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FUNCTION

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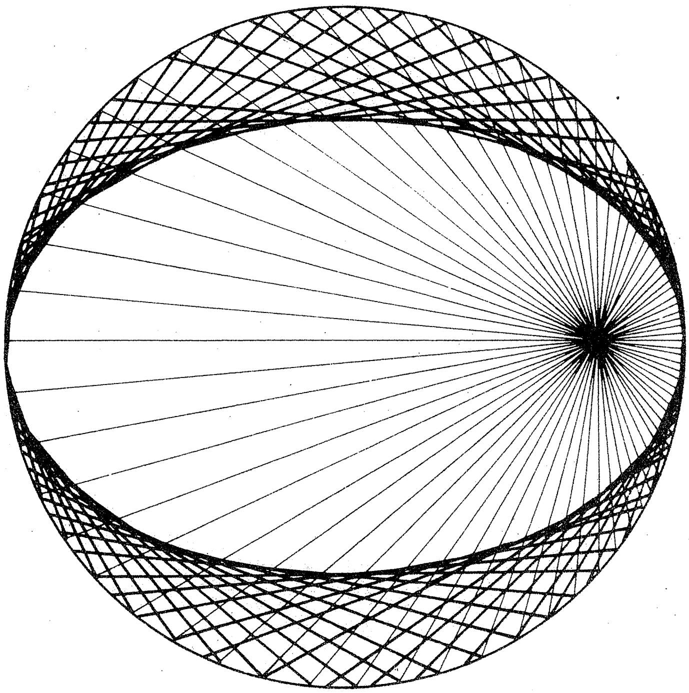

A SCHOOL MATHEMATICS MAGAZINE

Published by Monash University

Function is a mathematics magazine addressed principally to students in the upper forms of schools. Today mathematics is used in most of the sciences, physical, biological and social, in business management, in engineering. There are few human endeavours, from weather prediction to siting of traffic lights, that do not involve mathematics. *Function* contains articles describing some of these uses of mathematics. It also has articles, for entertainment and instruction, about mathematics and its history. Each issue contains problems and solutions are invited.

It is hoped that the student readers of *Function* will contribute material for publication. Articles, ideas, cartoons, comments, criticisms, advice are earnestly sought. Please send to the editors your views about what can be done to make *Function* more interesting for you.

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Articles, correspondence, problems (with or without solutions) and other material for publication are invited. Address them to:

The Editors,
Function,
Department of Mathematics,
Monash University,
Clayton, Victoria. 3168.

Alternatively correspondence may be addressed individually to any of the editors at the addresses shown above.

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This year, had he lived, Albert Einstein would have celebrated his hundredth birthday. This centenary year has focussed attention on his work and has occasioned careful evaluation of his thought. Einstein was not, in the strict sense of the word, a mathematician. He was rather a theoretical physicist, or even a philosopher, who made much use of the mathematics known, but not widely so, in his day, introducing a standard of mathematical rigour that has greatly enriched theoretical physics. His concept of *geometrisation* of physics brought much of that subject into mathematics, although it remains, to some extent, controversial.

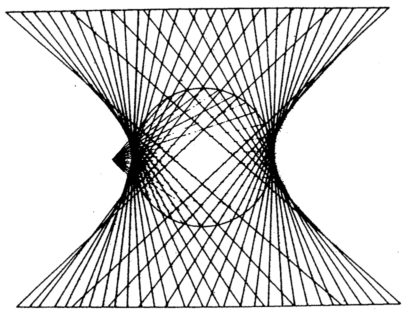
A prediction arising from his work is that there can exist objects known as "black holes". Dr C.B.G. McIntosh writes of these in this issue. Our previous issue dealt briefly with another - the bending of light-rays as they pass a star, such as the sun.

CONTENTS

THE FRONT COVER

J.O. Murphy
Monash University

Conics can be constructed geometrically in many varied ways and the line-envelope method illustrated here for both figures utilizes a circle and a given point rather than the normal focus (point) - directrix (straight line) property usually associated with the form of these curves. Initially, straight lines are drawn, radiating from a selected point, to intersect the circumference of a circle and then further lines are erected normal to these lines, at their points of intersection, to form the envelope. The only difference between the construction of the ellipse on the front cover and the hyperbola on this page is the location of the point in respect to the circle. The selected point is then one of the focal points of the resulting curve and the perpendicular lines are tangents to the curves. In addition, three special cases are evident - they arise, when the point is located at the centre of the circle, on the circumference of the circle or at infinity. The results for the first and third cases could be anticipated as special cases of the diagrams generated here. However, the second case, although trivial, is rather interesting in this context inasmuch that the result is not a parabola.



Some simple and related paper folding procedures associated with a point and a circle can also generate conics. Here the curves are enveloped by the creases folded into the paper. For an ellipse, fold a selected point P, located within the circle, on to the circumference of the circle. A simple geometric argument will establish that the focal points of the resulting ellipse are now P and the centre of the circle. If P is taken outside the circle a hyperbola results and special cases arise again when P is on the circumference of the circle or located at its centre.

BLACK HOLES

C.B.G. McIntosh, Monash University

Black holes are objects which exert gravitational influences but cannot be seen because they do not emit light signals. The first suggestion that such objects exist was made by the French mathematician Pierre Simon de Laplace in 1798. Laplace said "A luminous star, of the same density as the Earth, and whose diameter should be two hundred and fifty times larger than that of the Sun, would not, in consequence of its attraction, allow any of its rays to arrive at us; it is therefore possible that the largest luminous bodies in the universe may, through this cause, be invisible."

Laplace's prediction was made on the basis of Newton's theory of gravitation; he suggests that the gravitational field of the concentrated object is so strong that light emitted by that object would not have sufficient energy to escape the surface of that object. Thus the object would be invisible to an external observer.

It is Albert Einstein's gravitational theory, the general theory of relativity, dating from 1915, that fully predicts and describes black holes; thus it is in terms of this theory that black holes are discussed in this article.

General relativity describes gravitation in terms of the geometry of 4-dimensional spacetime and Einstein gave a set of equations, the "Einstein field equations" of general relativity, which describes the geometry for the spacetime. These field equations have the form

$$\begin{bmatrix}\text{Function of the geometry} \\ \text{of a given region of} \\ \text{spacetime}\end{bmatrix} = \begin{bmatrix}\text{Function of the matter-} \\ \text{energy content of that} \\ \text{region of spacetime}\end{bmatrix}$$

A solution of these equations in a given region thus gives a description of the geometry of that region for a given type of matter-energy content. One of the first solutions of these equations is one given by Karl Schwarzschild in 1916. This describes the geometry of the region of spacetime in vacuum (i.e. the matter-energy content of the region is zero) surrounding a spherical star. Even though this solution has been known for such a long time, it was not until the 1960's that many of its basic physical properties were understood; indeed it was not until the 1960's with the discovery of quasars, cosmic background radiation and neutron stars that much work went into understanding general relativity and its implications for astrophysics and other areas of physics.

In 1963, Schwarzschild's solution of Einstein's equations was generalized by Roy Kerr (a New Zealand mathematician) to include rotation; the geometry is no longer spherical but it is symmetric about the axis of rotation. This was generalized in 1965 by Ezra Newman and co-workers to account for the possibility of the star having a net electric charge. It can be shown, under reasonable assumptions, that this solution with three parameters, M, a and e (for mass, angular momentum and electric charge) is the most general solution of Einstein's equations which has the properties of a black hole.

Many questions now arise: In what ways do these solutions represent black holes? How big are black holes? What other properties do they have? How do they form? How can a black hole be detected? Has one been detected?

Mathematically, then, a black hole is a certain type of solution of Einstein's field equations. Physically, a large black hole is formed when an object such as a star has undergone complete gravitational collapse. Small black holes may have been formed in the big bang at the beginning of the Universe. No light can be emitted by a black hole. No matter can be ejected. Anything that falls into the hole loses its identity.

Consider a black hole of mass M but without angular momentum and electric charge (this is just Schwarzschild's solution). Surrounding this black hole there is a spherical surface of radius $r = 2MG/c^2$ (where G is the gravitational constant and c is the speed of light); this radius is known as the Schwarzschild radius. The formula shows that a black hole of the mass of the sun would be about four miles across! The surface at this radius is known as the event horizon. The word horizon is used because objects such as light rays, radio signals, rocket ships or other stars can cross this surface from the outside to the inside but nothing can cross in the other direction. If you are watching a spaceship go towards the black hole, you will see it disappear from sight; you will never see it again! It goes over the horizon! There is no way in which it could turn round and escape. The people in the spaceship cannot tell you what it is like inside the event horizon because their radio or other signals cannot cross back over the horizon. The spaceship and its occupants would however be acted upon by the extremely strong gravitational field from the black hole. They would be torn apart by the force from this field in the direction of their motion and then the constituent parts would eventually be crushed by the extremely large forces near the centre of the black hole. Thus the occupants could not examine the black hole for very long.

