,-1,10) and chooses the minimum (i.e. the "minimax") which occurs in column 2. Both *A* and *B* should play strategy 2.

Note that here minimax = maximin = -1; such a game is said to be strictly determined. For the above game, player *B* can guarantee an expected win of 1 unit per game regardless of what his opponent may do. However, it should be noted that this is only the value towards which the average tends; if the game is played only a few times, luck may increase or decrease the payoff. We also note that game theory leads to very conservative strategies.

As a trivial example, let us consider the following game:

Rules: You choose one side of a coin i.e. either *H* or *T*.
I select one of the 4 aces from a deck of cards.

Payoffs: (i) If you select *H*: I pay you 15¢, 4¢, -5¢, and 1¢.
(ii) If you select *T*: I pay you 1¢, -8¢, -6¢, and -2¢ if I have chosen the spade, heart, diamond or club ace respectively for both (i) and (ii).

Do you agree to play the game?

This is an example of a zero-sum game as my payouts equal my gains for the strategies.

The payoff matrix (representing payments from me) to you is

You \ Me	Spade	Heart	Diamond	Club	Min	
H	15	4	-5	1	-5	Maximin
T	1	-8	-6	-2	-8	
Max	15	4	-5	1		
			Minimax			

Since minimax = maximin, the game is strictly determined. Here, I will play the ace of diamonds all the time and I can be guaranteed a win of at least 5¢ per game. You should not play this game!

Nurse rostering

Hospital administrators are becoming increasingly concerned. Staff and students from the Mathematics Department here at Footscray have conducted projects for the Western General Hospital relating to (i) a better internal courier system, (ii) a bed demand for emergencies, (iii) a survey of people attending the outpatients department and (iv) a study of waiting times in both the outpatients and the accident and emergency departments.

The Western General Hospital conducts a large accident and emergency department where people either receive emergency treatment or are referred to the department's clinic (where "a wait of 24 hours or more would not affect the patient".). Nursing and medical care demand can be forecast using the previous year's data. However, the demand on public holidays can fluctuate greatly due to unpredictable variables such as weather. Duckett [4] has suggested the use of game theory to assist in the determination of the level of nursing manpower to be provided on such days. In most hospitals, rostering is done by guesswork leading to either overstaffing (i.e. under-utilization of staff) or understaffing resulting in poor quality attention from the nursing staff.

Since the demand cannot be forecast, let us assume that it can be classified as low, medium or high. Also, let us assume that between 2 and 5 nurses can be rostered on any shift. These represent the strategies for our game leading to a 4 × 3 payoff matrix. The payoffs in this game are a measure of the value of a particular combination of strategies to the hospital. These measures can be quantified by discussions between the director of Accident and Emergency and the nursing administration. Duckett suggests a possible payoff matrix by allocating a maximum of 10 points for each payoff; for example, high demand and 5 nurses would have a payoff of 10 whilst high demand and 2 nurses would earn 1 point.

A possible payoff matrix is:

Nurses per shift	DEMAND		
	Low	Medium	High
(R_1) 5	4	8	10
(R_2) 4	7	10	8
(R_3) 3	10	5	2
(R_4) 2	5	3	1

Unlike the games mentioned in the previous section, we cannot select a demand strategy - we cannot say that the best strategy is low demand! We can select the number of nurses per shift and one method used is to select the *maximin* i.e. calculate the minimum payoff per row (4,7,2,1) and then select the maximum (7,row 2). This method of solution leads to 4 nurses being rostered per shift. Thus, each roster is appraised by looking at the worst payoff for each roster and the "optimal choice" is the one with the best worst payoff.

The maximin and minimax criteria are each ultraconservative in that they concentrate upon the state having the worst consequence. Why not look at the best state or a combination of the best and worst? This is the essence of what is called the Hurwicz criterion [2]. For an act R_i, let m_i be the minimum and M_i be the maximum of the payoffs. Let a fixed number α ($0 < \alpha < 1$), called the pessimism-optimism index, be given such that to each R_i we associate an index $H = \alpha m_i + (1 - \alpha)M_i$, called the Hurwicz α-criterion. Of two acts, the one with the higher α-index is preferred.

For the above,

$$H(5) = 4\alpha + 10(1 - \alpha) = 10 - 6\alpha$$

$$H(4) = 7\alpha + 10(1 - \alpha) = 10 - 3\alpha$$

$$H(3) = 2\alpha + 10(1 - \alpha) = 10 - 8\alpha$$

$$H(2) = \alpha + 5(1 - \alpha) = 5 - 4\alpha.$$

Since $0 < \alpha < 1$, $H(2) < H(3) < H(5) < H(4)$, and hence 4 nurses per shift results in the optimum solution. The maximin made the same decision.

Another criterion is based on the "principle of insufficient reason". Since we are not able to forecast the level of demand, let us assume that all are equally likely. For each act (R_i) let us calculate the expected or average payoff, and select the act with the largest average.

For the problem above, we have:

Act	1	2	3	4
Average	7·33	8·33	5·67	3·00

Again the second act, 4 nurses per shift, is the one to be selected.

References

1. Robert Singleton and William Tyndall, *Games and Programs*, W.H. Freeman and Co., 1974.

2. Duncan Luce and Howard Raiffa, *Games and Decisions*, J. Wiley and Sons, 1967.

3. J.D. Williams, *The Compleat Strategyst*, McGraw-Hill, 1966.

4. Stephen Duckett, Nurse Rostering with Game Theory, *Journal of Nursing Administration*, Jan. 1977.

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GREAT CIRCLE NAVIGATION

Capt. John Noble,

8 Agnes Ave. North Balwyn

Great circles are circles on a sphere, whose centres are the centre of the sphere. In navigation, courses and distances along arcs of great circles are determined by spherical trigonometry. The alternative to great circle navigation is to follow a *rhumb line* - a path which cuts all meridians of longitude at the same angle. Rhumb line course and distance are determined by plane trigonometry. Meridians of longitude, and the equator, are both great circles and rhumb lines. On east and west courses in high latitudes, a considerable distance can be saved by following a great circle track in preference to a rhumb line. On Mercator's projection, rhumb lines are straight, but great circles curve toward the poles, the curve increasing with latitude.

Sailing ships on the voyage from England to Australia kept to rhumb lines in the Atlantic, making the best progress permitted by the prevailing winds and the variables of Cancer until they picked up the north east trade winds - the permanent area of high pressure with anticyclonic circulation that persists between the tropic of Cancer and the equator. Bowled along by this favourable and reliable wind, they sought the doldrums at their narrowest, favouring the coast of South America for this purpose. They then picked up the south east trade winds for a close-hauled trek almost due south into the variables of Capricorn - a more significant area of variable winds in the South Atlantic (known as the Horse Latitudes).

Progress was uncertain in this area until the westerlies were encountered between 30° and 40° south latitude†. From this position, advantage could be taken of great circle navigation, the curve to the south providing a shorter distance as well as taking advantage of the more persistent westerlies in higher latitudes. The course commenced as south-easterly, gradually changed to easterly as the apex of the great circle was reached at about 60° latitude and came slowly back to northeast as the destination came closer.

But, at 60° south latitude, there were increased hazards from extreme cold, violent storms and icebergs. Prudent ship-masters determined their maximum latitude, probably 50° to 55°, and followed a "composite great circle": great circle to maximum latitude, rhumb line along this latitude to a longitude where the original great circle intersected the latitude again, then great circle again to the destination.

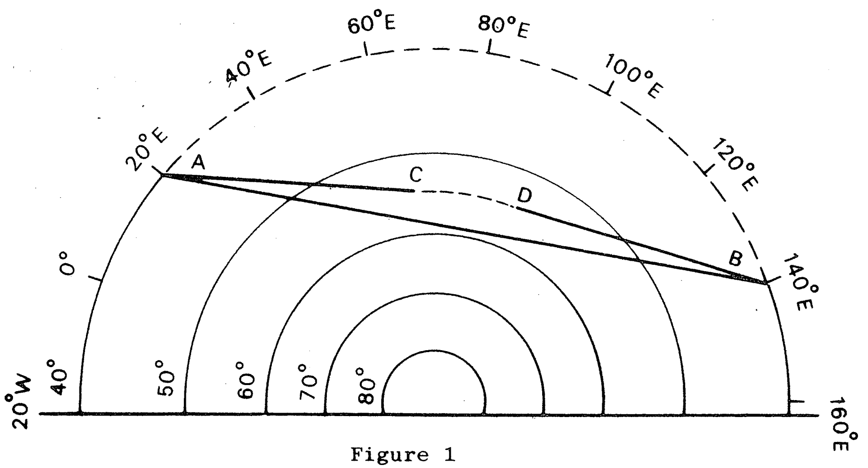

Figure 1

In this *polar gnemonic* projection, great circles appear as straight lines and latitude circles (dotted) are seen as circular. *A* is a point somewhat south of the Cape of Good Hope and *B* is close to Port Phillip. The path *ACDB* is a composite great circle route.

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† See *Function*, Vol.2, part 3.

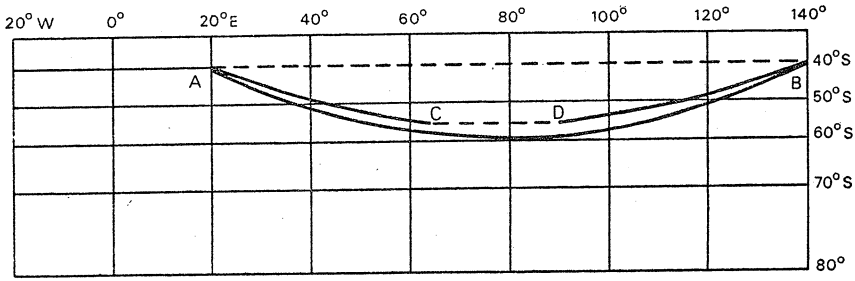


Figure 2

The paths drawn in Figure 1 reappear in the more familiar *Mercator* projection. Note that the great circle no longer appears shorter, due to the distortion introduced by the map.