Black holes are thus invisible to someone looking through a telescope. Not only are they black, but their small size means that their resulting angular diameter in the sky would be of the order of magnitude of a million millionth of a second of arc.

The only qualities of a black hole that can be measured are its mass, its angular momentum and its electric charge. We cannot even ask, in a meaningful way, from what elements it is made. Two different stars of equal mass, but of differing composition, can form identical black holes.

The physics of a black hole to an observer thus depends on where he or she is. An observer who chooses to follow matter through the horizon will see it crushed to indefinitely high density; but will also be crushed by indefinitely high forces. This crushing will take place at a finite time (as measured by that observer) after the matter (or observer) has crossed the horizon and is inevitable. The observer has no more power to return to a larger r value (outside the black hole r is the radial distance from the centre of the hole) than we have the power to turn back the hands of the clock of life. The tidal gravitational forces experienced by an object or observer resulting from the black hole are proportional to $1/r^3$. (Tides, even on earth, result from forces of this character.) For a black hole of one solar mass, the force at the event horizon (or Schwarzschild radius $r = 2MG/c^2$) is about a thousand million times the force due to gravitational acceleration on the surface of the Earth. An observer who was near a black hole of this size would be stretched lengthwise in the direction of motion, and crushed sideways by such tidal forces well before he or she reached the event horizon. For a massive black hole (many orders of magnitude greater than the mass of the sun) an observer can cross the event horizon and experience very little force - but can never escape! Inside a black hole of about ten billion times the mass of the sun, the observer could last about a day as measured by an atomic clock he or she might be carrying. But then crushing is inevitable. This crushing means that objects and then the atoms that once formed those objects would be ripped apart by the tidal forces.

On the other hand, an observer who stays a long way from the black hole to watch an object falling through the horizon does not actually see it cross the horizon; he or she measures that it would take an infinite time to do so! However the object does almost suddenly disappear from sight (and from other means of contact) after a finite time as the light from the object is red-shifted enormously and can no longer be seen by the distant observer. (This means that the wavelength of the light emitted becomes progressively longer, and therefore the light appears redder, until the wavelength is so long that the light cannot be seen. This process can take place in extremely short times.)

Black holes more massive than the sun are formed from the gravitational collapse of large stars. When the sun will have used up a lot of its hydrogen in thermonuclear reactions, present theory suggests that it will expand into a red giant and later, after using up more fuel, will contract into a white dwarf of about one hundredth of its present radius. A larger star, say one of twice the solar mass, will probably eventually explode as a supernova and its core will collapse into a neutron star. A neutron star of mass equal to that of a solar mass would have a radius of about one seventy thousandth of the sun's radius!

However there is no stable equilibrium state for stars of more than about three solar masses. So after gravitational collapse it is expected that such a star will collapse into a black hole. The mass of such black holes may increase by the accretion of other material, by the swallowing up of other stars, or after the collision with another black hole; but it may never decrease.

Small black holes may however have been formed in the big bang at the beginning of the Universe. A black hole of about the mass of a fair-sized mountain would have a radius of about 10^{-13}cm! Such black holes cannot be created today as there is no way that there could be the necessary forces to compress material to form such an object. Stephen Hawking in 1975 showed that such black holes will radiate like a black body due to quantum mechanical processes and that the smallest black holes would have radiated away by now. However this is too complicated a story to go into here. Large black holes also radiate but at such a slow rate that they virtually seem not to radiate at all.

And detection? Black holes cannot be detected in isolation; but one hope is that one can be found as a partner of a "live" star in a binary system in which the hole and the star rotate round each other. There is good reason to believe that the X-rays from a source known as Cygnus *X*-1 result from material being torn apart just before it plunges into a black hole, an unseen companion of the massive star HDE 226868. The probable black hole has a mass of about four solar masses; big enough for a collapsed star of this mass to have formed a black hole. There is also evidence that there is a black hole at the centre of the galaxy M87.

The theory of black holes is thus an extremely interesting one. Some ideas of further properties can be found by reading some or all of the following references. More properties remain to be discovered. We must also wait for news that one or more black holes have been discovered. Theory predicts them; there is a very good chance that they exist - but don't go and visit one!

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ALSO RECOMMENDED

Reno Ruffini and John A. Wheeler. "Introducing the black hole". Physics Today, Jan. 1971, 30.

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ANGLE TRISECTION –EXACT AND APPROXIMATE

John Mack, University of Sydney

A recent issue of *Function* (Vol.2, No.5) contains an article on angle trisection. Its sister journal in New South Wales, *Parabola*, also treated this topic recently.† I was reminded of two "angle trisection" constructions sent me over the years. As far as I know, neither of these methods has been used before and they provide some nice mathematics (rather, some nice geometry) for the interested reader.

According to the rules of the game of plane geometry, as played by the ancient Greeks, the equipment to be used consisted of *compasses* and an *unmarked straight edge* (of indefinite length). This equipment could be used to

(a) draw a straight line of indefinite length through two *given* distinct points, and

(b) construct a circle with centre at a *given* point and passing through a second *given* point.

The three games most beloved of the ancients (and played using the above rules) were the *trisection of an arbitrary angle*, the *squaring of a circle* (that is, the construction of the side of a square with area equal to that of a given circle), and the *duplication of a cube* (that is, the construction of the side of a cube with volume equal to twice that of a given cube). Discoveries made by nineteenth century mathematicians have shown that no one can win any of these games. This has not stopped the games being played. For some, it is of interest to see how to change the rules in order to win. For others, interest lies in seeing how close one may get to winning without bending the rules.

All the players provide something interesting to mathematics, and my purpose here is to offer some different examples of *exact* or *approximate* angle trisection.

By introducing co-ordinates into plane geometry, the geometrical problem of angle trisection is transformed into an algebraic problem - namely that of solving a certain cubic equation using the usual operations of arithmetic and the extra operation of taking square roots. (This is developed slowly and

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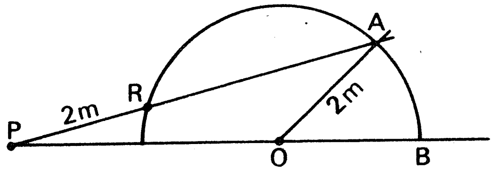
†See reference [1] at the end of the article. For information on *Parabola*, see our April 1979 issue.

clearly in the first chapter of *The Trisection Problem*, a marvellous little book [2] by Robert C. Yates, published by the U.S. National Council of Teachers of Mathematics.) When this is done, it turns out that one can play the game and trisect infinitely many angles, but there are infinitely many angles that cannot be trisected in the required manner. One such impossible angle is 60°. In fact, *one cannot play the game and construct an angle of* 1° - thus the fundamental unit of angle measure that one first meets cannot be constructed with "ruler and compass"!

Although the word *ruler* was used in the line above, it is important to realise that the exact term is *straight edge*. This term is meant to imply that there are no measuring marks on the instrument. Indeed, if we allow just two marks on the edge, we (flout the rules of the game and) effect any trisection! Here is one method of doing it, taken from Yates' book (p.33).

Suppose the two marks are labelled P and R, and are distant $2m$ units apart. Take the given angle AOB to be trisected, and with centre O and radius $2m$ draw a circle to cut the rays of the angle at A and B respectively:



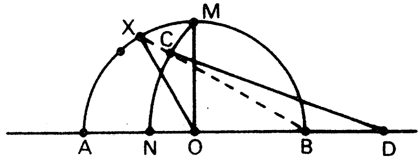
Take the ruler and slide it through A so that P lies on BO produced, and R lies on the circle. Then the angle APO is one-third of the angle AOB. (Prove it!)

Thus weakening the rules to allow markers on the ruler is enough to solve the problem. The problem can also be solved if we allow the use of a single hyperbola or parabola. (There is a curve called the quadratrix which, if allowed, will trisect angles *and* square the circle!)

Let us return to the original game. What sort of accuracy can be achieved by constructions made according to the rules? The simple answer is that remarkably good constructions are known, and new accurate constructions continue to be discovered. The famous German artist Albrecht Dürer in 1525 described a method which is astonishingly good - for angles less than 60% the error is at most *one second* (and so less than 0·0003%), and is at most 18 seconds (0·005%) for any acute angle. This may be found in Yates' book.