Great circle navigation required a continuous monitoring of longitude. Determination of longitude presented a major problem for navigation at this time. Ships bound for Port Phillip (whose longitude had not been accurately calculated) faced, after a 5 000 mile trek across the Southern Ocean, the reef-strewn southwestern shore of King Island, if they were only a few miles east of their estimated longitude. A similar discrepancy to the west and their landfall would be out of range of Cape Otway lighthouse, in the region where the *Loch Ard* and a number of other ships foundered.

Longitude discrepancies, in fact, led to disastrous ends for many sailing ship voyages. Adelaide-bound ships, for example, came to grief off Kangaroo Island.

Homeward bound from Australia, well-found ships continued eastwards round Cape Horn. The true great circle route in this case goes well into the Antarctic and a composite route was invariably followed. Cape Horn lies at latitude 55° south, and once this was reached, a rhumb line was followed due east to the landfall.

Determination of longitude by ships at sea had been a problem since world trading for merchant ships had been made possible by the defeat of the Spanish Armada in 1588 and the opening of the sea route around Africa 100 years earlier by the Portuguese navigator Diaz.

Pope Alexander VI had decreed that the Portuguese explore eastwards and the Spanish to the west, but Magellan, a Portuguese who had defected to Spain, discovered and named

Magellan Straits in 1520 and then sailed westwards to the Spice Islands.

World trade by British and Dutch ships started in 1595 when four ships of the Dutch East India Company rounded the Cape of Good Hope to consolidate their East Indian Empire. London merchants then petitioned Queen Elizabeth I to authorise a British expedition in a similar direction. The British East India Company sent four ships round the Cape in 1601, and both companies developed fleets of slow and cumbersome ships requiring fifteen to eighteen months to complete a single voyage.

While latitude could be determined readily from mid-day readings of the sun's altitude, longitudinal differences between landfalls could not be determined. Generally, therefore, these ships followed either coastlines or parallels of latitude to their destinations.

The East Indiamen, after rounding the Cape, followed the African coast north to the eighth parallel of south latitude and then maintained this to a landfall on the coast of Java. This course took them through the doldrums, whose light winds and frequent calms were responsible for the slow voyages.

In 1611, Hendrik Brouwer, in a Dutch East Indiaman, varied this procedure by continuing due east from the Cape of Good Hope for an estimated 3 000 miles, after which he sailed northwards to Java. This gave him the advantage of the high latitude westerlies, and he was able to complete his voyage in a record seven months.

Seven years later, Dirk Hartog in the *Eendracht*, overshot the turning point and made a landfall at what was later named Shark Bay by Dampier. Other Dutch navigators followed Brouwer's example also, although some came to grief off West Australia's inhospitable coast. The most disastrous wreck was that of the *Batavia* on the Abrolhos Islands in 1629.

Although world trade developed during the ensuing 150 years, navigators still had no means of determining their longitude. One step in this direction was the determination of the longitude of known landfalls. Lieutenant James Cook, during his service in North America, prepared a paper on the deduction of longitude from simultaneous observations of an eclipse of the sun. By comparing the local times of the eclipse as recorded in Newfoundland, with those recorded (simultaneously) at Greenwich, he established the longitude of Newfoundland.

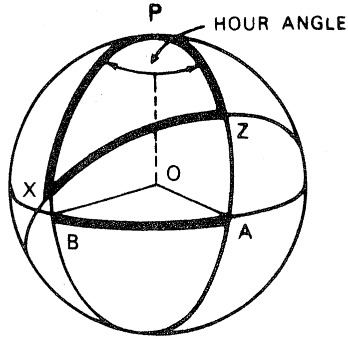
The planet Venus was to transit across the sun in 1769 and the Royal Society successfully petitioned George III to send an expedition to Tahiti. Cook's method was to use this event to establish the longitude of Tahiti, by then a focal point of Pacific exploration.

Cook was appointed leader of the expedition and sailed from Plymouth in HMS *Endeavour* on the 26th of August, 1768. After rounding Cape Horn he reached Tahiti on the 11th of April, 1769, completing his observations there on the 13th of July. He then continued westwards and, after charting the coasts of New Zealand and Australia, reached Batavia (modern Jakarta) on the 11th of October, 1770. (Cook, in fact, recorded the date as the 10th of October, as he did not skip a day on crossing the 180th meridian - now the International Date Line.)

The other major advance in determination of longitude was the introduction of *chronometers*. These were reliable time-pieces which enabled ship's local time, as determined by astronomical observations, to be compared with Greenwich time. Every four minutes in the difference corresponds to one degree of longitude (so that 180° of longitude produces a twelve hour discrepancy).

Figure 3

Three great circles intersect at *P,Z,X* the vertices of a *spherical triangle*. The angle *ZPX* is termed the *hour angle* of a heavenly body above *X*, the observer being at *Z*. Spherical trigonometry is the branch of mathematics used to analyse such "triangles" as *ZPX*, and is used in navigation.



In Britain, a "Board of Longitude" had been established, and a reward of £20,000, a very large sum in those days, was offered for a timepiece which would allow longitude determinations accurate to within 30 miles at the end of a six weeks' voyage.

In 1728, John Harrison produced a chronometer similar to a grandfather clock. However, its pendulum mechanism was useless at sea. He allowed for this in his second model by replacing the pendulum with a pair of straight bar balances so arranged that the motion of the ship accelerated the period of one balance in the same proportion as it retarded the other. Springs controlled the balances and these were so constructed as to compensate for the effects of changing temperatures.

In his later models, he made these coupled balances circular and used a spiral balance spring. His fourth chronometer was only five inches in diameter, and proved to be reliable within the required limits. He did not, however, receive the full reward until a further instrument was produced by his son in 1770. This instrument proved accurate to within 4½ seconds over ten weeks.

Cook took a duplicate of this on his second voyage in HMS *Resolution* (1772 - 1775). This performed well in all climates from the Tropics to the Antarctic.

Thereafter, chronometers became an essential navigation instrument. They were soon supplemented by a system of time signals at every major harbour.

The solution of the "longitude problem" made possible accurate great circle navigation and allowed a considerable reduction in the time taken to sail from England to Australia.

Further Reading

G. Blainey: *The Tyranny of Distance* (Sun Books).

D. Charlwood: *Wrecks and Reputations* (Angus and Robertson).

D. Charlwood: *Settlers under Sail* (Vic. Govt Printer).

P. Mason: *From Genesis to Jupiter* (Aust. Broadcasting Commission).

These books should be available from your school or municipal library.

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THE MODEL MAKER AND THE OOLIGOOJI HIGH DAM

N.S. Barnett,
Footscray Institute of Technology

People become mathematicians, as with other professions, for a variety of reasons. Some enter the profession by design and others because of circumstances. Most enjoy the problem solving situation and some delight in the construction of abstract systems. A few have had their interest in mathematics fired by the enthusiasm of a teacher. Some see mathematics as a means of solving practical problems whilst yet others use practical situations as a stimulus for mathematical modelling, without ever really tackling real life problems. In order to understand this better and to perhaps gain some understanding of the difficulties of mathematical modelling in its practical context, consider the following story. The story is not true; neither have the names been changed to protect the innocent.

Suppose you live in an arid region of Ooligooji and your major concern as you eke out your existence from year to year is water - will there be enough to feed your camel - to water your meagre patch of maize let alone sufficient for your three wives and fifteen children. Your worries are alleviated when your country decides to employ the services of some 'crack' overseas scientists and engineers who, filled with love and concern for the inhabitants of Ooligooji, embark on a scheme to eliminate its nagging water shortage. A major development in this scheme will be the construction of the Ooligooji High Dam, a vast concrete monolith to which boggle-eyed tourists will flock annually, not to mention envious engineers and politicians from around the globe. Plans are drawn up, contracts tendered and vast numbers of locals employed at twice their usual salary (and at half the salary of those brought in from overseas).

Prior to commencement of construction much work has gone on 'behind the scenes'. A survey of the possible sites has yielded, as most appropriate, a location adjacent to the presidential palace, providing the added benefit that El-Presidente can follow closely the progress of construction of this giant edifice. All agree (all, that is, who are deemed worthy of comment) that the dam should be big - large - gigantic if necessary, so as to completely eliminate the dependence of Ooligooji on the whims of mother nature. Just how big should the dam be? El-Presidente has objected to the exhorbitant cost even though 95% of the financing is coming from foreign aid; after all, 5% of a colossal sum, everyone knows, is a large amount. "Never in a hundred years will there be enough water to flood over such a dam!" shouts El-Presidente. "The capacity could be reduced by a half and still meet all our requirements in addition to facilitating new irrigation programs" The engineers, however, have done their sums. They insist that to remove any reasonable possibility of flooding and to permit the proposed irrigation development the dam must be constructed its intended size. You see, the Ooligooji monsoon dumps most of its nation's precipitation in three short weeks of the year and the loss from the proposed dam through evaporation and stagnation during the rest of the year would be enormous.