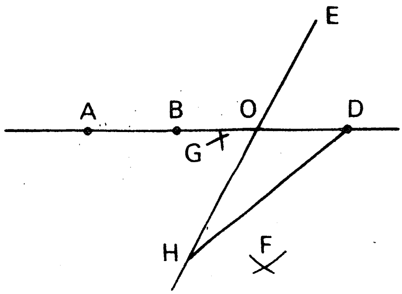
The German mathematician Heinrich Tietze, in his book *Famous Problems of Mathematics* [3] describes a construction found by Eugen Kopf, a tailor who lived in Ludwigshafen early this century. The maximum error in using this has been shown to be always less than eight minutes twelve seconds (less than

0·137%). The method is simple to describe;



AOX is the given acute angle. Construct the semicircle *AMB* and the perpendicular *OM* to *AB*. With centre *B*, radius *MB*, draw the arc *MN*, and let *BX* meet this arc at *C*. With centre *M* and radius *AB*, draw an arc to cut line *AB* at *D*. Then the angle *CDO* is approximately one-third the angle *AOX*. (Try it out.)

About twelve years ago, Mr Alan Martin, of Greenwich, N.S.W., sent me a construction which he thought gave a trisection. This construction is simple, and gives an accurate approximation. I am grateful to Mr Martin for his permission to allow me to describe it.



EOD is the given acute angle. Extend *EO*, *DO* as shown. Locate *B*, *A* so that *DO* = *OB* = *BA*. With centres *B* and *D*, and radius *BD*, locate *F* as shown. With centres *A* and *F*, and radius *FO*, locate *G* as shown. With centre *G* and the same radius *FO*, locate *H* on *EO*. Then the angle *OHD* is approximately one-third of the angle *EOD*.

Remarks 1. This is easy to do. Draw several angles *EOD*, carry out the method and measure *EOD* and *OHD* with a protractor.

2. An appealing feature of this construction (and one which is also found in Kopf's) is that the *angle constructed does not have its vertex at the vertex of the given angle*. Would you have thought of doing that?

3. Mr Martin thought his method was exact, for the following reason. Let the circles with centres *O* and *H* and radius *OD* intersect at *K* (on the same side of *OH* as *D*). Then *H*, *K*, *D* are collinear. Prove that if this is true, then the construction is exact.

4. Using remark 3 as a key, we analyse the construction by letting *H** be the point on *EO*, near *H*, for which

the angle OH^*D is the trisector of EQD. Introducing axes Oxy, with Ox along OD and $D = (1,0)$, it turns out that $H^*(x,y)$ satisfies the equation

$$r^4 - 3r^2 + 2x = 0,$$

where $r^2 = x^2 + y^2$. Regarding the construction as providing an approximation H to H^*, we see that as the given angle EOD varies from 0° to 90°, the locus of H is a circular arc with centre G and radius FO, and that this arc is being used as an approximation to the more complicated locus of H^*. Thus one way of gauging the accuracy of the construction is to plot the paths of H and H^* for varying sizes of EOD, and to see how close they are.

Using axes Oxy as described above, with $D = (1,0)$, some direct calculation produces the following coordinate values.

(a) $x_F = 0$, $y_F = -\sqrt{3}$.

(b) $x_G = \dfrac{\sqrt{105}}{14} - 1$, $y_G = -\dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{35}}{7}$.

(c) $x_H = (x_G + my_G - \sqrt{\{3(1 + m^2) - (y_G - mx_G)^2\}})/(1 + m^2)$,

$y_H = y_G - \sqrt{\{3 - (x_H - x_G)^2\}}$,

where m is the slope of OE (that is, the tangent of EOD).

Letting angle $OED = 3\phi$, angle $OHD = \psi$, and noting that x_H, y_H are both negative for $0° < \phi < 30°$, we find

$$\psi = 3\phi - \tan^{-1}\frac{y_H}{x_H - 1}.$$

From these formulae, the absolute and relative errors can be calculated in terms of ϕ, using a programmable calculator or a computer. Some interesting questions arise:

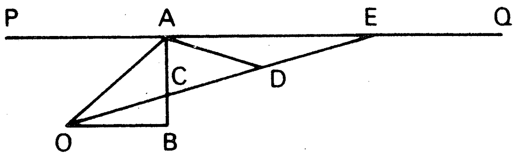
A. For what values of ϕ ($0° \leqslant \phi \leqslant 30°$) is the method exact?

B. Is the error of constant sign?

C. Can the error be estimated, without exact calculation?

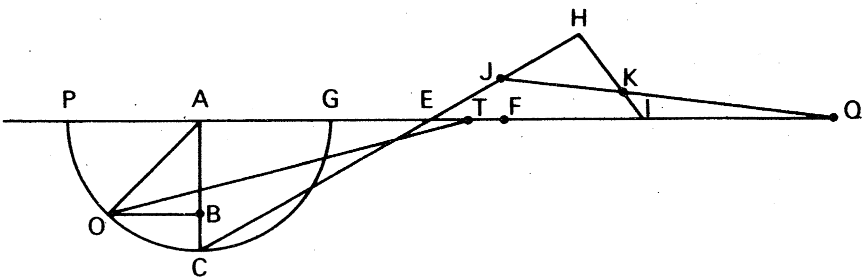
D. How large is the maximum absolute error?

A second trisection construction was posed to me last year by Mr Joseph Keating, of Sydney, N.S.W., who has kindly permitted me to describe it here. This time, however, I shall leave you to develop your own analysis. The idea behind this method is as follows.



Starting with the angle *AOB*, suppose *PQ*, *OB* are parallel and *AB* is perpendicular to *OB*. Suppose *E* can be found on *PQ* so that $AO = AD = CD = DE$. Then angle *COB* trisects angle *AOB*.

Mr Keating's method of approximating the point *E* proceeds thus:



1. With centre *A*, radius *AO*, draw the semicircle *PCG*.

2. Construct *E* on *PG* so that $CE = PG$, and *H* on *CE* with $EH = AO$.

3. Construct *F* on *PG* with $GF = AO$, *I* on *PG* with $FI = AO$, and *Q* on *PG* with $IQ = PC$. (The steps so far are independent of angle *AOB*.)

4. Mark *J* on *EH* so that $HJ = OB$.

5. Let *JQ* meet *HI* in *K*.

6. Construct *T* on *EF* so that $KT = AO$.

7. Angle *TOB* approximately trisects angle *AOB*.

Here are some hints to help you analyse the construction. *E* is exact for trisecting the right angle obtained when *O* moves to *C*. *F* is the limiting position of *T* as *O* approaches *F* (and angle *AOB* approaches 0°). The maximum error in trisecting any acute angle is not greater than 12 minutes approximately.

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TESTING FOR DIVISIBILITY

T.M. Mills, Bendigo C.A.E.

We all know that

- a number is divisible by 2 if its last digit (the one in the units column) is even,
- a number is divisible by 5 if its last digit is 0 or 5, and
- a number is divisible by 3 if the sum of its digits is divisible by 3.

Fewer people know that a number is divisible by 9 if the sum of its digits is divisible by 9. Fewer still know that these are all examples of a single rule. In this article we shall discover this rule. Then you could invent tests for divisibility by 6, or 11, or even 53.

First, we must state the general problem. Let

$$n = d_k\ d_{k-1} \ldots d_2 d_1 d_0$$

be a $(k+1)$ - digit number: d_0 is in the "units column", d_1 is in the "tens column" and so on. Let b be some integer larger than 1. The problem is to determine whether or not b divides n.

Second, we write n in a more convenient form:

$$n = d_0 + 10d_1 + 10^2 d_2 + \ldots + 10^k d_k.$$

Now when we divide n by b we obtain

$$(1) \quad \frac{n}{b} = \frac{1}{b}.d_0 + \frac{10}{b} d_1 + \frac{10^2}{b} d_2 + \ldots + \frac{10^k}{b} d_k.$$

Third, we study the terms $10^j/b$. Suppose that when we divide 10^j by b we are left with a remainder r_j. That is,

$$(2) \quad \frac{10^j}{b} = q_j + \frac{r_j}{b} \qquad j = 1,2,3,\ldots, .$$

Fourth, we use equation (2) in equation (1) and obtain

$$\frac{n}{b} = \frac{1}{b}.d_0 + (q_1 + \frac{r_1}{b})d_1 + (q_2 + \frac{r_2}{b})d_2 + \ldots + (q_k + \frac{r_k}{b})d_k$$

which becomes, after a little rearranging,

$$n/b = (q_1 d_1 + q_2 d_2 + \ldots + q_k d_k) + (d_0 + r_1 d_1 + r_2 d_2 + \ldots + r_k d_k)/b.$$

However, $q_1 d_1 + q_2 d_2 + \ldots + q_k d_k$ is an integer because each q_j and d_j is an integer. Our rule, then, is

RULE: b divides $n = d_k d_{k-1} \ldots d_2 d_1 d_0$ if, and only if

b divides $d_0 + r_1 d_1 + r_2 d_2 + \ldots + r_k d_k$

where the r_j are given by equation (2).