Whilst the debate rages, deep in the dark recesses of the mathematics department of the Ooligooji National University, a lone mathematician is working industriously (as do all mathematicians). He has contemplated the operation of the proposed Ooligooji High Dam and has broken it down into three fundamental components. At any time the dam will contain a certain volume of water, it will of course have an input (largely dumped in three short weeks) and its operation will involve a draught or release schedule. "Wouldn't it be great," he surmises, "if by suitably characterising these components I could perform the appropriate calculations and present figures to El-Presidente which would establish that the dam could safely be built a fraction of the proposed size." He immediately sets to work but quickly finds that if he is to obtain suitably compact results and if he is not to spend hour upon hour working a computer he will have to make some simplifying assumptions. One of these assumptions is that the amount of input into the dam in any one week (through run-off, rainfall and stream input) is totally independent of the amount of input in any previous week. Moreover, if the release is a fixed amount each week then with some simplifying input assumptions he can calculate, for a given size dam, the probability of the dam flooding over or emptying within a certain number of years. If he further supposes that the reservoir operation quickly settles down to a steady mode of behaviour he can calculate the probability of the dam ever overflowing or going empty. With such steady state behaviour the effect of the initial content of the dam is negligible. Chuckling with excitement he presents his findings before the contracting engineers and of course El-Presidente. The derision which greets his findings reverberates around the presidential palace. "What nonsense," he is told, "to assume that dam inputs are independent from month to month; what about the three-week wet season, at least there should be some regard in the calculations for this."

The assumption that dam release is constant," he is told, "is totally and utterly ludicrous. In reducing the dam operation to the fundamentals of input, content and release you are forced to include in the draught, not only the scheduled release but all losses from the dam through seepage, evaporation, etc. ... and to say such losses are constant is extreme stupidity. The supply for domestic purposes, within bounds, is basically a supply and demand process which to some extent has an inverse relationship to the amount entering the dam. "How then," he is asked, "does he account for the fact that he assumes, in his model, that the release is independent of input? Anyway, if the level of the water in the dam is depleted sufficiently, obviously output will be restricted," - and so his demise continues ad nauseum.

Dejected he returns to his quiet corner of the National University to try and improve on his model. In fact improving his model in accordance with the engineers' comments is no great problem but what it does to his calculations is nobody's business. He is quickly bogged down and it looks as though he will be forced to spend hours computing. It isn't that he has anything against computing but the main advantage of his analytical approach is that the structure of the problem is preserved and he is able to observe how the probabilities are dependent on the input and output characteristics. As he struggles he finds that he can introduce seasonality without too great a difficulty but the interdependence of input and release, that really fouls things up. "But, wait," he says "how have the engineers traditionally proceeded - how have they decided in times past on the size of dams to be built? Of course! By observation of rainfall, experience, intuition, intelligent guesswork and a large margin for error. Anyway, mother nature tends to be unpredictable." "The engineers," he reasons, "are not after exact answers so let me derive some approximate results for the quantities previously alluded to." Carefully avoiding heavy calculations but favouring, rather, elegant mathematics, he proceeds. As he does so he finds that he is able to satisfy a number of the engineers' quibbles regarding assumptions. With his new-found approximations concerned with the effect of dam size on the likelihood of flooding he presents himself once more at the presidential palace. The engineers are quick to inform him that his approximations are still dependent on a number of simplifying assumptions which just cannot be justified in practice. Moreover, and even more damning, they say that approximations without some idea of the degree of approximation are of little use.

Arriving back at his office he telephones a fellow-mathematician who works in the University of Upper Ooligooji. This individual is much more of a pragmatist - "The lesson is simple," he states, "the engineers aren't interested in elegant mathematical argument but rather in simulations and computations." "Look!" he says, "Your dam model and its operation is an interesting stochastic process, doubtless it has a number of interesting properties which make it well worthy of further investigation." So he sits down and manipulates assumptions to suit his mathematical analysis rather than to meet practical demands. Thus a new theory is born, a theory which had its origin in a practical problem yet develops quickly into a problem of sophisticated mathematics. Many more mathematicians subsequently join the team and so the theory (and the learned papers) increase ad infinitum. And what about the engineers? Well, they are free to argue their case with El-Presidente who eventually (with the promise of further financial aid) succumbs to pressure and the mighty edifice begins its construction.

This little story is of course fiction and it certainly isn't meant to ridicule mathematicians or engineers, but rather to point out that their approach to problems frequently differs. There is an area of mathematics called dam theory which treats problems of the sort here mentioned. It is also true to say that this theory developed (largely in Australia) from an effort to model and analyse a practical situation but much of subsequent work in this area has been of purely theoretical interest. Another area of mathematical pursuit which has developed similarly is called queueing theory. This has some similarities with dam theory and much of the original work was centred around the problems of telephone calls arriving at an exchange. Many life situations involve queuing so the study of the operation of such queues so as to minimise waiting time, for example, would be of great interest. Certain production line problems and maintenance problems can be catagorised as queues. A vast theory has developed over the years but only a relatively small amount has found practical usage. Such studies as these have the involvement of three types of person. (i) Those who are basically interested in the practical issues - usable answers are what counts, after all its a real world. (ii) Those who are interested in the theoretical aspects; and the usually small category (iii) those who realise that both (i) and (ii) have something to offer and endeavour to bridge the gap between them.

Just because an area of study is called "applied" doesn't always mean that it is applied! Queueing theory and dam theory are classified as areas of applied probability. It would perhaps be safer to conclude from usage of the word "applied" that the study in question had its origins in an applied problem. Many theoretical mathematicians point to the fact that much that was once accounted of just theoretical interest, many years later has found practical use, so perhaps it is appropriate to conclude with a comment by William Feller, an eminent twentieth century mathematician:

"It has been claimed that the modern theory of probability is too abstract and too general to be useful ... only yesterday the practical things of today were decried as impractical, and the theories which will be practical tomorrow will always be branded as valueless games by the practical men of today!"

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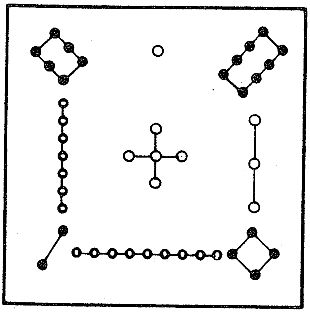
MAGIC SQUARES

P. Greetham, Boronia Technical School

A magic square is a square array of numbers which has the property that all the rows, diagonals and columns add up to the same number.

It seems to have been the Chinese who developed magic squares in approximately the 12th Century, and called them *lo-shu*. Shown below is the lo-shu as it was traditionally drawn by the Chinese, and translated into Arabic numerals. Legend has it that the Emperor Yu spotted the square pattern on the back of a tortoise while walking beside the river Lo.



6	1	8
7	5	3
2	9	4

The lo-shu above is the simplest form of 3 × 3 magic square since it is constructed with each of the integers from 1 to 9. It is called the standard square. Seven (7) other minor versions of this can be formed by reflecting or rotating the square. Ignoring these minor variations, this standard 3 × 3 square is unique. Why?

To answer this question, let the rows, columns and diagonals each add up to m. For the standard square, $m = (1 + 2 + 3 + \ldots + 9)/3 = 15$. Let the central number be a, the top right number be c and the middle number in the top row be b. The square then becomes:

$15 - b - c$	b	c
$a + b + 2c - 15$	a	$15 + a - b - 2c$
$15 - c - a$	$15 - a - b$	$b + c - a$

But the sum of the numbers in the second row is $3m$, which must be equal to 15. Hence $a = 5$. The square now becomes:

$15 - b - c$	b	c
$b + 2c - 10$	5	$20 - b - 2c$
$10 - c$	$10 - b$	$b + c - 5$

(This argument may proceed by a number of alternative routes.)

We may now set up a number of inequalities based on four considerations:

1. Every entry must be positive;
2. Every entry must be less than 10;
3. We may so orient the square that c is the largest corner entry;
4. We may so orient the square that $b < 20 - b - 2c$.

The resulting inequalities reduce to these:

$$b < 5 \qquad \text{(i)}$$
$$c > 5 \qquad \text{(ii)}$$
$$b + c < 10 \qquad \text{(iii)}$$
$$15 < b + 2c < 20 \qquad \text{(iv)}$$

By inequality (i), $b = 1$ or 2 or 3 or 4, and
by inequality (ii), $c = 6$ or 7 or 8 or 9.

But $b \neq 4$, as this would violate inequality (iii), and $c \neq 9$, for the same reason. This leaves nine possible combinations. Of these, four violate inequality (iv), and three others violate inequality (iii). We are left with two possibilities:

$$b = 1,\ c = 8 \text{ or } b = 2,\ c = 7.$$

It is simple now to test these to show that only the first works. (The second gives $15 - b - c = b + 2c - 10 = 6$.)

If the restrictions applying to the standard square are relaxed, somewhat more general squares may be obtained. You may care to investigate these for yourself.

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LETTER TO THE EDITOR

To construct a right-angled triangle given the length of one of the sides (other than the hypotenuse) it is possible, using the following rules, to find suitable lengths for the hypotenuse and the other side.

Let x be the length of the given side,
z be the length of the hypotenuse,
y be the length of the other side,
a be any real number, other than x or 0.

Then

$$y = \left|\frac{(x^2/a) - a}{2}\right| = \left|\frac{x^2 - a^2}{2a}\right| ,$$

$$z = \left|\frac{x^2/a - a}{2} + a\right| = \frac{x^2 + a^2}{2a} .$$

For example, let $a = 12$, $x = 6$. Then $y = 4{\cdot}5$, $z = 7{\cdot}5$. The triangle of sides 6, 4•5, 7•5 is right-angled.

Negative values of a are also possible. E.g. let $a = -18$, $x = 6$. Then $y = 8$, $z = 10$. The 6, 8, 10 triangle is right-angled.