Now let us see how this general rule works. Take $b = 3$, because the rule is well known in this case.

To see if 3 divides $n = d_k d_{k-1} \ldots d_2 d_1 d_0$, we compute the r_j's from equation (2).

$$10^1/3 = 3 + 1/3,$$

$$10^2/3 = 33 + 1/3,$$

$$10^3/3 = 333 + 1/3,$$

and so on. Thus $r_j = 1$ for every j.

Our general rule tells us that

3 divides $n = d_k d_{k-1} \ldots d_2 d_1 d_0$ if, and only if,

3 divides $d_0 + d_1 + d_2 + \ldots + d_k$

which is the usual rule.

A nice test for divisibility by 11 is the following:

11 divides $n = d_k d_{k-1} \ldots d_2 d_1 d_0$ if, and only if,

11 divides $d_0 - d_1 + d_2 - d_3 + - \ldots + (-1)^k d_k$.

Can you derive this by a slight modification of the above discussion?

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NEW MATHS

Computer Science had the absolute highest pass rate due to the large numbers in the course.

University of Melbourne, Faculty of Science, Board of Review Minutes, 5.12.1978.

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MAD MAX

Stunt work nowadays is getting very physics oriented - and very mathematical.

A stuntman, interviewed by Peter Couchman, ATVO, 23.4.1979.

TRAPPING ANIMALS

G.A. Watterson, Monash University

In a forest live two species of animals. A zoologist friend of mine is interested in finding out the numbers of animals of each species, or perhaps some other measures of their relative abundances. Unfortunately in these times of financial stringency, a Monash zoologist can only afford one trap, and even that trap is so small that it holds only one animal. So, every day at noon - because of their nocturnal behaviour, zoologists seldom emerge early in the day - my friend visits the trap in the forest and observes which species of animal has been caught during the last 24 hours, if any. He then releases any trapped animal, and resets the trap for the next 24 hours.

A typical set of results from this experiment might be that in 30 days, species A was caught on 7 days, species B was caught on 11 days, and on 12 days no animal was caught. What can we do with data like these?

I thought of a mathematical model which I knew would lead to easy results. I also knew that he had so little data that there was no way of him finding out whether my model was a good approximation to the real situation! My model was as follows. Suppose there were n_A animals of species A in the forest, and each wandered about the forest randomly. Let p_A be the probability that any animal of species A would come close enough to the trap during a particular day that it would be caught in the trap, unless something was already in the trap. Then the number of species A animals doing this in a particular day would have a binomial probability distribution with mean $\mu_A = n_A p_A$. But n_A is very large and p_A is very small, so that the number of animals of species A coming close to the trap would have approximately a Poisson distribution, mean μ_A, and in fact the probability that *no* species A animal came close to the trap during a particular day would be approximately

$$e^{-\mu_A}.$$

Similarly, suppose that μ_B denotes the expected number of species B animals coming close enough to the trap during a particular day to be caught (if the trap allowed multiple catchings). The actual number would have approximately a Poisson distribution, and the probability that *no* species B animal came near the trap in a particular day would be approximately

$$e^{-\mu_B}.$$

The probability that no animal, of either species, came near the trap on a particular day is

$$e^{-\mu_A} \times e^{-\mu_B} = e^{-(\mu_A+\mu_B)},$$

assuming each species behaves independently of the other. If the observed proportion of days in which no animal was caught is, say, 12/30, then we could estimate $\mu_A + \mu_B$ by solving

$$\frac{12}{30} = e^{-(\mu_A+\mu_B)},$$

that is,

$$\mu_A + \mu_B = -\ln(\tfrac{12}{30}) = 0{\cdot}92. \qquad (1)$$

On each day, we expect $\mu_A + \mu_B$ animals to be near the trap, of which μ_A are of species A and μ_B of species B. On the days when *something* is caught, we might suppose that the probability that it is of species A rather than species B is $\mu_A/(\mu_A + \mu_B)$. For instance, if there were observed 18 days when something was caught, and of these, 7 yielded species A, we could estimate

$$\frac{\mu_A}{\mu_A + \mu_B} \quad \text{by} \quad \frac{7}{18}\,. \qquad (2)$$

From (1) and (2) we now have the estimates

$$\mu_A = \frac{7}{18} \times 0{\cdot}92 = 0{\cdot}36$$

and (3)

$$\mu_B = 0{\cdot}92 - 0{\cdot}36 = 0{\cdot}56.$$

A more detailed study could tell us how accurate (or inaccurate!) the estimates in (3) might be. Note that these estimates are not telling us how many animals of the two species there are in the population, but only how many, on average, could be trapped per day at the site of the trap, assuming that the trap was immediately emptied after each catch (unlike the actual situation). Thus μ_A and μ_B are some measures of species abundance, and we can't do much better than just estimating them with the available data. Moreover, μ_A and μ_B would be important to indicate the average returns to a commercial trapper who might wish to set up his traps at that position in the forest. Of course, his trapping might gradually deplete the species abundances, and his long term average returns may be lower than those initially.

If one wished to estimate the actual numbers n_A and n_B of animals in the forest, a different experiment would help. We could catch m_A and m_B animals of the two species, mark them so that we could recognise them again, and release them. After sufficient time has elapsed for the marked animals to mingle back into the population, we could catch a sample of animals. If, among the species A animals in the sample a proportion π_A were marked, then we could use π_A as an estimate of m_A/n_A.

Thus, $$n_A \approx m_A/\pi_A.$$

Similarly, n_B could be estimated by m_B/π_B, where π_B was the proportion of marked animals among the species B animals in the sample.

When asked whether my analysis of his data was of any help, my zoologist friend replied: "Yes. Tremendous help." I am not one to quarrel with the expert!

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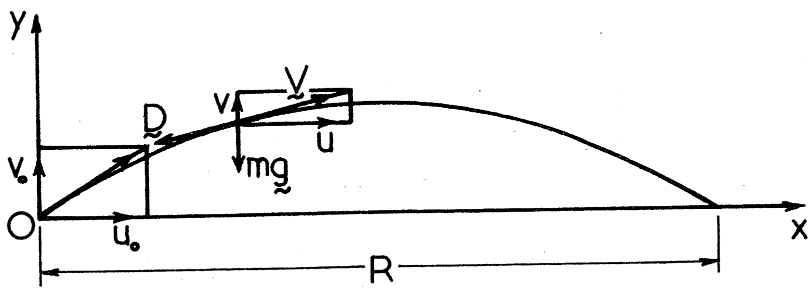
THE LONG JUMP AT MEXICO CITY

M.N. Brearley, R.A.A.F. Academy

A remarkable event occurred in the long jump at the 1968 Olympic Games in Mexico City. The Games record (Harlan, [1]), which had crept up since 1904 by five increments totalling 0·76m, was raised 0·80m by a single mighty leap. At his first (and only) attempt, R. Beamon of the U.S.A. lifted the record from 8·10m to 8·90m, an increase of 9·9%. His achievement may be gauged by noting that a similar improvement in the mile world record time of about 3 min 49 sec would cut it by nearly 23 seconds to about 3 min 26 sec.

It has been suggested that Beamon's leap owed much to the reduced air drag resulting from the lower air density at the high altitude (2260m) of Mexico City. To test this hypothesis it is sufficient to regard the athlete as a projectile in free flight, and to compare the horizontal ranges achieved in the presence of air drag at sea-level and at Mexico City. Only the path of his centre of mass G need be considered, the complications of take-off and landing being irrelevant to the comparison.



In the notation of the figure, the vector equation of motion at the point (x,y) in the presence of air drag $\underset{\sim}{D}$ is

$$m\dot{\underset{\sim}{V}} = m\underset{\sim}{g} + \underset{\sim}{D}. \tag{1}$$

The direction of $\underset{\sim}{D}$ is opposite to that of $\underset{\sim}{V}$, and its magnitude D is known from experimental work of Nonweiler [2] to be

$$D = k\rho V^2, \tag{2}$$

where k is a constant, ρ is the air density, and $V = |\underset{\sim}{V}|$. Then (1) may be written

$$\dot{\underset{\sim}{V}} = \underset{\sim}{g} - KV\underset{\sim}{V}, \tag{3}$$

where

$$K = k\rho/m, \tag{4}$$

which is a constant for any fixed value of ρ.