To prove the formula is correct, we form $x^2 + y^2$.

$$\begin{aligned} x^2 + y^2 &= x^2 + \left(\frac{x^2 - a^2}{2a}\right)^2 \\ &= \frac{1}{4a^2}(4a^2x^2 + x^4 + a^4 - 2a^2x^2) \\ &= \left(\frac{x^2 + a^2}{2a}\right)^2 \\ &= z^2, \end{aligned}$$

so that Pythagoras' Theorem is satisfied.

Genet Edmondson,
Year 9 (1979), Presbyterian Ladies' College.

[*A nice observation. It is usually given in mathematics courses in a slightly different, but equivalent, form. If* $x^2 + y^2 = z^2$, *then all possible triples* x,y,z *are given by* $x = 2uv$, $y = u^2 - v^2$, $z = u^2 + v^2$. *A cuneiform tablet known as* Plimpton 322 *allows us to deduce that this result was known to the Babylonians, but we do not know how they arrived at it. Eds.*]

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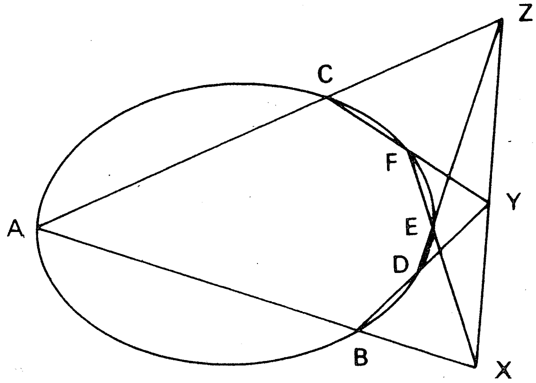
CORRECTION

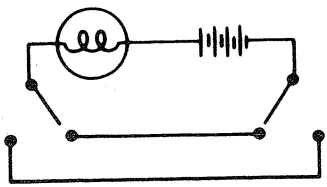
In trying to re-draw Fig.1 in the article by Dame Kathleen Ollerenshaw (*Function* Vol.3, Part 5, p.18), we goofed! The correct figure is



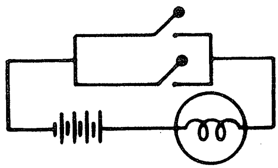
SOLUTION TO PROBLEM 3.4.2

Many hallways have light switches at either end, allowing the light to be operated from each. How can the wiring be arranged to achieve this?

Magnus Cameron, Year 7 (1979) at Glen Waverley High School, Victoria, and Stephen Tolhurst, Year 12 (1979) at Springwood High School, N.S.W., both provided a circuit diagram which solves the problem.



Ravi Sidhu, Grade 10 (1979) Ignatius Park College, Townsville and Stephen Tolhurst noted that two ordinary switches (simpler than those above), if wired in parallel, allow the light to shine if either switch is turned on, but if one switch *is* on, the other doesn't control the light.



Stephen also discussed what happens if the switches are in series. Our correspondents mentioned the importance of these switching circuits in computer logic. Would someone write us an article on that subject?

SOLUTION TO PROBLEM 3.4.3

The following solution was contributed by Andrew Mattingley, Science I (1979) Monash University.

Problem: Let Δ_n denote the number of ways of putting 'n' letters in 'n' addressed envelopes so that every letter goes into a wrong envelope. Derive a formula for Δ_n.

Solution:

Recursion formula: $\Delta_n = (n-1)(\Delta_{n-1} + \Delta_{n-2})$.

$\Delta_1 = 0$

$\Delta_2 = 1$

by formula $\Delta_3 = 2$

$\Delta_4 = 9$

$\Delta_5 = 44$ and so on.

Sample proof:

Let E_i denote the ith envelope and L_i denote the ith letter. If the mailing was to be done properly L_i should go into E_i for all $i \in \{1,2,\ldots,n\}$.

Now for n letters and n envelopes we take L_1 and place it in one of the other envelopes E_j, say, (note that there are '$n-1$' possibilities). Now if L_j goes into E_1 then the remaining letters can all be put into the wrong envelope in Δ_{n-2} ways. The other possibility is that L_j does not go into E_1. Now we have '$n-1$' letters and $n-1$ envelopes where all the letters must go into the wrong envelopes where in this case the right envelope for L_j is E_1. The number of ways in which these letters can be arranged is Δ_{n-1}. So if L_1 goes into E_j the number of ways the letters can all be put in the wrong envelope is

$$\Delta_{n-1} + \Delta_{n-2}.$$

And so for the '$n-1$' possible envelopes L_1 may go into we obtain for Δ_n

$$\Delta_n = (n-1)(\Delta_{n-1} + \Delta_{n-2}) \quad (1)$$

Furthermore with one letter and one envelope it is not possible to enclose the letter incorrectly, hence

$$\Delta_1 = 0,$$

and with two letters and two envelopes

$$\Delta_2 = 1.$$

Then, from (1), $\Delta_3 = 2$, $\Delta_4 = 9$, $\Delta_5 = 44$, etc.