The Cartesian components of (3) are easily seen to be

$$\dot{u} = -Ku(u^2 + v^2)^{\frac{1}{2}}, \tag{5}$$

$$\dot{v} = -g - Kv(u^2 + v^2)^{\frac{1}{2}}. \tag{6}$$

These non-linear simultaneous differential equations must be solved subject to the initial conditions

$$t = 0,\ x = y = 0,\ u = u_0,\ v = v_0. \tag{7}$$

The equations (5), (6) have no exact solutions in terms of elementary functions. They may be approximated without significant loss of accuracy by using the obvious long jump feature that $v^2 < u^2$ at all times. If we simply neglect v^2 compared with u^2, (5) and (6) become

$$\dot{u} = -Ku^2, \tag{8}$$

$$\dot{v} = -g - Kuv. \tag{9}$$

It is easily verified that (7) and (8) are satisfied by the solution

$$x = (1/K)\log(1 + Ku_0t). \tag{10}$$

It is also easy to find u as a function of t, and this could be used for u in (9) and the resulting equation solved for v and y. Numerical checks show that results which are very nearly as accurate can be obtained by replacing (9) by the cruder approximation

$$\dot{v} = -g. \tag{11}$$

By integrating (11) twice, using the initial conditions (7), and then setting the resulting form for y equal to zero, we obtain the total time of flight as

$$t_1 = 2v_0/g. \tag{12}$$

The range R is shown by (10) and (12) to be

$$\begin{aligned} R &= (1/K)\log\{1 + (2u_0v_0/g)K\}, \\ &= (1/K)\log(1 + KR_0), \end{aligned} \tag{13}$$

where R_0 is the range in the absence of air resistance.

Later it will be apparent that $KR_0 << 1$, so that (13) may be well approximated by the first two terms of the logarithmic series,

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots .$$

This gives the approximation,

$$\begin{aligned} R &\approx (1/K)(KR_0 - \tfrac{1}{2}K^2R_0^2) \\ &\approx R_0 - \tfrac{1}{2}KR_0^2 , \end{aligned} \tag{14}$$

which is a formula for comparing the range R in the presence of air with the range R_0 in vacuo.

As a numerical illustration, let us take

m = 80kg (approximately the mass of Beamon),

$k = 0{\cdot}181\text{m}^2$ (from Nonweiler's data for his Subject C),

$\rho_1 = 1{\cdot}225\text{kg/m}^3$ = air density at sea-level, see [3],

R_0 = 7·9m (for a jump length of about 8·90m, as the feet are about 1m ahead of the centre of mass).

Then (4) yields

$$K_1 = k\rho_1/m = 2{\cdot}772 \times 10^{-3}\ m^{-1},$$

and (14) gives

$$R_1 = R_0 - \tfrac{1}{2}K_1R_0^2 = 7{\cdot}813\text{m}.$$

The reduction in range due to air resistance at sea-level is

$$R_0 - R_1 = 0{\cdot}087\text{m}.$$

At the Mexico City altitude of 2260m, the air density (Gray, [3]) is

$$\rho_2 = 0{\cdot}984\text{kg/m}^3.$$

Then (4) and (14) give

$$K_2 = 2{\cdot}226 \times 10^{-3}\ m^{-1},$$

$$R_2 = 7{\cdot}830\text{m}.$$

The increase in range beyond that at sea-level is

$$R_2 - R_1 = 0{\cdot}017\text{m}$$

which is a trivial gain. The altitude contributed virtually nothing to Beamon's record leap. Even the complete absence of air resistance would have produced an increase of only about 8·7cm in the length of the jump.

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FROM A FORTHCOMING PLAY

The play *Language Takes a Holiday* by Monash mathematician, Aidan Sudbury, will be premiered by the Philosophy Department as part of the Open Day festivities at Monash on August 4th. This dialogue concludes Scene Two. The participants are Professor Fist, editor of the journal *Counterfactual*, and the manager of the Paradise Hotel.

MANAGER To make up for the trouble we have caused you, we shall give you a party today.
FIST That's very kind of you. When will this be?
MANAGER You won't know. This is to be a surprise party.
FIST That's impossible. You agree that if I expect the party in the hour before it occurs that it couldn't be called a surprise.
MANAGER Yes.
FIST So you couldn't give it to me in the last hour of the day, because at eleven o'clock I'd know it was coming in the next hour.
MANAGER 'mm.
FIST And it couldn't be between ten and eleven, because at ten I'd know it couldn't be between eleven and twelve, so I'd expect it in the next hour.
MANAGER 'mm.
FIST And so I can go through the day and eliminate every hour. You can't surprise me now you've told me.
MANAGER The management always keeps its word. Your shower, sir.

The manager does keep his word. Can you explain the paradox?

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ITERATED ARITHMETIC AND GEOMETRIC MEANS

J.K. Mackenzie
C.S.I.R.O. Division of Chemical Physics

Given any two positive numbers a, g the *arithmetic mean*

$$A = \tfrac{1}{2}(a + g) \tag{1}$$

is a familiar and frequently calculated combination; a slightly less familiar combination is the *geometric mean*

$$G = \sqrt{(ag)}. \tag{2}$$

This article will show how these combinations may be used to evaluate certain integrals. The method involves successive calculation of arithmetic and geometric means alternately until a prescribed degree of numerical agreement has been attained. The essentials of these results were discovered by C.F. Gauss in 1797† and have attracted attention again in recent years since they suggest methods for the rapid calculation of inverse trigonometric functions by means of an electronic calculator.

We begin with a discussion of some simple properties of the two means and of their relative magnitudes.

First note that $a = g$ implies $A = G = a = g$, a case that will no longer interest us.

Now, remembering that a, g are positive, we may assume

$$a > g > 0. \tag{3}$$

On expansion

$$(\sqrt{a} - \sqrt{g})^2 = a - 2\sqrt{(ag)} + g = 2(A - G). \tag{4}$$

But the left-hand side is always positive (being a perfect square) so that the arithmetic mean is always greater than the geometric mean. Now put

$$Q(x) = (x - a)(x - g) = x^2 - 2Ax + G^2. \tag{5}$$

$Q(x)$ is negative only if $a > x > g$. But $Q(A) = -(A^2 - G^2)$ and $Q(G) = -2(A - G)G$. Equation (4) tells us that both of these

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† When hc was 20 years old.

expressions are negative, so both means lie between a and g. Thus combining these results

$$a > A > G > g \tag{6}$$

and it is clear that the difference $A - G$ is less than $a - g$. In fact, from equation (4),

$$A - G = \tfrac{1}{2}(\sqrt{a} - \sqrt{g})^2 = \frac{(a - g)^2}{2(\sqrt{a} + \sqrt{g})^2} < \tfrac{1}{2}(a - g), \tag{7}$$

where the last inequality follows from the relation

$$1 - \frac{a - g}{(\sqrt{a} + \sqrt{g})^2} = \frac{2\sqrt{g}}{\sqrt{a} + \sqrt{g}} > 0.$$

A clearer understanding of the relation between $A - G$ and $a - g$ is obtained by regarding a as near g and writing

$$a = g(1 + \varepsilon) \tag{8}$$

with $\varepsilon > 0$. If we replace $\sqrt{a} + \sqrt{g}$ in the third part of relation (7) by the smaller quantity $2\sqrt{g}$, we find that

$$A - G < \frac{1}{8}g\varepsilon^2. \tag{9}$$

Thus, if $\varepsilon < 10^{-s}$ so that the decimal expansions of a and g agree to at least $s - 1$ significant figures, it follows that A agrees with G to at least $2s - 1$ significant figures. Successive repetition of the calculation, replacing A by a and G by g and so on, quickly narrows down the difference to a very small quantity indeed.

This leads naturally to a scheme for calculating what is called the arithmetico-geometric mean $M(a,g)$. Starting from

$$a_0 = a, \; g_0 = g,$$

with, as before, $a > g > 0$, the values

$$a_{n+1} = \tfrac{1}{2}(a_n + g_n), \; g_{n+1} = \sqrt{(a_n g_n)} \tag{10}$$

are calculated in succession for $n = 0,1,2,\dots$.

Equation (4) shows that a_n is always greater than g_n and relation (6) shows that

$$a_n > a_{n+1} \tag{11}$$

$$g_{n+1} > g_n. \tag{12}$$

Thus, the numbers a_n form a decreasing sequence and the numbers g_n an increasing one. By using the final inequality in (7) n times we may prove

$$a_n - g_n < 2^{-n}(a - g), \tag{13}$$

which tends to zero. Thus by (11), (12), and (13), both a_n and g_n tend to the same limit which is denoted by $M(a,g)$, and is called the arithmetico-geometric mean of a and b.

Note that, while the estimate in (13) shows that the iterative scheme in (10) always converges, it only establishes a slower initial rate of convergence than the very rapid ultimate rate of convergence established by (9).

We may use these considerations to calculate values for certain integrals. Write

$$I(A,G) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(A^2 \cos^2\phi + G^2 \sin^2\phi)}} . \tag{14}$$

This is one of the so-called *elliptic integrals* which arise in mechanics, in geometry and in many other branches of mathematics. In mechanics they arise particularly in the theory of the simple pendulum, of the gyroscope and of planetary motion. They cannot be evaluated by elementary means.

On making the substitutions, using Equations (1),(2),

$$\tan\phi = \pm \frac{(a+g)\tan\psi}{a - g\tan^2\psi} \tag{15}$$

into the integral in (14), we can derive (with considerable difficulty) another integral of the same form but with A,G replaced by a,g respectively. Thus

$$I(A,G) = I(a,g) \tag{16}$$

and this is Gauss' version of a transformation discovered by Landen† in 1755.

If we use Equation (16) backwards, successively for $n = 1,2,3,\ldots$, we find, in the notation of Equation (10),

$$I(a,g) = I(a_n,g_n)$$

for all n, and so in the limit as n tends to infinity, and a_n, g_n both converge to $M(a,g)$,

$$I(a,g) = I(M(a,g),M(a,g)) = \frac{\pi}{2M(a,g)} . \tag{17}$$

This equation was used by Legendre†† in 1825 to calculate elliptic integrals. If T is the period calculated by using the simple harmonic formula, valid for small amplitudes, then

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†English mathematician, 1719 - 1790.

††French mathematician who made major contributions to many areas of mathematics, 1752 - 1833.

the period of a simple pendulum, whose amplitude is 2α (i.e. it swings through an angle 4α, 2α each side of the vertical), is given by the formula $T/M(1,\cos\alpha)$. Consider the case $\alpha = 45°$ for which the value $M(1,\frac{1}{\sqrt{2}}) \simeq 0{\cdot}84721$ is rapidly calculated as follows

n	a_n	g_n
0	1·00000	0·70711
1	0·85355	0·84090
2	0·84722	0·84720

These figures are accurate to five decimal places.

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LETTER TO THE EDITOR

GIRLS AND MATHEMATICS

For girls, there is still a good deal of conflict over their role in life - are they going to have a career, or become wives and mothers, or combine the two? There is still the feeling in society that it is primarily the man's role to be the bread-winner, and that if a woman has children she should give up any former career to stay home and look after them. It is extremely difficult, as I have found, to try and combine both. There is still a strong community attitude that maths is not a girls' subject and that it is unfeminine to be good at maths and science.

There also seems to be a general feeling of disillusionment amongst young people over the results of science and technology - wars and armaments, inhuman factories, harmful environmental effects of certain types of development - I'm sure this reaction has led to young people wanting to do subjects like sociology and psychology.

According to a recent survey, about 35% of girls in year 11 take no maths, and this rises to 65% in year 12. These figures don't surprise me. For 3 years recently, I was teaching at an independent girls' school in Melbourne. All girls did some type of maths up to year 10. In year 11, those who were good at maths did maths 1 and 2. This would amount to about 20 girls out of a total of 120 at that level. This group would then do either general maths or pure and applied in year 12. In addition to this, 75 girls out of 120 at year 11 level, did a subject called maths A. This was not nearly as rigorous as maths 1 and 2, and did not demand a very great ability in maths. There were topics such as maths in art, maths in geography, consumer maths, maths in nursing. There was clearly a need for this type of subject, but personally I was not happy with the high proportion of girls doing it. This subject ended at the end of year 11. A number of girls would leave at this stage. Some firms (and hospitals) asked for girls to have a maths subject at year 11 level, and although the actual maths standard was relatively low, this subject (maths A) was the best that could be provided for a large number of girls not particularly gifted mathematically.

Mrs M.G. Tassicker, 15 Wolseley St, Mont Albert.

MATHEMATICAL SWIFTIES

Around the turn of the century, American teenagers thrilled to the exploits of the schoolboy hero, Tom Swift, whose creator, Victor Appleton, churned out story after story in which this redoubtable youngster fought crime, put down revolutions single-handed, and overcame enormous odds to advance the inseparable causes of right and U.S. interest. Tom's laconic speech patterns and Appleton's inept writing style ("Yes, it's an emergency all right", returned Tom slowly) led to a vogue for an esoteric form of humour in the mid-60's - the so-called *Tom Swiftie*:

"Lovely lettuce", remarked Tom crisply,
or better:

"My feet hurt", declared Tom flatly;
"Tough!", Jane replied callously.

The journal *American Mathematical Monthly* pioneered a variant - the "Mathematical Swiftie". Some of the examples below are theirs, others not.

"$x^2 + y^2 = a^2$", Tom stated roundly.

"$|x| + |y| = 1$", declared Tom squarely.

"In the equation $ax + by = c$, neither a nor b is zero", Tom remarked obliquely.

"The tangents to this surface all lie within the surface itself", Tom stated planely.

"Every recurring decimal may be expressed as a fraction", said Tom rationally.

"$0 < x < 1$", said Tom openly.

"This angle is less then $\pi/2$", Tom noted acutely.

"I can't describe the set $\{x|4 < x < 1\}$", Tom muttered emptily.

"That line is perpendicular to the surface", Tom stated normally.

"Is that expansion Taylor-made?", inquired Tom seriously.

"The sets have no elements in common", Tom remarked disjointedly.

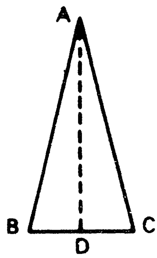
"Did we just cross the event horizon of a black hole?" asked Tom densely.

"The derivative is continuous", went on Tom smoothly.

Many others are, of course, possible. You may care to send us some of your own invention - the editors suggest *Function*ally.

THE ASSES' BRIDGE

If *ABC* is an isosceles triangle, for which *AB* = *AC*, then its base angles *ABC* and *ACB* are equal. See the diagram to the right. This thoerem is Proposition 5 of Book I of Euclid's *Elements*. Before about 50 years ago, it was best known by another name: *Pons Asinorum*, the Asses' Bridge. Coxeter, in his *Introduction to Geometry*, speculates that the name derives from the "bridge-like appearance" of the diagram (especially as drawn in the version of the proof used by Euclid). "Anyone unable to cross this bridge must be an ass" seemed to be the attitude.



Most texts on geometry (at least until recently) proved the result by bisecting the angle *BAC* by the line *AD* and demonstrating the congruence of the triangles *ABD*, *ACD*. The Greek geometer *Pappus* produced an elegant alternative proof around the year 340 A.D. We know of this from the *Commentaries of Proclus*, a Greek text, whose author lived about 100 years later.

Pappus proved the congruence of the triangles *ABC*, *ACB*. (He omitted the line *AD*.) I.e. he took the original triangle and its mirror image, and showed that these could be superimposed. Although British schoolteachers of the period 1850 - 1950 seem to have favoured the angle bisector proof, Pappus' competing one never disappeared from circulation. C.L. Dodgson (Lewis Carroll) discussed it (somewhat disparagingly) in his 1885 text *Euclid and his Modern Rivals*. Heath's influential version of Euclid's *Elements* (now available in a Dover reprint) gives it (Vol.1, p.254). Todhunter, in his version of Euclid's book, reproduces Euclid's original proof, which is different again, but from which the Pappus proof can easily be made to emerge. Euclid, in fact, proved a generalisation, of which the *Pons Asinorum* is a ready corollary. Modern versions of Pappus' proof are given by Coxeter and also by Harold Jacobs in his *Geometry* (published by Freeman in 1974, widely available, and well worth a long read). Jacobs' account is particularly clear and elegantly witty.

An interesting claim appeared about fifteen years ago. This went to the effect that the Pappus proof was invented by a computer. You will find this claim in, for example, Albert Battersbee's *Mathematics in Management* (Pelican, 1966). Similar statements are to be found in writings by I.J. Good, a statistician (in his paper "The Social Implications of Artificial Intelligence", appearing in an anthology, *The Scientist Speculates*, which he edited in 1962), and R.W. Hamming, a computer scientist, whose account (in *American Mathematical Monthly*, Jan. 1963) is rather more sceptical.