SOLUTION TO PROBLEM 3.5.1

Magnus Cameron also tackled this problem, and found that in Australian Rules Football, the number of points, p, is correctly given by multiplying the number of goals, g, by the number of behinds, b, in five cases. He thought these were the only possible cases, and he was right. We seek non-negative integers g, b, p such that both equations

g	b	p
0	0	0
2	12	24
3	9	27
4	8	32
7	7	49

$$p = 6g + b \qquad (1)$$

$$p = g \times b \qquad (2)$$

hold. Equating (1) and (2) we find

$$g(b - 6) = b \qquad (3)$$

so that g, and $b - 6$, both divide b exactly. Let us suppose that $b = kg$ where k is an integer. Then we must have, from (3),

$$g(kg - 6) = kg,$$

so that either $g = 0$ (leading to Magnus' first solution) or

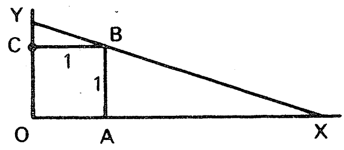
$$kg - 6 = k, \text{ that is, } g = 1 + \frac{6}{k}.$$

Trying $k = 1,2,3,4,5,6$ yields respectively $g = 7,4,3,\frac{5}{2},\frac{11}{5},2$, and $b = 7,8,9,10,11,12$, which include the other cases Magnus found, and two impossible (fractional) cases. Higher values of k only lead to fractional cases.

We still seek solutions to problems 3.3.2, 3.3.5, 3.4.1, 3.5.2, 3.5.3, 3.5.4. Here are some further problems.

PROBLEM 4.1.1 (stolen from the Gazette of the Australian Mathematical Society).

OABC is a square box, of side 1 unit in length. *XY* is a ladder touching the box at *B*. Find conditions under which the length *XY* of the ladder, and its height *OY*, are both rational numbers.

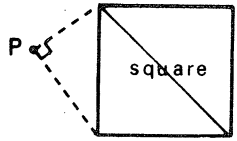


PROBLEM 4.1.2

x,y,z are positive numbers such that each is the product of the other two. What are the values of x,y,z? Is it possible to find six different positive numbers such that each is the product of two of the others?

PROBLEM 4.1.3

Find the locus of the points, *P*, such that when viewed from *P*, a given square subtends an angle of 90°.



PROBLEM 4.1.4

A man seeks a wife. The marriage-agency will present him with three ladies, in random order. But if he wishes to marry one of the ladies, he must make her an offer just after they have been introduced. Except for the last lady, he cannot wait until he has met them all and then decide which is best.

In order to maximize his probability of choosing the best lady, should he

(a) choose the first lady,

(b) ignore the first and choose the second,

(c) ignore the first, and choose the second if she is better than the first. If not, choose the third,

(d) ignore the first two ladies and choose the third?

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

WHAT IS THE CORRECT PERCENTAGE?

A mammoth 92.92 per cent of Sunday's cricket crowd at the MCG behaved themselves (33 arrests; crowd 39 183).

The Age, 12.12.79.

∞ ∞ ∞ ∞

A PREDICTION

After the metric system has been adopted ... and our people have become accustomed to its use, we would no more dream of going back to the present system of weights and measures than we would think of carrying on the processes of arithmetic through the medium of the old Roman letters in place of the Arabic numerals we now employ.

Alexander Graham Bell before the U.S. Congress, February, 1906.

MONASH SCHOOLS' MATHEMATICS LECTURES 1980

Monash University Mathematics Department invites secondary school students studying mathematics, particularly those in years 11 and 12 (H.S.C.) to a series of lectures on mathematical topics. The first lecture of 1980 will be given by the chairman of the Mathematics Department, Professor P. Finch, who will speak on

"Statistical Problems in Medicine"

Friday, March 21, 1980

at 7.00 p.m. in the Rotunda Lecture Theatre R1.

The lectures are free, and open also to teachers and parents accompanying students. Each lecture will last for approximately one hour and will not assume attendance at other lectures in the series. Lecture theatre R1 is located in the "Rotunda", which shares a common entry foyer with the Alexander Theatre. For further directions, please enquire at the Gatehouse in the main entrance of Monash in Wellington Road, Clayton. Parking is possible in any car park at Monash.

Further lectures in the series, all at 7.00 p.m. in R1, will be:

28th March	"How Euclid didn't solve quadratic equations". Professor J. Crossley.
11th April	"Stonehenge and ancient Egypt: the mathematics of radiocarbon dating". Dr R. Clark.
18th April	"Exploring the world with Newtonian mechanics". Professor B. Morton.
2nd May	"Black holes". Dr C. McIntosh.
9th May	"The sundial". Dr C. Moppert.

We hope to publish some of the talks in *Function* later this year, for students who are unable to attend in person.

Please tell your friends at school about the lectures. They don't all subscribe to *Function*!