Usually, such stories can be traced back to their source via the references supplied by the author of each article. In this case, such a retracing leads only to strong clues as to the tale's beginning. The next part of the account is therefore to some extent speculative. Hamming, and Good (and, through Good, Battersbee) all seem to have been influenced by an IBM project of the late 1950s and early 1960s, devoted to the computer proof of Euclidean theorems. This study was led by H. Gelernter and N. Rochester. Their published papers do not recount the *Pons Asinorum* story, although one (in the *IBM Journal of Research and Development*, Oct. 1958) comes very near.

This paper describes two problem-solving routines. One, applied to the problem in question here, necessarily produces the angle-bisector proof. The other, equally necessarily, gives the Pappus demonstration.

Opinions differ as to the status of such proofs. Hamming is unimpressed, seeing the result as due to routine computer grind. Gelernter and Rochester, on the other hand, see it as genuine, non-routine, machine creativity. As Wang noted in the IBM Journal (1961), it depends on how you define the word "routine".

Battersbee makes a further interesting point. He inquires (in effect) how the Pappus proof would be received by a teacher "wired-up" for the angle-bisector one. There is an interesting piece of documentary evidence on exactly this question. An (in view of the outcome, fortunately) anonymous examiner sent Pappus' proof, which he had found on an examination paper, as a "Howler" to *Mathematical Gazette*. It may still be found there, as *Gleaning* No.1153, p.278 in Vol.21 (1937).

The anonymous scoffer drew a well-merited rebuke in a later issue of *Mathematical Gazette* (p.299, Vol.22, 1938) from C. Dudley Langford.

"I feel that the pupil ... possibly had something rather sound in his mind. Was he not trying to prove that the triangle *ABC* was congruent with its own mirror image? He might have made it more obvious ... but it seems to me to be perhaps not so foolish as it looks at first sight."

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Peregrine ... began to read Euclid ... but he had scarce advanced beyond the *Pons Asinorum*, when his ardor abated.

Peregrine Pickle,
Tobias Smollett, 1751

... the machine did what it was told to do. But then are we so different?

R.W. Hamming.

PROBLEM SECTION

MORE ON PROBLEM 2.4.4

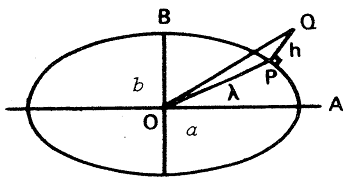
This problem was the one concerning the two marbles dropped from a New York skyscraper. The solution appeared in Vol.3, No.1. John Barton writes that the answer can be found by use of relative displacements and velocities, and the fact that the relative acceleration is zero. This gives the result $s - s' \approx 2\sqrt{s\Delta s}$ rather more easily, and allows us to see the origin of the distance increase in the initial relative velocity of the marbles, a relative velocity maintained throughout the flight.

SOLUTION TO PROBLEM 2.3.2

The question asked was: "What point on the earth's surface is farthest from the earth's centre?"

A correct answer, slightly incomplete, has been received from David Lumsden, 4th Form, Scotch College. The answer below is based on his.

We first note that the heights of mountains are measured above the height mean sea level would occupy at that latitude and longitude. This extended mean sea level defines a shape known as the *geoid*. (Nowadays this can be computed very accurately using satellite orbital data.) Although the geoid is irregular, it may be approximated to very good accuracy by an *oblate spheroid* - the shape produced by rotating an ellipse about its minor (short) axis. For the earth the distances a, b (see diagram at right) are 6378·2km and 6357·8km respectively. The oblate spheroid approximation is accurate to within about 30 metres. The heights of mountains are measured along the perpendicular to the geoid. Hence we require a formula for the length OQ in the figure. λ is the latitude and h the height of the mountain.



An accurate formula for OQ in terms of h, λ is very difficult to obtain, but we may find an excellent approximation to it using the fact that h/a and $(a - b)/a$ are very small. This means that the angle $\angle POQ$ is also very small. We first find the distance OP.

If (x,y) is the position of P, with O as origin and x,y axes along OA, OB respectively, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(This is the equation of the ellipse.) We also have

$$y = x \tan \lambda \quad \text{and} \quad OP = x \sec \lambda,$$

We now find

$$x = ab/\sqrt{b^2 + a^2 \tan^2\lambda}.$$

Hence

$$OP = \frac{ab \sec \lambda}{\sqrt{b^2 + a^2 \tan^2\lambda}} \qquad (*)$$

$$= \frac{a}{\sqrt{\{1 + \frac{a^2 - b^2}{b^2} \sin^2\lambda\}}}.$$

Since $a^2 - b^2$ is much less than b^2, this may be approximated to good accuracy as

$$OP \simeq a(1 - \frac{a^2 - b^2}{2b^2} \sin^2\lambda)$$

i.e.

$$OP \simeq a - \tfrac{1}{2}a\left(\frac{a^2 - b^2}{b^2}\right) \sin^2\lambda.$$

Since $\angle POQ$ is very small, we now have

$$OQ \simeq a + h - \tfrac{1}{2}a \left(\frac{a^2 - b^2}{b^2}\right) \sin^2\lambda.$$

We thus need to find the maximum value of

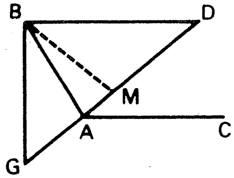
$$h - \tfrac{1}{2}a \left(\frac{a^2 - b^2}{b^2}\right) \sin^2\lambda,$$

or, using the figures given, $h - 21{\cdot}4 \sin^2\lambda$, where h is to be measured in kilometres.

We now need a table of the metric heights of mountains. The latest *Encyclopedia Britannica* has one in its article on mountaineering. The latitudes λ may be found in any good atlas. Only a small number of mountains need be checked. We need to pay close attention to those near the equator (for which λ is very small). We quickly find that all but two mountains can be discounted. These are *Chimborazo* in Ecuador and *Huascarán* in Peru. For Chimborazo, $\lambda = 1° \; 28{\cdot}5'$, $h = 6{\cdot}267$km, while for Huascarán, $\lambda = 9° \; 4{\cdot}8'$, $h = 6{\cdot}768$km. Thus $h - 21{\cdot}4 \sin^2\lambda$ works out to be 6·257km for Chimborazo and 6·233km for Huascarán. (Other mountains are a long way back - two other plausible candidates, Everest and Kilimanjaro, give 4·21km and 5·85km respectively.) The small difference of 24 metres means that we need to check our approximations. When the calculation is carried out more exactly, we still find a comparable difference. The biggest source of error is the irregularity of the geoid, which is about 5 metres higher under Huascarán than it is under Chimborazo. Thus Chimborazo just wins. This fact seems to have been first noted by Isaac Asimov in 1966.

SOLUTION TO PROBLEM 2.5.1

The diagram to the right, for which BD and AC are parallel, and BG is perpendicular to BD, and $GD = 2BA$, arose in connection with Gordon Smith's note "How to Trisect an Angle". The problem was to show that $\angle DAC = \frac{1}{3}\angle BAC$.



To solve the problem, let M be the mid-point of GD and join BM. Now as $\angle DBG$ is a right angle, a semi-circle on DG as diameter passes through B. Clearly, M is the centre of this semi-circle and hence MB is a radius. Thus $MG = MD = BA = BM$. Then the triangle BAM is isosceles, as is the triangle BMD. Then $\angle BAM = \angle BMA$. But $\angle BMA = \angle MBD + \angle MDB$ (as it is exterior to the triangle BMD). But $\angle MBD = \angle MDB = \angle DAC$ (as BD and AC are parallel). Hence $\angle BAM = 2\angle DAC$, and the result follows immediately.

You may have heard that it is impossible to trisect an angle by classical methods. Our method is not classical, as, although it involves only ruler and compass, it requires marks on the ruler; classical constructions do not allow this. (See John Mack's article in this issue.)

SOLUTION TO PROBLEM 2.5.4

The problem read:

Let P be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that $[P(k)]^2 = 1$, prove that $n(P) - \deg(P) \leqslant 2$ where $\deg(P)$ denotes the degree of the polynomial P.

No one solved this delightful problem. Here is a solution.

Clearly, the result holds true if $\deg(P)$ is equal to 1 or 2. Suppose $m = \deg(P) \geqslant 3$. The equation to be solved is $(P(k) - 1)(P(k) + 1) = 0$ i.e. $P(k) = 1$ or $P(k) = -1$. Let u be any integral root of the first equation and v any integral root of the second. Form $P(u) - P(v)$. This clearly has the value 2. But $P(u) - P(v)$ has the form $a_m(u^m - v^m) + \ldots + a_1(u - v)$, which is divisible by $u - v$ as all the coefficients are integers. Hence $u - v$ is a factor of 2 and u,v differ by either 1 or 2. We now examine all the possible cases. If $P(k) = 1$ has no roots, $P(k) = -1$ can have m. But if $P(k) = 1$ has one root, u, $P(k) = -1$ can have at most 4, $u \pm 1$, $u \pm 2$, for a total $n(P) = 5$. Other cases produce even fewer roots. We find the following results:

deg(P)	max $n(P)$
1	2
2	4
3	5
4	5
$m(\geqslant 5)$	m

which shows the formula to hold for all values of deg(P).

SOLUTION TO PROBLEM 3.1.3

This problem required us to show that A,B,C in the diagram opposite are collinear. A number of valid proofs have been submitted. Otis Wright, Year 10, Davidson High School, N.S.W., sent two, and Bob Hale, writing on behalf of a problem solving group at Deakin University, sent a total of four. Both correspondents produced variants of an elegant three-dimensional proof. We print a composite of their letters.

Visualise, instead of three circles, three spheres, whose common tangents make up three interpenetrating cones. The diagram shows the mid-plane of this object. Suppose a plane is tangential to all three spheres (e.g. place the object on a table). By symmetry, there are two such planes. Let them meet in a line ℓ. Obviously both planes are tangential to the cones, which therefore all have their apices on ℓ. Hence the mid-plane of the three cones gives three pairs of straight lines all meeting on ℓ.

The other four proofs all employed only plane methods. All involved some additional construction, and although some were ingenious, none had the immediacy of the one above. This proof is believed to have been invented by a Professor Sweet, an American engineer, early in this century.

We conclude this section with some new problems.

PROBLEM 3.3.1

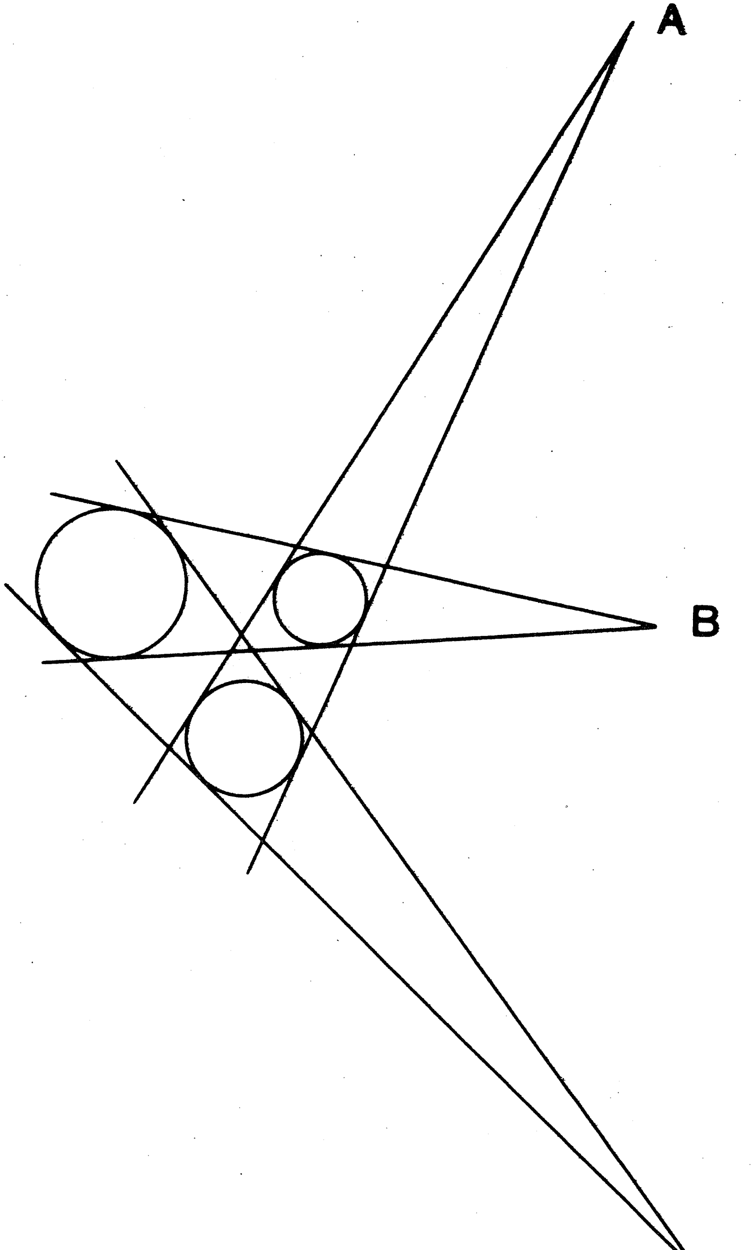
It is easy to see that if two unbiassed dice are tossed, their total can be one of

2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12,

with respective probabilities

$$\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}.$$

Is it possible to construct two dice, individually biassed in such ways that the eleven possible totals are equally likely?


A
B
C

PROBLEM 3.3.2

The hour hand, the minute hand and the second hand of a standard 12-hour clock are all together on the twelve at noon. If the clock keeps perfect time, they are all together again at midnight. Do they coincide at any other time? If so, when? If not, when do they most nearly do so? When do the hands come closest to trisecting the clock-face?

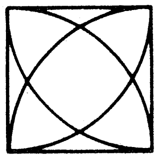
PROBLEM 3.3.3

Before logarithm tables, slide rules and pocket calculators were invented and produced, there were some trigonometric tables available. Multiplication of numbers can be reduced to addition of numbers using such tables. Give an algorithm for multiplying any two numbers (given, say, to four significant figures) using your school trigonometric tables).

PROBLEM 3.3.4

(Submitted by Lindsay Pope, Motueka High School, New Zealand.)

In the diagram at the right, the four circular arcs are quadrants tangent to the sides of the square. Find the area enclosed at the centre between the four quadrants.



PROBLEM 3.3.5

Consider the set $\{2^n$, where $0 \leqslant n \leqslant N\}$ - i.e. the first $N + 1$ powers of 2. Let $p_N(a)$ be the proportion of numbers in this set whose first digit is a. Find $\lim\limits_{N\to\infty} p_N(a)$. Is the first digit of 2^n more likely to be 7 or 8?

∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞ ∞

[Mrs Forrester, an impoverished gentlewoman] sat in state, pretending not to know what cakes were sent up; though she knew, and we knew, and she knew that we knew, and we knew that she knew that we knew, she had been busy all the morning making tea-bread and sponge cakes.

Cranford, Elizabeth Gaskell, 1853

They do not know for certain whether we have recognised them, whether we know that they are murderers. And even if, to be on the safe side, they reckon with it, they cannot be sure that we know that they know that we know that they are murderers!

The Angelic Avengers, Pierre Andrézel, 1946.

MATHEMATICS LECTURES

The series of lectures for 5th and 6th form students at Monash University continues. They are on Friday evenings, from 7 p.m. to 8 p.m. Your school has received a detailed programme.

Lectures are held in the Rotunda Lecture Theatre R1 (enquire at the main gate). The remaining lectures are:

July 6 Choosing the Site of a School, to Minimize the Distance to Three Villages. Dr E. Strzelecki.

July 20 Prime Numbers. Dr R.T. Worley.

August 3 Laputa or Tlön - How Real is the Imaginary? Dr M.A.B. Deakin.

∞ ∞ ∞ ∞ ∞

OPEN DAY

Monash University will hold its Open Day on Saturday, August 4. The Mathematics department will run talks, films, TV interviews with mathematicians, displays (including the popular Foucault pendulum) and will also have counsellors available for consultation.

Also of interest will be computing displays.

These activities will all be located in the Mathematical Sciences building near the North-West corner of the campus.

Note also Aidan Sudbury's play, *Language Takes a Holiday*. Undoubtedly, other activities will also be interesting and informative.

∞ ∞ ∞ ∞ ∞

BACK ISSUES

Did you miss the *Function* article on the Four Colour Problem? On Catastrophe Theory? Did Sir Richard Eggleston's account of Mathematics and Law pass you by? Are you still baffled by the Petersburg paradox? Do you need a full account of Boolean Algebra? Can you solve the baffling problem of the three prisoners? Have you seen the kaleidoscopic curves?

These highlights and much besides are to be found in Volumes 1 and 2 of *Function*. Back issues are still available at $3.50 per complete volume, 90¢ per single copy.